Fuzzy sets. A way to represent ambiguity and subjetivity

Conjuntos difusos. Una forma de representar la imprecisión y la subjetividad

José Rubén Niño Quevedo[1,a]

Abstract. Mathemathical modeling seeks to describe in a formal way a phenomenom but we can encounter two inconveniences, namely, the complexity and the uncertainty by vagueness. In order to take vagueness into consideration, the fuzzy set theory formalized by Zadeh in 1965 intends to give a mathematical treatment to the subjective topics. Additionally, it is considered as an important tool for getting a better understanding of some real situations. This is why we are motivated to give in this paper some of the basics notions of this branch of mathematics which has been in a continuous development for the latest fifty years.

Keywords: fuzzy sets theory, linguistic variable, fuzzy numbers.

Resumen. La modelación matemática busca describir de manera formal un fenómeno pero podemos encontrar dos inconvenientes, a saber, la complejidad y la incertidumbre por "ambigüedad". Para considerar la ambigüedad, la teoría de los conjuntos difusos formalizada por Zadeh en 1965 pretende dar un tratamiento matemático a los temas subjetivos. Adicionalmente, se le considera una herramienta importante para obtener un mejor entendimiento de algunas situaciones reales. Éste en el por qué se motivó a presentar, en este escrito, algunas de las nociones básicas de esta rama de las matemáticas que ha estado en continuo desarrollo durante los últimos cincuenta años.

Palabras claves: teoría de los conjuntos difusos, variable lingüística, números difusos.

Mathematics Subject Classification: 03E72, 94D05, 03B52.

Recibido: mayo de 2016 Aceptado: marzo de 2017

1. Introduction

When it is heard the word fuzzy in mathematics, for instance fuzzy logic, fuzzy arithmetic, fuzzy algebra, fuzzy analysis, fuzzy differntial equations, people often joke about it by saying that it is unclear, probably it is no elaborated

[1]Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia
[a]jrninoq@unal.edu.co

enough, it is blurry or foggy but "Fuzzy sets theory have never been an invitation to fuzzy thinking!" [32]. Fuzzy math is suitable for imprecision and aproximate reasoning because it is an effort to formalize the capability to perform a wide variety of mental tasks without any measurements or any computations [3]. Fuzzy sets were presented in 1965 by Lofti Zadeh [25] in order to give a representation of vague or imprecise concepts expressed in natural languages. Their theory is frecuently confused with probability theory but, the speciality of the fuzzy sets is to capture the idea of partial membership [9] and so, they can be seen like a generalization of the classic sets.

In the mathematical modeling of real world phenomena, the idea is to build transparent models to help people to understand, to get results or conclusions, even to justify the decisions taken. It is known that if the complexity of system increases then our ability to make precise its behaviour decrease [25] and, it is quite probable that we must face with the complexity and our inability to exactly differentiate the involved events in the situation, i.e., there are states which are described in fuzzy terms and where non-crisp boundaries or a subjective judgment is more appropiate because statistical tools are insufficient for building an accurate probabilistic representation [8]; however, nowdays it is possible to consider statistical analysis and fuzzy variables simultaneausly [4, 12]. In these cases, a fuzzy model is adequate because it allows us to convert complex systems into simpler ones and, in several cases, to avoid the complex mathematical modeling.

Fuzzy sets let us to deal with mathematical measure of a wide variety of phenomena and applications tied to human thinking [25]. At first, the impact of fuzzy sets started with the fuzzy logic which provides an approximate but effective way to describe the systems that are not easy to give a precise description. Then, it turned out that problems which considered concepts without a clear definition or were restricted by bivalued logic were successfully modeled through fuzzy sets and fuzzy logic techniques [17]. Nowdays, Fuzzy sets theory has been applied in a huge number of fields and now is a very important field of investigation, as much their mathematical implications as their practical applications [3, 31, 18] at the point of become in a suitable tool to deal with complex or non-linear processes, when it is wanted to represent, even to operate with, concepts which have imprecision or uncertainty like imitating human pattern recognition, judgement or common sense [5]; however, fuzzy sets theory have disadvantages too: Once the rules and membership functions are determinated, they remain fixed and accuracy is improved only by trial and error [14].

The main objective of this paper is to present the fundamentals of fuzzy sets theory because it is wanted to show that fuzzy mathematics is not fuzzy and it might be used for the mathematical representation and modeling of uncertainty. The document has four sections. The second section introduces the notion of linguistic variable; the third one presents basic notions about fuzzy sets and in the last section, fuzzy numbers, their arithmetic and topology are considered. The autor apologizes for English mistakes and grammar errors in the present document.

2. Linguistic Variables

The treatment of precision when the complexity is cosidered implies to explore the use of what might be called linguistic variables, that is, variables whose values are not numbers but words or sentences in a natural or artificial language [29]. A linguistic variable is regarded to degrees of membership and this idea is the centre of fuzzy modeling which, in many cases, reduces the complexity of the model [3, 11, 18, 30, 31], for example in models where the system being considered involves multiple-experts.

Introducing linguistic values for the quantification of a variable is motivated by the possibility of "computing with words" [10, 13, 33], i.e., using words or sentences rather than numbers because these linguistic characterizations are, in general, less specific than numerical ones [26]. Sometimes these (fuzzy) linguistic terms are composed of two parts: A **fuzzy predicate** or base (a word or sentence either you can say to affirm or negate an object or which meaning can contain ambiguity: young, smart, small, tall, low, etc.) and **hedges** (modifiers that change the meaning of the predicate: very, likely, unlikely, extremely, almost, quite, more or less, mostly, few, all, usually, so on).

Based on both the Zadeh's definition [26, 27, 28] and the definition given by Bede [1], a linguistic variable can be defined as follows.

Definition 2.1. A **linguistic variable** x is a linguistic term characterized by the quintuple

$$x := (x, U, T(x), M, \mathcal{G})$$

where,

- x is the **name of the linguistic variable** (linguistic term).
- U is the **universe of discourse**, that is, where the characteristics of the variable can be defined (Also it is denoted by X).
- $T(x)$ is the set of **labels of linguistic values** of x.
- $M : T(x) \to \mathcal{F}(U)$ is a function called **semantic rule** (This function assigns to each label in $T(x)$ a mathematical object which will be called a fuzzy (sub)sets of U).
- $\mathcal{G}$ is named **syntactic grammar**. It produces linguistic values for x from composition of fuzzy sets and a certain type of functions called hedges.

Hedges may be interpreted as a composition between a given function and a basic membership function [1] since they are formalized through functions $h : [0, 1] \to [0, 1]$ that satifies $h(0) = 0$ and $h(1) = 1$.
The following example clarifies the previous definition.

Example 2.2. The linguistic term "Human age" can be transform in a linguistic variable if the quintuple $(Humanage, T, U, G, M)$ is considered. Here:

$x =$ is Human age.

$U = [0, 120]$ is the universe of discourse for Human age.

$T(x) = \{$old, young, very old, very young, very very young, ...$\}$ is the set with the linguistic values taken into account for Human age.

$M : T(x) \longrightarrow \mathcal{F}(U)$ is a function such that $M(young) = Y$ and $M(old) = O$, where $Y = (0, 18, 40)$ and $O = (35, 60, 80)$ (These representation correspond to fuzzy triangular numbers and will be discussed later).

G : The syntax rules can be expressed as follows: If $y \in T(x)$ and *very* has associated the function $h = x^2$, then *very young*, with membership function $h(M(y))$, belongs to $T(x)$.

From Definition 2.1 and Example 2.2 we can see that a linguistic variable works as a kind of translator that assigns to **linguistic terms** a**fuzzy sets** [1]. Fuzzy sets offer a natural interaction between linguistic representations and numerical ones [9], they provide a suitable tool for handling imprecise or ambiguous words.

The following two sections are basically based on the ideas we found in the books written by Bede's [1], Lakshmikantham's [15], Dubois' [9], and Lee's [16]. Basic definitions, theorems and concepts are presented in order to give a background about fuzzy sets and particularly the fuzzy numbers.

3. Fuzzy Sets

The notion of set is one of the most important ones, used frequently in every day life as well as in mathematics [29]. Therefore, it is important to understand what a fuzzy set is. Fuzzy sets are based on classical sets, from now we call them **Crisp (sets)**. The term 'crisp' means not fuzzy and it was introduced by Buckley [2]; so, here in the present text, a crisp number is just a real number, a crisp matrix is a matrix whose elements are real numbers, a crisp solution to a problem is a solution involving crisp sets, crisp numbers, crisp functions and so on.

Example 3.1 (**Crisp Sets**)**.** Let $X := [0, 10]$ be the referential universe. Then, the following are (crisp) subsets of X

- $A := [3, 5]$.
- $B := \{2,\ 3,\ 5,\ 7\}$.
- $C := [0, 10] \cap \mathbb{N}$.
- $D := \left[0, \frac{\pi}{2}\right] \cup \left\{\frac{3}{2}\pi, \frac{5}{2}\pi\right\}$.

In order to define a fuzzy set it is necessary to consider a referential universe and a function. Its formal definition is the following.

Definition 3.2. Let X be a referential universe. We say that the pair $(F, F(x))$ is a **fuzzy (sub)set** of X if F is a subset of X endowed with a fuction

$$\begin{aligned} F : X &\longrightarrow [0, 1] \\ x &\longmapsto F(x) \end{aligned} \tag{1}$$

The function F is usually called **membership function** and sometimes it is denoted by μ_F. The **grade of membership** of x in the fuzzy set F is denoted by $F(x)$ and the **collection of all fuzzy subsets of** X by $\mathcal{F}(X)$.

The value $F(x)$ clasify the element x as: A total included member (if $F(x) = 1$), a no included member (if $F(x) = 0$) and a fuzzy member (if $0 < F(x) < 1$). To ilustrate the previous definition and this observation, the following example is given.

Example 3.3 (**Fuzzy Sets**). Let $X := [0, 10]$ be the referential universe. Then, the following are (fuzzy) subsets of X

- The pair $(A, A(x))$, where $A := [3, 5]$ and

$$A(x) := \begin{cases} 0.3 & x = 3 \\ 0.2 & x \in (3, 4) \\ 0.7 & x = 4 \\ 0.5 & x \in (4, 5) \\ 0.901 & x = 5 \\ 0 & \text{otherwise.} \end{cases}$$

- The pair $(C, C(x))$ where $C := [0, 10] \cap \mathbb{N}$ and

$$C(x) := \begin{cases} 1 & x = 0 \\ \frac{1}{x} & (0, 10] \cap \mathbb{N} \\ 0 & x \notin [0, 10] \cap \mathbb{N}. \end{cases}$$

- The pair $(B, B(x))$ where $B := \{2,\ 3,\ 5,\ 7\}$ and

$$B(x) := \begin{cases} 1 & x \in B \\ 0 & \text{in other case.} \end{cases}$$

- The pair $(D, D(x))$ where $D := \left[0, \frac{\pi}{2}\right] \cup \left\{\frac{3}{2}\pi, \frac{5}{2}\pi\right\}$ and

$$D(x) := \begin{cases} \cos(x) & x = 0 \\ 1 & x \in (0, \frac{\pi}{2}) \\ |\sin(x)| & \text{in other case.} \end{cases}$$

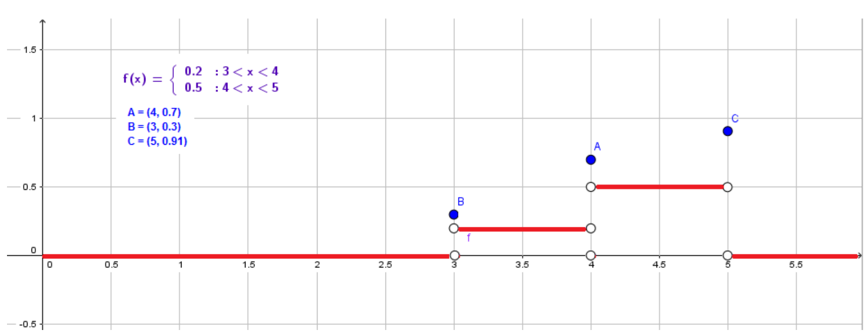


Figure 1: Graph of A(x).

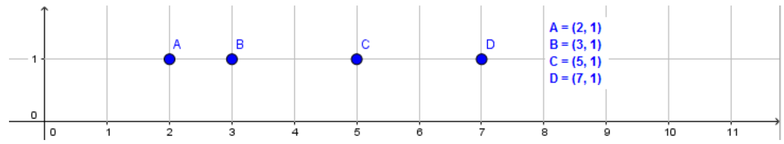


Figure 2: Graph of B(x).

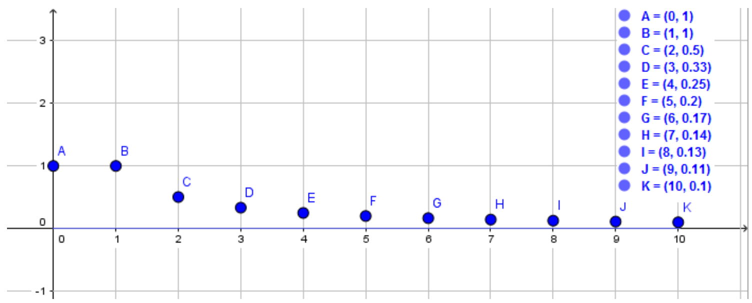


Figure 3: Graph of C(x).

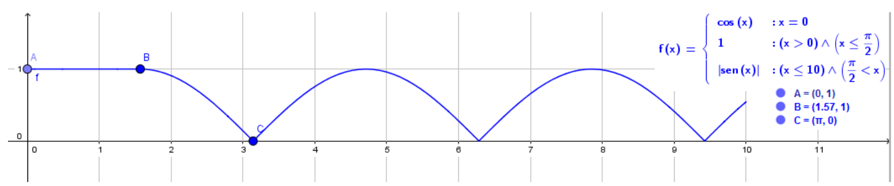


Figure 4: Graph of D(x).

Although the membership is important for crisp sets, its meaning is different in the fuzzy context because **the idea behind fuzzy sets is to capture the idea of partial membership**. In practice, what is really important is to represent correctly the knowledge provided by an expert and capture the meaning he intends to give to his own words [9], it implies that the membership function might change from one person to another and that accuracy to be improved by trial and error [14].

Remark 3.4. The domain of a fuzzy set can be any class and its codomain can be extended to any lattice or poset [1] and the collection $\mathcal{F}(X)$ is an analogous of the crisp power set, indeed, $\mathcal{F}(X) = [0, 1]^X$.

Remark 3.5. The notation $(F, F(x))$ was employed by Zadeh in order to refer to a fuzzy subset F [25] but $\{(x, F(x))\}_{x\in X}$ can be used too. If it is wanted to emphasize the cardinality of the referential universe use the following notation:

A finite set we will denote it by $\dfrac{F(x_1)}{x_1} + \cdots + \dfrac{F(x_n)}{x_n}.$	An enumerable set we will denote it by $\displaystyle\sum_{n\in\mathbb{N}} \frac{F(x_n)}{x_n}.$	A continuous set we will denote it by $\displaystyle\int \frac{F(x)}{x}.$

Remark 3.6. Fuzzy sets can be characterized by its membership function; therefore, hereinafter fuzzy sets will be dealt as functions $F: X \to [0,1]$, where X is a referential universe and will be denote them simply by F instead of $(F, F(x))$. The empty set is defined by the map $\emptyset(x) = 0$ and we will denote it by $\emptyset$ and the total set X is defined by the map $X(x) = 1$ and will be denoted by X.

Remark 3.7. **Every crisp set is also a fuzzy set!** (see Example 3.3). So, it is natural to think that fuzzy sets are a kind of generalization of crisp sets and the membership functions might be treated as generalizations of the traditional characteristic function of a crisp subset of X.

The following is an useful concept to work with fuzzy sets.

Definition 3.8. Let A be a fuzzy set of X, then we define for $\alpha \in [0,1]$, an $\alpha-$**cut** of A as the crisp set

$$A_\alpha := \{x \in X \mid A(x) \geq \alpha\}$$

and, a **strong** $\alpha-$**cut** of A as the crisp set

$$A_{\alpha^+} := \{x \in X \mid A(x) > \alpha\}.$$

The **support** of A is defined as the crisp set A_{0^+} and the **core** of A as the crisp set A_1.

Note that we have that $A_\gamma \subseteq A_\alpha \subseteq A_\beta$ whenever $1 > \beta > \alpha > \gamma > 0$, as it can be seen in Figure 5 but it does not imply that the $\alpha-$cuts must be connected as it is shown in Figure 6

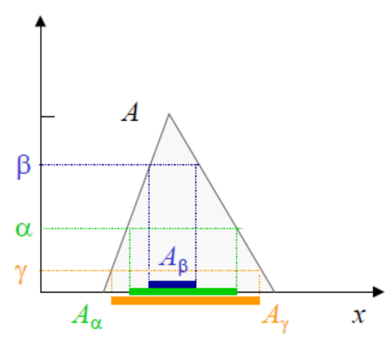


Figure 5: Fuzzy set $\alpha-$cuts.

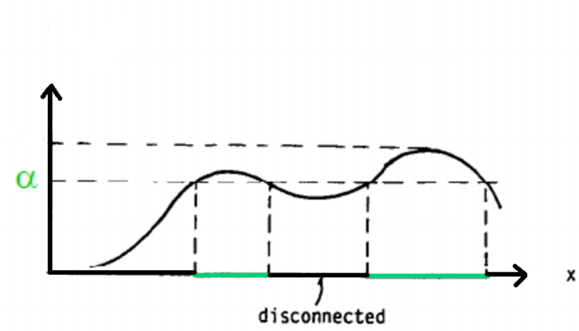


Figure 6: Non connected fuzzy set $\alpha-$cuts.

The following example shows another difference between crisp sets and fuzzy ones.

Example 3.9. Considering the sets given in Example 3.1 and Example 3.3, we have that in the crisp case the core, the support and all the $\alpha-$cuts coincide but, in the Fuzzy case:

- **The** $0.5-cut$ **is:**
  $A_{0.5}=[4,5]$.
  $B_{0.5}=\{2,3,5,7\}$.
  $C_{0.5}=\{0,1,2\}$.
  $D_{0.5}=[0,\frac{5}{6}\pi]\cup[\frac{7}{6}\pi,\frac{11}{6}\pi]\cup[\frac{13}{6}\pi,\frac{17}{6}\pi]\cup[\frac{19}{6}\pi,10]$.

- **Support**:
  $A_{0^+}=[3,5]$.
  $B_{0^+}=\{2,\ 3,\ 5,\ 7\}$.
  $C_{0^+}=[0,10]\cap\mathbb{N}$.
  $D_{0^+}=[0,10]\setminus\{\pi,2\pi,3\pi\}$.

- **Core**:
  $A_1=\emptyset$.
  $B_1=\{2,\ 3,\ 5,\ 7\}$.
  $C_1=\{0,1\}$.
  $D_1=[0,\frac{\pi}{2}]\cup\{\frac{3}{2}\pi,\frac{5}{2}\pi\}$.

Now we will talk about the basic connectives and some operations we can apply on fuzzy sets.

3.1. Basic Connectives

Connectives over fuzzy sets are defined and studied through pointwise operations over the interval $[0,1]$, but defining this kind of operators depends on the membership functions nature. The following are some important operations found out in literature and they are considered as generalizations of those in crisp sets.

Definition 3.10. Let $A,B\in\mathcal{F}(X)$, the following are operations over fuzzy sets of X.

- **Inclusion** For all $x \in X$, $A \subseteq B := A(x) \leq B(x)$.

  The iguality among two fuzzy subsets A, B is valid if and only if for all $x \in X$, $A(x) = B(x)$.

- **Intersection** For all $x \in X$, $A \wedge B(x) := \min\{A(x), B(x)\}$.
- **Union** For all $x \in X$, $A \vee B(x) := \max\{A(x), B(x)\}$.
- **Complementation** For all $x \in X$, $A^c(x) := 1 - A(x)$.

The following example can help the reader to understand the latest definition.

Example 3.11. Considering again the sets given in Example 3.1 and Example 3.3, we have that in the **Crisp case**: $B \subset C$, $A \cap B = \{3, 5\}$ and $A \cup B = [3, 5] \cup \{2, 7\}$. However, in the **Fuzzy case**: $B \nsubseteq C$ because $C(2) = 0.5 < 1 = B(2)$.

- **Intersection.**

$$A \cap B(x) = \begin{cases} 0.3 & x = 3 \\ 0.901 & x = 5 \\ 0 & \text{in other case.} \end{cases}$$

- **Union.**

$$A \cup B(x) = \begin{cases} 1 & x \in \{2, 3, 5, 7\} \\ 0.2 & x \in (3, 4) \\ 0.7 & x = 4 \\ 0.5 & x \in (4, 5) \\ 0 & \text{otherwise.} \end{cases}$$

We can observe that many properties for the crisp sets are preserved for fuzzy ones but the laws of contradiction and excluded middle ("tertio non datur") do not hold for fuzzy sets [1], since that

$$A \wedge A^c(x) \neq \emptyset \qquad \text{and} \qquad A \vee A^c(x) \neq X.$$

For example, if we take in account the fuzzy set A from the Example 3.3, we can observe that $\emptyset \neq A \wedge A^c$ because $A \wedge A^c(3) = 0.3$ and $X \neq A \wedge A^c$ since that $A \vee A^c(3) = 0.7$.

In general, it is no difficult to see that are true for every fuzzy set:

$$0 \leq A \wedge A^c(x) = \min\{A(x), 1 - A(x)\} \leq 0.5$$

and

$$0.5 \leq A \vee A^c(x) = \max\{A(x), 1 - A(x)\} \leq 1.$$

Aditionally, there are operations that depend on the membership function. These operations are unary operations that modify the membership function of the Fuzzy Set. In practice they are known as hegdes and the most common are **Concentration type** and **Dilatation type**. The first one can be related to words like 'very ... ' and reduces the membership grades like $A_C(x) := A(x)^2$ does; the second one can be related to words like 'probably ... ' and increases the membership grades like $A_D(x) := \sqrt{A(x)}$ does.

In addition, we must remark that Zadeh proposed extensions which are important tools in fuzzy set theory and its applications.

Definition 3.12 (**First Zadeh's Extension Principle of** f)**.** Given a function $f : X \longrightarrow Y$, where X and Y are crisp sets, it can be extended to a (**fuzzy**) **function** $F : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ such that $V = F(U)$, where

$$V(y) := \begin{cases} \sup\{U(x) \mid x \in X \text{ and } f(x) = y\} & f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The first Zadeh's extension principle serves for extending a real-valued function into a corresponding fuzzy function. Now, two examples are presented to clarify how the ZEP-1 works.

Example 3.13. Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Now, consider the function $f : X \to Y$ given by $f(a) = f(b) = 1$, $f(c) = f(d) = 2$. Then the fuzzy set $B(x) = \frac{1}{a} + \frac{0.4}{b} + \frac{1}{c} + \frac{0.7}{d}$ in $\mathcal{F}(X)$ is extended to the fuzzy set $F(B) \in \mathcal{F}(Y)$ with values:

- $F(B)(1) = \max\{B(x) \mid x \in f^{-1}(1)\} = \max\{B(a), B(b)\} = \max\{1,\ 0.4\} = 1$,
- $F(B)(2) = \max\{B(x) \mid x \in f^{-1}(2)\} = \max\{B(c), B(d)\} = \max\{1,\ 0.7\} = 1$,
- $F(B)(3) = 0$, because $f^{-1}(3) = \emptyset$.

Hence,

$$F(B) = \frac{1}{1} + \frac{1}{2},$$

that corresponds to the crisp subset $\{1,\ 2\}$ of Y.

Example 3.14. Let $f : [0, 10] \to [0, 1]$ given by $f(x) = 1 - \frac{x}{10}$ and the fuzzy set $A(x) = \frac{x}{10}$ in $\mathcal{F}([0, 10])$. Then, for all $z \in [0, 1]$,

$$F(A)(z) = \sup\{A(x) \mid f(x) = z\} = \sup\left\{\frac{x}{10} \mid 1 - \frac{x}{10} = z\right\} = 1 - z.$$

The second Zadeh's extension principle is a two dimensional case of the first one, i.e., it allows a crisp mapping $f : X \times Y \longrightarrow Z$, where X, Y, and Z are nonempty sets, to be extended to a mapping on fuzzy sets and it is very important because it allows us to extend operations between real numbers to the fuzzy case.

Definition 3.15 (**Second Zadeh's Extension Principle of** f)**.** Given $f : X \times Y \to Z$ and A and B fuzzy sets, of X and Y respectively, it is possible to build a function $f : \mathcal{F}(X) \times \mathcal{F}(Y) \longrightarrow \mathcal{F}(Z)$ by:

$$g(A, B)(z) := \begin{cases} \sup\limits_{z = f(x,y)} \min\{A(x), B(y)\} & z \in \operatorname{Ran} f, \\ 0 & z \notin \operatorname{Ran} f. \end{cases} \tag{3}$$

Remark 3.16. Zadeh's extension is well defined for any fuzzy set $A \in \mathcal{F}(X)$. Indeed, when $f^{-1}(y) \neq \emptyset$, the set $\{A(x) \mid x \in X, f(x) = y\}$ is non-empty and bounded and so, it has a least upper bound.

The following subsection is about how to convert crisp into fuzzy and viceversa and these ideas were taken from Uhrig's and Pedrycz' books [20, 23].

3.2. Fuzzification and Defuzzification

Fuzzification is a kind of process of changing a real scale value into a fuzzy value (that is to give a membership function to a set (or universe)) and it is usually done by experience and analysis; so, a wrong fuzzification of the input variables might cause instability and error of the modeled system [23]. Types of fuzzifiers are **singleton fuzzifier, characteristic fuzzifier, triangular fuzzifier, trapezoidal fuzzifier, gaussian fuzzifier** (see Definitions 4.10, 4.11 and 4.12).

On the other hand, **defuzzify** consists of replacing the fuzzy variable for a crisp one [21]. In order to do that we can use different techniques. The most useful are:

- **Maximum Defuzzification Technique (MDT):** This method gives the output with the highest membership function. If the fuzzy set has membership function μ then the picked element x^* satisfies for all x in the universe:
$$\mu(x^*) \geq \mu(x).$$
- **Centroid Defuzzification Technique (CoG - Center of Gravity):** This method purposed by Sugeno in 1985 is the most commonly used technique and it is very accurate [21]. If $U \in \mathcal{F}(X)$
$$x^* = \frac{\int_W \mu_i(x)x dx}{\int_W \mu_i(x) dx},$$
where $W := U_0$.
- **Weighted Average Defuzzification Technique (WAD):** It is other of the most frequently used in fuzzy applications since it is one of the more computationally efficient methods. Unfortunately, it is usually restricted to symmetrical output membership functions [21]. We can defuzzify by doing:
$$x^* = \frac{\sum \mu_i(x)x dx}{\sum \mu_i(x) dx}$$
- **Center of Area (CoA):**. If $U \in \mathcal{F}(R)$ then this number is defined as the point of the support of U that divides the area under the membership function into two equal parts. If that number is a then it satisfies:
$$\int_{-\infty}^{a} U(x)dx = \int_{a}^{\infty} U(x)dx.$$
- **Expected value and expected interval (EV and EVI)**. If U is a continuous fuzzy number then the expected value is given by
$$EV(U) := \frac{1}{2}\int_0^1 (U_r^- + U_r^+)dr.$$

and the expected interval is

$$EVI(U) := \left[\int_0^1 U_r^- dr, \int_0^1 U_r^+ dr \right].$$

The expected value is the midpoint of the expected interval [21].

The Figure 7 shows an idea where is located the chosen point to represent the defuzzification.

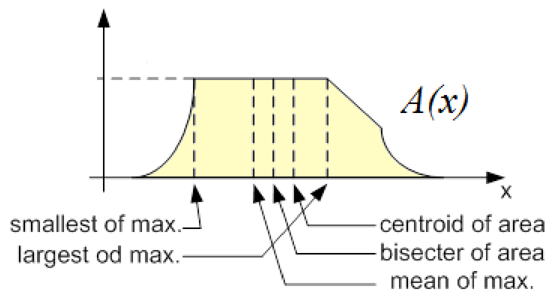


Figure 7: Points that represent some of the defuzzification processes.

4. Fuzzy Numbers

Fuzzy numbers are special fuzzy subsets of the real numbers which are of great importance in fuzzy systems. That is why in this section the fuzzy numbers are defined, described and characterized. In applications, continuous fuzzy numbers are used and, among various shapes of them, triangular (shaped) fuzzy numbers and the trapezoidal (shaped) fuzzy numbers are the most popular ones.

Definition 4.1. A **fuzzy number** is an element of $\mathcal{F}(\mathbb{R})$ whose membership function $U : \mathbb{R} \to [0,1]$ satisfies the following:

1. Exists $x_0 \in \mathbb{R}$ such that $U(x_0) = 1$. (**Normality**).
2. Given $x,\ y \in \mathbb{R}$ and $t \in [0,1]$

$$U(tx + (1-t)y) \geq \min\{U(x), U(y)\}. \text{ (\textbf{Fuzzy Convexity}).}$$

3. For any $x_0 \in \mathbb{R}$, it holds that

$$U(x_0) \geq \lim_{x \to x_0^{\pm}} U(x). \text{ (\textbf{Upper Semi-Continuity}.}$$

4. $\overline{U_{\underset{0}{+}}} = \overline{\{x \in \mathbb{R} | U(x) > 0\}}^{\mathbb{R}}$ is a compact set. (**Compact Support**).

The set of all fuzzy numbers will be denoted by $\mathcal{F}_C(\mathbb{R})$

According to the definition of fuzzy number given above, the fuzzy set represented by the Figure 8 is a fuzzy number. To see example of fuzzy sets which are not fuzzy numbers see Figure 4.

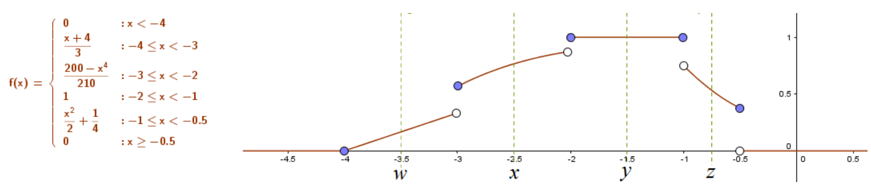


Figure 8: Fuzzy Number.

Remark 4.2. Another ways to define the upper-continuity are [6]:

- For any non-decreasing sequence $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \ldots$ whose limit is α,

$$U_\alpha = \lim_{n \to \infty} \bigcap_{i=1}^{n} U_{\alpha_i}. \text{ In particular,} U_\alpha = \bigcap_{\beta < \alpha} U_\beta.$$

- For all $\epsilon > 0$, exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $U(x) - U(x_0) < \epsilon$

Remark 4.3. **Every real number is a fuzzy number**, i.e. $\mathbb{R} \subset \mathcal{F}_C(\mathbb{R})$, In effect, $\mathbb{R}$ can be identified as

$$\mathbb{R} = \{\chi_x \mid x \in \mathbb{R}\} \qquad \text{where} \qquad \chi_x(y) := \begin{cases} 1 & x = y, \\ 0 & x \neq y. \end{cases}$$

Furthermore, fuzzy numbers generalize closed intervals and the following set may be taken into account:

$$\mathbb{I}_{\mathbb{R}} = \{\chi_{[a,b]} \mid [a, b] \text{is an usual closed interval in } \mathbb{R}\}.$$

Before giving the characterizations of fuzzy numbers, the following lemma is presented.

Lemma 4.4. *If U is fuzzy convex, then U_α is convex for each $\alpha \in I$.*

Proof. [9] Let U be fuzzy convex and $x, y \in U_0$ for some $\alpha \in (0, 1]$, so $U(x) \geq \alpha$ and $U(y) \geq \alpha$. Then, for any $\lambda \in [0, 1]$,

$$U(\lambda x + (1 - \lambda)y) \geq \min\{U(x), U(y)\} \geq \alpha.$$

So, $\lambda x + (1 - \lambda)y \in U_\alpha$. Hence, for all $\alpha \in (0, 1]$, U_α is a convex subset of $\mathbb{R}$. □

4.1. Characterization of Fuzzy Numbers Theorems

The following theorems are employed to characterize the fuzzy numbers and their proofs can be seen in the Bede's book [1].

Theorem 4.5 (**Stacking Theorem**). *If $U \in \mathcal{F}_C(\mathbb{R})$ is a fuzzy number and for $\alpha \in [0,1]$ the sets U_α are its $\alpha-$cuts, then:*

(i) U_α is a closed interval, i.e., for any $\alpha \in [0,1]$:

$$U_\alpha = [U_\alpha^-, U_\alpha^+],$$

where $U_\alpha^- := \inf U_\alpha$ and $U_\alpha^+ := \sup U_\alpha$.

(ii) If $0 \leq \alpha \leq \beta \leq 1$, then $U_\beta \subset U_\alpha$.

(iii) For any sequence α_n which converges from below to $\alpha \in (0,1]$ we have:

$$\bigcap_{n\geq 1} U_{\alpha_n} = U_\alpha.$$

(iv) For any sequence α_n which converges from above to 0 we have:

$$\overline{\bigcup_{n\geq 1} U_{\alpha_n}} = U_0.$$

Proof. Let $U \in \mathcal{F}_C(\mathbb{R})$ be a fuzzy number and U_α, for $\alpha \in [0,1]$, its $\alpha-$cuts, then:

(i) First, note that every set U_α is nonempty and bounded since $U_1 \neq \emptyset$ and the fact $\overline{U_{\underset{0}{+}}}$ is a compact set in $\mathbb{R}$ implies $U_{\underset{0}{+}}$ is bounded. Let U be a fuzzy number and $\alpha \in (0,1]$. If $a, b \in U_\alpha$, then $U(a) \geq \alpha$ and $U(b) \geq \alpha$. Then from the fuzzy convexity, if $x \in [a,b]$ the $x \in U_\alpha$ since

$$U(x) \geq \min\{U(a), U(b)\} \geq \alpha.$$

As a conclusion U_α contains any closed interval $[a,b]$ and so U_α is a convex set. All is left to be proven is that U_α is closed.

From Upper Semicontinuity, if $U(x_0) < \alpha$ then there is an open interval W with $x_0 \in W$ such that $U(x) < \alpha$, for all $x \in W$. Then, the set $\{x | U(x) < \alpha\}$ is open and then its complement is a closed set, i.e., U_r is closed. Therefore, U_α is a closed interval for any $\alpha \in [0,1]$ because on the real line, closed convex sets are closed intervals.

(ii) if $0 < \alpha_1 \leq \alpha_2 \leq 1$ then, if $x \in U_{\alpha_2}$ then $U(x) \geq \alpha_2 \geq \alpha_1$ and so, $x \in U_{alpha_1}$. Onthe otherhand, if $\alpha_1 = 0$ or $\alpha_2 = 0$, the the result is immediate.

(iii) Consider a non-decreasing sequence (α_n) such that converges to α. Then $U_{\alpha_n} \subseteq U_{\alpha_{n-1}}$, is a descending sequence of closed intervals $U_{\alpha_n} = [U^-_{\alpha_n}, U^+_{\alpha_n}]$. By the Nested Interval Theorem[22], we can achieve $U^-_{\alpha_n}$, $U^+_{\alpha_n}$ converge that is $U^-_{\alpha_n} \to a, U^+_{\alpha_n} \to b$ and consequently

$$[a,b] = \bigcap_{n\in\mathbb{N}} U_{\alpha_n}.$$

So, it is enough to show that $U(a) \geq \alpha$ and $U(b) \geq \alpha$. Suppose that $U(a) < \alpha$, then since U is upper semicontinuous, there is a neighborhood W of a, such that $U(x) < \alpha$. This implies the existence of a rank $N \in \mathbb{N}$ with $U(U^-_{\alpha_n}) < \alpha$ for any $n \geq N$. Then since $\alpha_n \to r$ we obtain that there exists $n \in \mathbb{N}$ such that $U(U^-_{\alpha_n}) < \alpha_n$ which is a contradiction. Then it follows that $U(a) \geq \alpha$. Similarly we can show that $U(b) \geq \alpha$ so, $U(x) \geq \alpha$ and then $[a,b] \subseteq U_\alpha$. Additionally, from (ii), we have $U_\alpha \subseteq U_{\alpha_n}$ and it applies $U_\alpha \subseteq [a,b]$. Then finally we get $[a,b] = U_\alpha$, that is,

$$U_\alpha = \bigcap_{n\in\mathbb{N}} U_{\alpha_n}.$$

(iv) Since U_0 is a closed set and $\bigcup_{n\geq 1} U_{\alpha_n} \subseteq U_0$, we have that $\overline{\bigcup_{n\geq 1} U_{\alpha_n}} \subseteq U_0$. Reciprocally, $x \in U_0$ implies that there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \{x \in \mathbb{R} \mid U(x) > 0\}$ that converges to x. Without loss of generality we may assume that $x_n \in U_{\alpha_n} \subseteq \bigcup_{n\geq 1} U_{\alpha_n}$. Then, we obtain $x \in \overline{\bigcup_{n\geq 1} U_{\alpha_n}}$.

So, we have completely proved ths Stacking Theorem. □

The following theorem is the reciprocal of the Theorem 4.5.

Theorem 4.6 (**Negoita - Ralescu Characterization Theorem**). *Given a family of subsets $\{M_\alpha\}_{\alpha\in[0,1]}$ that satisfies the following conditions:*

(i) M_α is a non-empty closed interval for any $\alpha \in [0,1]$.

(ii) If $0 \leq \alpha \leq \beta \leq 1$, we have $M_\beta \subset M_\alpha$.

(iii) For any sequence α_n which converges from below to $\alpha \in (0,1]$ we have:

$$\bigcap_{n\geq 1} M_{\alpha_n} = M_\alpha.$$

(iv) For any sequence α_n which converges from above to 0 we have:

$$\overline{\bigcup_{n\geq 1} M_{\alpha_n}} = M_\alpha.$$

Then there exists a unique $U \in \mathbb{R}_{Fuzzy}$*, such that* $U_\alpha = M_\alpha$*, for any* $\alpha \in [0,1]$.

The following is a characterization through monotonous functions.

Theorem 4.7 (**L - U Representation Theorem**). *Let* U *be a fuzzy number and let* $U_\alpha = [U^-_\alpha, U^+_\alpha] = \{x \mid U(x) \geq \alpha\}$*. Then the functions* $U^-, U^+ : [0,1] \longrightarrow \mathbb{R}$*, defining the endpoints of the* $\alpha-$*cuts, satisfy the following conditions:*

(i) $U^-(\alpha) := U^-_\alpha \in \mathbb{R}$ *is a bounded, non-decreasing, left-continuous function in* $(0,1]$ *and it is right-continuous at* 0.

(ii) $U^+(\alpha) := U^+_\alpha \in \mathbb{R}$ *is a bounded, non-increasing, left-continuous function in* $(0,1]$ *and it is right-continuous at* 0.

(iii) $U^-_1 \leq U^+_1$.

Proof. For a given $U \in \mathbb{R}_{Fuzzy}$, and given $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, from the Stacking Theorem 4.5 we obtain $U_{\alpha_2} \subseteq U_{\alpha_1}$. Then we have, for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$:

$$U^-_{\alpha_1} \leq U^-_{\alpha_2} \leq U^-_1 \leq U^+_1 \leq U^+_{\alpha_2} \leq U^+_{\alpha_1},$$

which implies immediately the monotonicity properties and (*iii*). Left continuity at $\alpha \in (0,1]$ follows from property (*iii*) of the Stacking Theorem 4.5. Indeed, let $\alpha_0 \in (0,1]$ be fixed and (α_n) a increasing sequence converging to α_0, i.e., $\alpha_n \to \alpha_0$. Then from the property (*iii*) of the Stacking Theorem 4.5 we obtain

$$\bigcap_{n \in \mathbb{Z}^+} U_{\alpha_n} = U_{\alpha_0},$$

which immediately implies $U^-_{\alpha_n} \to U^-_{\alpha_0}$ and $U^+_{\alpha_n} \to U^+_{\alpha_0}$, i.e., both functions are left continuous at arbitrary $\alpha_0 \in (0,1]$. In order to prove right continuity at 0 we consider a decreasing seguence (α_n) such that converges to 0. We have

$$U_0 = \overline{\{x \mid U(x) > 0\}} = \overline{\bigcup_{n \in \mathbb{Z}^+} \{x \mid U(x) \geq \alpha_n\}} = \overline{\bigcup_{n \in \mathbb{Z}^+} U_{\alpha_n}}.$$

□

The functions $U^-, U^+ : [0,1] \longrightarrow \mathbb{R}$, defining the endpoints of the $\alpha-$cuts, are denoted in the context of fuzzy numbers by L and R respectively. Additionally, the reciprocal of the Theorem 4.7 is the next result.

Theorem 4.8 (**Goetschel - Voxman Characterization Theorem**). *Let us consider the functions* $U^-, U^+ : [0,1] \longrightarrow \mathbb{R}$*, that satisfy the following conditions:*

(i) $U^-(\alpha) := U^-_\alpha \in \mathbb{R}$ *is a bounded, non-decreasing, left-continuous function in* $(0,1]$ *and it is right-continuous at* 0.

(ii) $U^+(\alpha) := U^+_\alpha \in \mathbb{R}$ *is a bounded, non-increasing, left-continuous function in* $(0,1]$ *and it is right-continuous at* 0.

(iii) $U^-_1 \leq U^+_1$.

Then there is a fuzzy number $U \in \mathbb{R}_{Fuzzy}$ *that has* U^-_α , U^+_α *as endpoints of its* $\alpha-$*cuts,* U_α.

4.2. Types of Fuzzy Numbers

Several types of fuzzy numbers which are often used in applications are described below, we start by giving the most general fuzzy number then particular cases of them are shown.

Definition 4.9. A **L-R Fuzzy Number** is a fuzzy set $U : \mathbb{R} \to [0,1]$ whose membership degree fulfill the following rule:

$$U(x) = \begin{cases} 0 & x < a_0^- \\ L\left(\frac{x-a_0^-}{a_1^- - a_0^-}\right) & a_0^- \leq x < a_1^- \\ 1 & a_1^- \leq x < a_1^+ \\ R\left(\frac{a_0^+ - x}{a_0^+ - a_1^+}\right) & a_1^+ \leq x < a_0^+ \\ 0 & a_0^+ \leq x, \end{cases} \tag{4}$$

where $L, R : [0,1] \longrightarrow [0,1]$ are two continuous, increasing functions fulfilling $L(0) = R(0) = 0$, $L(1) = R(1) = 1$ and $a_0^- \leq a_1^- \leq a_1^+ \leq a_0^+$ are real numbers. The level sets of a L-R Fuzzy Number are given by

$$U_\alpha = [a_0^- + \underline{a}L^{-1}(\alpha), a_0^+ - \overline{a}R^{-1}(\alpha)], \text{ where } \alpha \in [0,1].$$

L-R fuzzy numbers are considered important in the theory of fuzzy sets and they are very useful in applicatons. Symbolically, we write $U = (a_0^-, a_1^-, a_1^+, a_0^+)_{L,R}$, where $[a_1^-, a_1^+]$ is the core of U, and $\underline{a} := a_1^- - a_0^-$ and $\overline{a} := a_0^+ - a_1^+$ which are called the **left spread** and the **right spread** respectively [1].
Particular cases of L-R fuzzy numbers are trapezoidal, triangular and gaussian fuzzy numbers. Reader can see their shapes in Figure 10.

Definition 4.10. A **Triangular Fuzzy Number (TFN)** is characterized by the membership function:

$$U(x) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & a \leq t < b \\ \frac{c-t}{c-b} & b < t \leq c \\ 0 & c < t. \end{cases} \tag{5}$$

It is noted by three numbers $a < b < c$ where the base of the triangle is the interval $[a, c]$ and its vertex is at $x = b$.

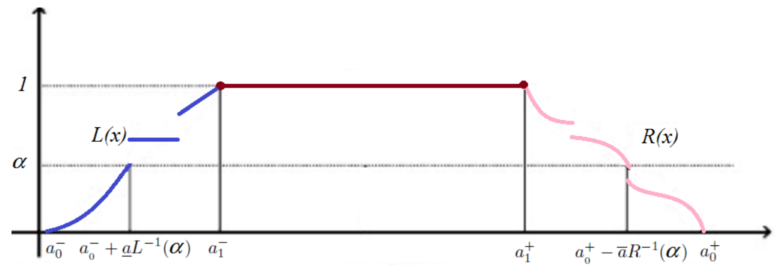


Figure 9: Graphical representation of a L-R fuzzy number.

Definition 4.11. A **Trapezoidal Fuzzy Number (TrFN)** is characterized by the membership function:

$$U(x) = \begin{cases} 0 & t < a \\ \frac{t-a}{b-a} & a \leq t < b \\ 1 & b \leq t \leq c \\ \frac{d-t}{d-c} & c < t \leq d \\ 0 & d < t. \end{cases} \tag{6}$$

It is noted by four numbers $a < b < c < d$ where the base of the trapezoid is the interval $[a, d]$ and its top is over $[b, c]$ and the endpoints of the $\alpha-$level sets are given by

$$U_\alpha^- := a + \alpha(b-a) \quad \text{and} \quad U_\alpha^+ := d - \alpha(d-c).$$

Definition 4.12. Gaussian Fuzzy Numbers (GFN) is characterized by the membership function:

$$U(x) = \begin{cases} 0 & x < x_1 - a\sigma_l \\ \exp\{-\frac{(x-x_1)^2}{2\sigma_l^2}\} & x_1 - a\sigma_l \leq x < x_1 \\ \exp\{-\frac{(x-x_1)^2}{2\sigma_r^2}\} & x_1 \leq x < x_1 + a\sigma_r \\ 0 & x_1 + a\sigma_r \leq x, \end{cases} \tag{7}$$

where x_1 is the core of the fuzzy number, σ_l, σ_r are the left and right spreads and $a > 0$ is a tolerance value.

Gaussian fuzzy numbers often are used in fuzzy control systems.

Remark 4.13. It does not matter if the universe of discourse is restricted to a closed compact interval, because all the fuzzy sets that fulfill normality, fuzzy convexity and upper-semicontinuity become in a fuzzy number.

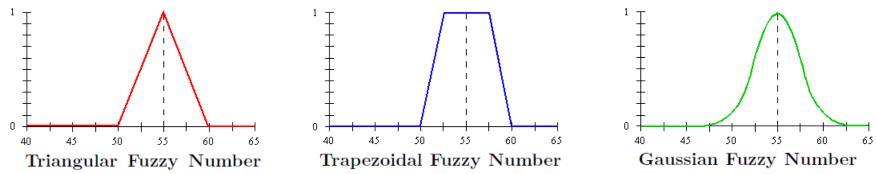


Figure 10: Types of fuzzy numbers the reader can use in applications.

4.3. Fuzzy Arithmetic

Let U and V be two fuzzy numbers then, it is natural to think about their arithmetic. Fuzzy numbers would be little use if there no answer for questions related to these issues [24]. Fortunately, from the Zadeh's extension principle we can connect fuzzy sets with operations and tools of the classical mathematics.

Thus, to define the membership grade of x of the fuzzy aritmethic operations, from the ZEP, the following formula arises:

$$U(x) \star V(y) = \sup_{x \star y = z} \min\{U(x), V(y)\},$$

where $\star \in \{+, -, \times, \div, \vee(\max), \wedge(\min)\}$. It is necessary to keep in mind that these fuzzy operations are not just pointwise operations. The ZEP is a very general result which help to define a fuzzy arithmetic of fuzzy numbers but it can be applied to many other situations, in fact, it can be applied to any crisp relation of function in mathematics to provide an anlogous fuzzy one [24].

Since a fuzzy number U is charactized by having a closed interval as support, i.e. $U_0 = [U_0^-, U_0^+]$ and by using a quadruple $(U_0^-, U_1^-, U_1^+, U_0^+)$ to represent itself, it is possible to combine both ideas with interval arithmetic in order to give a best understanding for fuzzy arithmetic.

From the interval analysis we have the following definitions.

Definition 4.14 (**Arithmetic of Interval Operations**). Given $[a, b]$ and $[c, d]$ in the set $\mathbb{I}_{\mathbb{R}}$ and $\lambda \in \mathbb{R}$,then [24]:

(i) $[a, b] + [c, d] := [a + c, b + d]$.

(ii) $[a, b] - [c, d] := [a - d, b - c]$.

(iii) $\lambda \cdot [a, b] = \begin{cases} [\lambda a, \lambda b] & \lambda \geq 0 \\ [\lambda b, \lambda a] & \lambda < 0 \end{cases}$

(iv) $[a, b] \times [c, d] := [\min A, \max A]$, where $A := \{ac, ad, bc, bd\}$.

(v) $[a, b] \div [c, d] := [a, b] \times [\frac{1}{d}, \frac{1}{c}] = [\min B, \max B]$, where $B := \{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}$. As long as zero does not belong to $[c, d]$ when we divide by this interval.

Hence, we can now characterize $W = U \star V$ through their $\alpha-$cuts, that is, for all $\alpha \in [0,1]$, $W_\alpha = U_\alpha \star V_\alpha$, where $\star = \{+, -, \times, \div\}$ is an arithmetic operation between intervals. The following example explains how fuzzy arithmetic works.

Example 4.15. Let $A, B \in \mathcal{F}_C(\mathbb{R})$ be TFN characterized by the following membership functions

$$U(x) := \begin{cases} 0 & x \leq -1;\ x > 3 \\ \frac{x+1}{2} & x \in [-1,1] \\ \frac{3-x}{2} & x \in (1,3]. \end{cases} \qquad V(x) := \begin{cases} 0 & x \leq 1;\ x > 5 \\ \frac{x-1}{2} & x \in (1,3] \\ \frac{5-x}{2} & x \in (3,5]. \end{cases}$$

Graphically, this two fuzzy sets are

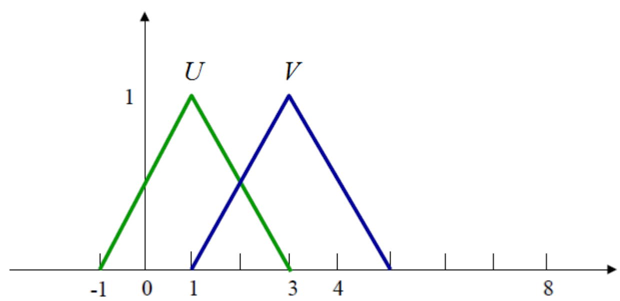


Figure 11: TFN A and B.

Their $\alpha-$cuts are of the form:

$$U_\alpha := [2\alpha - 1, 3 - 2\alpha] \qquad \text{and} \qquad V_\alpha := [2\alpha + 1, 5 - 2\alpha].$$

Hence:

- Addition is given by:

$$U_\alpha + V_\alpha = [4\alpha, 8 - 4\alpha].$$

- Subtraction is given by:

$$U_\alpha - V_\alpha = [4\alpha - 6, 2 - 4\alpha].$$

- Multiplication is given by:

$$U_\alpha \times V_\alpha = [-4\alpha^2 + 12\alpha - 5, 4\alpha^2 - 16\alpha + 15].$$

- and, division is:

$$U_\alpha \div V_\alpha = \left[\frac{2\alpha - 1}{2\alpha + 1}, \frac{3 - 2\alpha}{2\alpha + 1}\right].$$

Pictures of these operations are shown in Figure 12, Figure 13, Figure 14, Figure 15.

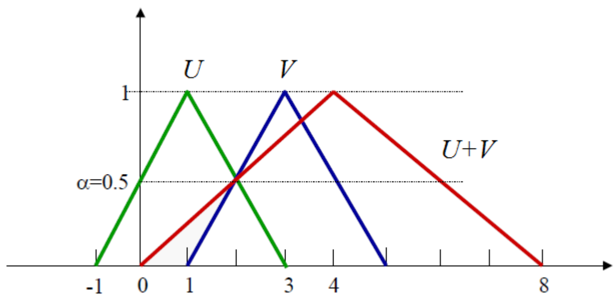


Figure 12: Adition of TFN.

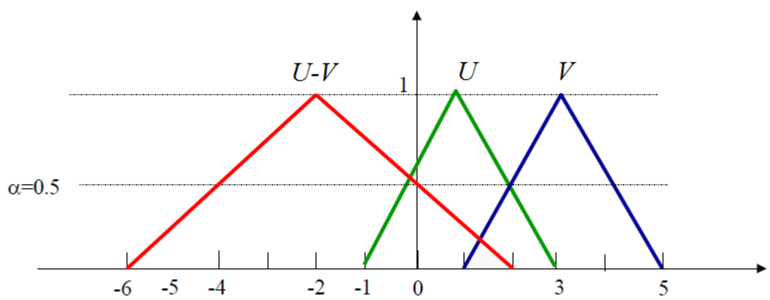


Figure 13: Subtraction of TFN.

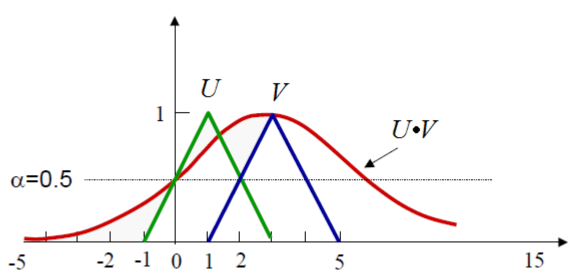


Figure 14: Multiplication of TFN.

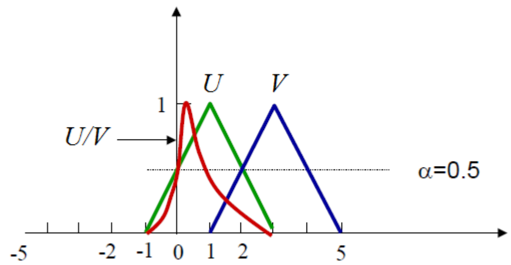


Figure 15: Division of TFN.

Remark 4.16. Addition of fuzzy numbers satifies commutativity, associativity, the existence of a neutral element but **None of** $U \in \mathcal{F}_C(\mathbb{R}) - \mathbb{R}$ **has an opposite in** $\mathcal{F}_C(\mathbb{R})$ (with respect to - w.r.t. - the operation +). Then, a conclusion is that **the space of fuzzy numbers is not a linear space**.

The next definitions are alternative forms to define the subtractions and are very useful to define differentiability in $\mathcal{F}_C(\mathbb{R})$.

Definition 4.17. The **Hukuhara difference (H-difference** $\ominus_H$**)** is defined by

$$U \ominus_H V = W \Longleftrightarrow U = V \dot{+} W,$$

being $\dot{+}$ the standard fuzzy addition.

The Hukuhara difference rarely exists, so several alternatives and generalizations were proposed like the generalized Hukuhara differentiability [1].

Definition 4.18 (($gH-$**difference**))**.** Given two fuzzy numbers $U, V \in \mathcal{F}_C(\mathbb{R})$, the **generalized Hukuhara difference (**$gH-$**difference)** is the fuzzy number W, if it exists, such that

$$U \ominus_{gH} V = W \Longleftrightarrow U = V \dot{+} W \text{ or } V = U - W.$$

Another type of difference used is the next one.

Definition 4.19 (($g-$**difference**))**.** The **generalized difference (**$g-$**difference)** of two fuzzy numbers $U, V \in \mathcal{F}_C(\mathbb{R})$ is given by its level sets as

$$(U \ominus_g V)_\alpha = \overline{\bigcup_{\beta \geq \alpha} U_\beta \ominus_{gH} V_\beta},$$

for all $\alpha \in [0, 1]$ where the $gH-$difference, $\ominus_{gH}$, is with interval operands U_β and V_β and it is well defined in this case.

We must note that for any fuzzy numbers $U, V \in \mathcal{F}_C(\mathbb{R})$ the $g-$difference $U \ominus_g V$ exists and it is a fuzzy number.

4.4. Topology of Fuzzy Numbers and Fuzzy Analysis

The objective of this section is to endow the set of fuzzy numbers with some metrics. Fuzzy sets and fuzzy numbers are more complicated objects than real vectors or real numbers, and we will see that metric distance and the topological development is closer to function spaces like $C([a,b])$ or $L_p([a,b])$ than for $\mathbb{R}^n$ [15]. Consequently, sophisticated functional analytic techniques can be applied to problems of fuzzy analysis [1].

This section is based on ideas which are found in the books written by Bede, Lakshmikantham and Dubois [1, 9, 15]. In order to understand the metric space of the fuzzy numbers some definitions and ideas are presented.

The most well known, and also the most employed metric in the space of fuzzy numbers is the Hausdorff distance for fuzzy numbers which is based on the classical Hausdorff-Pompei distance between compact convex subsets of $\mathbb{R}^n$ [15]. From Zadeh's extension principle we obtain for $\mathcal{F}_C(\mathbb{R})$ the following definition.

Definition 4.20. Given $U, V \in \mathcal{F}(\mathbb{R})$ and $x, c \in \mathbb{R}$; if we consider the functions $f : \mathbb{R}\times\mathbb{R} \to [0,1]$ and $g_c : \mathbb{R} \longrightarrow [0,1]$ defined as $f(x,y) := x+y$ and $g_c(x) := cx$ respectively, the following operations can be defined:

$$U \tilde{+} V(x) := \sup_{a+b=x} \min\{U(a), V(b)\}. \tag{8}$$

$$c \tilde{\cdot} U(x) := \begin{cases} U(x/c) & c \neq 0 \\ \chi_{\{0\}}(x) & c = 0. \end{cases} \tag{9}$$

Unfortunately, if the **space of fuzzy numbers**, $\mathcal{F}_C(\mathbb{R})$, is endowed with this operations then it does not be a linear space.

4.4.1. The space $\mathcal{E}^n$

We start this section by defining the space which will be used through all this subsection.

Definition 4.21. Let us denote by $\mathcal{E}^n$ the space of all fuzzy subsets U of $\mathbb{R}^n$ which satisfy the following:

(i) U maps $\mathbb{R}^n$ onto $I := [0,1]$;

(ii) U_0 is a bounded subset of $\mathbb{R}^n$;

(iii) U_α is a compact subset of $\mathbb{R}^n$, for all $\alpha \in I$;

(iv) U is fuzzy convex, that is, for all $s, t \in \mathbb{R}^n$ and $\lambda \in I$

$$U(\lambda s + (1-\lambda)t) \geq \min\{U(s), U(t)\}.$$

Note that when $n = 1$ we have $\mathcal{E}^n = \mathcal{F}_C(\mathbb{R})$. Moreover, the characterization theorems (Theorem 4.5, Theorem4.6, Theorem4.7, Theorem4.8) as well the properties for the sum and scalar multiplication of $\mathcal{F}_C(\mathbb{R})$ can be extended to $\mathcal{E}^n$ [15]. So, we are going to use them without any warning. To finish this subsubsection, we are going to define the following useful set.

Definition 4.22. For any nonempty subset K of $\mathbb{R}^n$ define the subset

$$\mathcal{E}^n(K) = \{U \in \mathcal{E}^n \mid U_0 \subset K\}.$$

4.4.2. The Hausdorff Metric

Metrics for elements in $\mathcal{F}_C(\mathbb{R})$ are needed for the calculus of fuzzy functions.

Definition 4.23. Let $x \in \mathbb{R}^n$ and A a nonempty subset of $\mathbb{R}^n$.
The **distance from** x **to** A is:

$$d(x, A) := \inf_{a \in A} \| x - a \| . \tag{10}$$

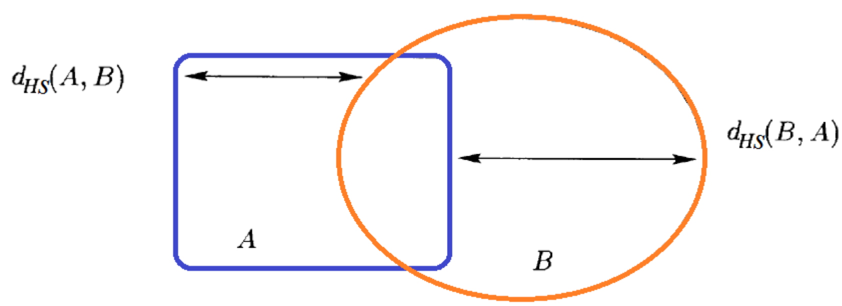
Now, if A and B be two nonempty subsets of $\mathbb{R}^n$ we will define:

Definition 4.24. The **Hausdorff separation** of B from A by

$$d_{HS}(B, A) = \sup_{b \in B} d(b, A). \tag{11}$$

We must to note that in general,

$$d_{HS}(A, B) \neq d_{HS}(B, A).$$



The Hausdorff separations of A and B.

In consequence, we define:

Definition 4.25 (**Crisp Hausdorff distance**)**.** Given A and B nonempty subsets of $\mathbb{R}^n$ then we defines the **Hausdorff distance** between A and B by

$$d_H(A, B) = \max\{d_{HS}(A, B), d_{HS}(B, A)\}. \tag{12}$$

Additionaly, the Hausdorff distance when $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are intervals is [9, 15]:

$$d_H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}. \tag{13}$$

From this, an extension for Lipschitzian condition is the following.

Definition 4.26 (**Lipschitzian Fuzzy Set**). A fuzzy set $U \in \mathcal{E}^n$ is called a **Lipschitzian fuzzy set** if it is a Lipschitz function of its membership grade in the sense that

$$d_H(U_\alpha, U_\beta) \leq K|\alpha - \beta| \tag{14}$$

for all $\alpha, \beta \in I$ and some fixed finite constant K.

An example of the previous definition is the next one.

Example 4.27. A triangular fuzzy number $U \in \mathcal{F}_C(\mathbb{R}) = \mathcal{E}^1$ is characterized by an ordered triple $(U_l, U_m, U_u) \in \mathbb{R}^3$ with $U_l \leq U_m \leq U_u$ such that $U_0 = [U_l, U_u]$ and $U_1 = \{U_m\}$, and

$$U_\alpha = [U_m - (1-\alpha)(U_m - U_l), U_m + (1-\alpha)(U_u - U_m)]$$

for any $\alpha \in I$. In addition

$$d_H(U_\alpha), U_\beta) = |\alpha - \beta| \max\{(U_m - U_l), (U_u - U_m)\},$$

so all the triangular fuzzy numbers are Lipschitzian, where, in this case, the constant is $K := \max\{(U_m - U_l), (U_u - U_m)\}$.

Then we define the distance between fuzzy numbers as follows.

Definition 4.28 (**Fuzzy Hausdorff distance**). Let
$D_{FH} : \mathcal{F}_C(\mathbb{R}) \times \mathcal{F}_C(\mathbb{R}) \longrightarrow \mathbb{R}^+ \cup 0$ be a function defined by,

$$\begin{aligned} D_{FH}(U, V) &:= \sup_{r \in [0,1]} d_H(U_r, V_r) \\ &= \sup_{r \in [0,1]} \max\{|U_r^- - V_r^-|, |U_r^+ - V_r^+|\}. \end{aligned}$$

where $U_r = [U_r^-, U_r^+]$, $V_r = [V_r^-, V_r^+]$ belong to $\mathbb{I}_\mathbb{R}$ and d_H is the classical Hausdorff-Pompeiu distance between real intervals. Then D_{FH} is called the **Fuzzy Hausdorff distance** between fuzzy numbers. Another notation for this metric is given in the next section.

An L_p- type distance can also be defined.

Definition 4.29 (L_p **distance**). Let $1 \leq p < \infty$. We define the L_p **distance** between fuzzy numbers as

$$\begin{aligned} D_p(U, V) &:= \left(\int_0^1 d_H(U_r, V_r)^p dr \right)^{\frac{1}{p}} \\ &= \left(\int_0^1 \max\{|U_r^- - V_r^-|, |U_r^+ - V_r^+|\}^p dr \right)^{\frac{1}{p}}. \end{aligned}$$

The next theorems give a property of the Hausdorff distance and L_p distance between fuzzy numbers.

Theorem 4.30 (**Diamond-Kloeden**). *The spaces* $(\mathcal{F}_C(\mathbb{R}), D_{FH})$ *and are metric spaces.*

Proof. (i) Note that $D_{FH}(U,V) = \sup_{r\in[0,1]} \max\{|U_r^- - V_r^-|, |U_r^+ - V_r^+|\} \geq 0.$ Also, $D_{FH}(U,V) = 0$ if and only if for $r \in [0,1]$ we have that $U_r^- = V_r^-$ and $U_r^+ = V_r^+$. Additionaly, it is not difficult to see that $D_{FH}(U,V) = D_{FH}(V,U)$. Finally, for the triangle inequality we have

$$|U_r^- - V_r^-| \leq |U_r^- - W_r^-| + |W_r^- - V_r^-| \leq D_{FH}(U,W) + D_{FH}(W,V)$$

and

$$|U_r^+ - V_r^+| \leq |U_r^+ - W_r^+| + |W_r^+ - V_r^+| \leq D_{FH}(U,W) + D_{FH}(W,V)$$

which implies

$$D_{FH}(U,V) \leq D_{FH}(U,W) + D_{FH}(W,V).$$

As a conclusion $(\mathcal{F}_C(\mathbb{R}), D_{FH})$ is a metric space.

(ii) Let us check that D_p satisfies the definition of to be a metric. Since that d_H is a metric, we have

(a)

$$D_p(U,V) = \left(\int_0^1 d_H(U_r, V_r)^p\, dr\right)^{\frac{1}{p}} \geq 0.$$

(b) Given $U, V \in \mathcal{F}_C(\mathbb{R})$, then

$$D_p(U,V) = \left(\int_0^1 d_H(U_r, V_r)^p\, dr\right)^{\frac{1}{p}} = \left(\int_0^1 d_H(V_r, U_r)^p\, dr\right)^{\frac{1}{p}} = D_p(V,U).$$

(c) Given $U, V \in \mathcal{F}_C(\mathbb{R})$,

$$\begin{aligned} D_p(U,V) = 0 &\iff \left(\int_0^1 d_H(U_r, V_r)^p\, dr\right)^{\frac{1}{p}} = 0 \\ &\iff \int_0^1 d_H(U_r, V_r)^p\, dr = 0 \\ &\iff d_H(U_r, V_r) = 0 \text{ for all } r \in [0,1] \\ &\iff U_r = V_r \text{ for all } r \in [0,1] \\ &\iff U = V. \end{aligned}$$

(d) Consider $U, V, W \in \mathcal{F}_C(\mathbb{R})$,

$$\begin{aligned}
D_p(U,V) &= \left(\int_0^1 d_{H}(U_r,V_r)^p\,dr\right)^{\frac{1}{p}} \\
\underset{\substack{\textbf{(Triangular}\\ \textbf{Inequality)}}}{\leq} & \left(\int_0^1 \left(d_{H}(U_r,W_r)+d_{H}(W_r,V_r)\right)^p dr\right)^{\frac{1}{p}} \\
\underset{\substack{\textbf{(Minkowski}\\ \textbf{Inequality)}}}{\leq} & \left(\int_0^1 d_{H}(U_r,W_r)^p dr\right)^{\frac{1}{p}} + \left(\int_0^1 d_{H}(W_r,V_r)^p dr\right)^{\frac{1}{p}} \\
= \quad & D_p(U,W) + D_p(W,V).
\end{aligned}$$

□

4.4.3. Other Metrics on $\mathcal{F}_C(\mathbb{R})$

The most commonly used metrics on $\mathcal{F}_C(\mathbb{R})$ involve the Hausdorff distance between the level sets of the fuzzy sets [7, 15, 9]. In this subsection, only are given some of the metrics we found in texts about Fuzzy Analysis.

Definition 4.31. For $A, B \in \mathcal{E}^n$, then:

(i) **Supremum metric (Hausdorff Metric).**

$$D_{\infty}(A,B) = \sup_{0\leq\alpha\leq 1} \max\{L(\alpha), R(\alpha)\},$$

where, $L(\alpha) = |a_1(\alpha) - b_1(\alpha)|$ and $R(\alpha) = |a_2(\alpha) - b_2(\alpha)$.

(ii) **Integral distance.** When $n = 1$,

$$D_{\int}(A,B) := \int_a^b |A(x) - B(x)|dx,$$

where $[a, b]$ is an interval contining $A_0 \cap B_0$.

(iii) **Minkowski distance.** For $n = 1$,

$$D_{Mw} := \left(\sum_{x\in\mathbb{R}} |A(x) - B(x)|^w\right)^{\frac{1}{w}}$$

(iv) L_p **metric.** For $1 \leq p < \infty$,

$$D_p(A,B) = \left(\int_0^1 d_H(A_\alpha, B_\alpha)^p d\alpha\right)^{\frac{1}{p}}$$

With these metrics $\mathcal{F}_C(\mathbb{R})$ is a metric space and it has the following properties [15].

Theorem 4.32. *$(\mathcal{F}_C(\mathbb{R}), D_\infty)$ is a complete metric space.*

Theorem 4.33. *$(\mathcal{F}_C(\mathbb{R}), D_p)$ is not a complete metric space.*

Proposition 4.34. *The closed unit ball of $(\mathcal{F}_C(\mathbb{R}), D_\infty)$,*

$$B_\infty[0,1] := \{u \in \mathcal{E}^1 \mid D_\infty(u,0) \leq 1\}$$

is not separable.

Proof. For a given $t \in [0,1]$ we define

$$U_t(x) \begin{cases} 0 & x \notin [0,1] \\ t & x \in [0, \frac{1}{2}) \\ 2(1-t)x + 2t - 1 & x \in [\frac{1}{2}, 1]. \end{cases}$$

The level sets of U_t can be written as

$$(U_t)_r \begin{cases} [0,1] & 0 \leq r < t \\ \left[\frac{r+1-2t}{2(1-t)}, 1\right] & t \leq r \leq 1. \end{cases}$$

Then the Hausdorff-Pompeiu distance "level-wise" between two elements U_t and U_s of the sequence, $t < s$ is given as

$$D_H((U_t)_r, (U_s)_r) = \begin{cases} 0 & 0 \leq r < t \\ \frac{r+1-2t}{2(1-t)} & t \leq r < s \\ \frac{r+1-2t}{2(1-t)} - \frac{r+1-2s}{2(1-s)} & s \leq r \leq 1. \end{cases}$$

We have

$$D_\infty(U_t, U_s) = \sup_{r \in [t,s]} \frac{r+1-2t}{2(1-t)} \geq \frac{1}{2} > \frac{1}{3}.$$

So the open balls $B(U_t, \frac{1}{3})$ are disjoint and uncountably many. So a countable dense subset if there would exist, would need to have an element in each such ball, which is impossible. So, the closed unit ball in $(\mathcal{E}^1, D_\infty)$ is not separable. □

Corollary 4.35. *$(\mathcal{F}_C(\mathbb{R}), D_\infty)$ is not a separable space.*

Theorem 4.36. *The space $\mathcal{F}_C(\mathbb{R}), D_p)$ is a separable space.*

Theorem 4.37. *$D_\infty-$convergence implies D_p-convergence, for $1 \leq p < \infty$. The converse is false.*

We have found that in fuzzy analysis literature the Hausdorff distance between the fuzzy numbers is usually used. Some autors say that it is suitable because, with this metric, the structure of the metric space $(\mathcal{F}_C(\mathbb{R}), D_\infty)$ is complete and it is near to the structure of a Banach space [1]. Although we have neither a linear space structure nor a Banach space, several properties which hold in Banach spaces also hold in $(\mathcal{F}_C(\mathbb{R}), D_\infty)$, see [1].

4.4.4. Norm of a Fuzzy Number

Let us denote $\| U \|_{F}= D_\infty(U,0)$, for all $U \in \mathcal{F}_C(\mathbb{R})$ the norm of a fuzzy number. Remember that $\mathcal{F}_C(\mathbb{R})$ is not a linear space.

Proposition 4.38 (Anastassiou-Gal)**.** $\| \cdot \|_{F}$ *has the following properties:*

(i) $U = 0$ *iff* $\| U \|_{F}= 0$.

(ii) $\| \lambda \cdot U \|_{F}= |\lambda| \, \| U \|_{F}$, *for all* $\lambda \in \mathbb{R}$ *and* $U \in \mathcal{F}_C(\mathbb{R})$.

(iii) $\| \lambda \cdot (U \dotplus V) \|_{F} \leq \| \lambda \cdot U \|_{F} + \| \lambda \cdot V \|_{F}$, *for all* $\lambda \in \mathbb{R}$ *and* $U, V \in \mathcal{F}_C(\mathbb{R})$.

(iv) $| \, \| U \|_{F} - \| V \|_{F} \, | \leq D_\infty(U, V)$, *for all* $U, V \in \mathcal{F}_C(\mathbb{R})$.

(v) For any a *and* b *having the same sign and any* $U \in \mathcal{F}_C(\mathbb{R})$ *we have*

$$D_\infty(a \cdot U, b \cdot U) = |b - a| \, \| U \|_{F} \, .$$

(vi) $D_\infty(U, V) = \| U \ominus_{gH} V \|_{F}$, *for all* $U, V \in \mathcal{F}_C(\mathbb{R})$.

Proof. Let U be a fuzzy number, since $\| U \|_{F}= D_\infty(u, 0)$ and D_∞ is a metric, we have that

(i)

$$\| U \|_{F}= 0 \Longleftrightarrow D_\infty(U, 0) = 0 \Longleftrightarrow U = 0.$$

(ii) For all $\lambda \in \mathbb{R}$ and $U \in \mathcal{F}_C(\mathbb{R})$ we have

$$\| \lambda \cdot U \|_{F}= D_\infty(\lambda \cdot U, 0) = D_\infty(\lambda \cdot U, \lambda \cdot 0) = |\lambda| D_\infty(U, 0) = |\lambda| \, \| U \|_{F} \, .$$

(iii) For all $U, V \in \mathcal{F}_C(\mathbb{R})$,

$$\begin{aligned} \| U \dotplus V \|_{F} &= D_\infty(U \dotplus V, 0) \\ &\underset{\textbf{Theorem 4.30}}{\leq} D_\infty(U, 0) + D_\infty(V, 0) \\ &= \| U \|_{F} + \| V \|_{F} \, . \end{aligned}$$

(iv) For all $U, V \in \mathcal{F}_C(\mathbb{R})$,

$$D_\infty(U, 0) \leq D_\infty(U, V) + D_\infty(V, 0)$$

and

$$D_\infty(V, 0) \leq D_\infty(U, V) + D_\infty(U, 0).$$

Hence,

$$D_\infty(U, 0) - D_\infty(V, 0) \leq D_\infty(U, V)$$

and

$$D_\infty(V, 0) - D_\infty(U, 0) \leq D_\infty(U, V).$$

So,

$$|D_\infty(U, 0) - D_\infty(V, 0)| \leq D_\infty(U, V),$$

and therefore,

$$| \, \| U \|_{F} - \| V \|_{F} \, | \leq D_\infty(U, V).$$

(v) Let $U \in \mathcal{F}_C(\mathbb{R})$, and $a > b > 0$. Then we have

$$D_\infty(a \cdot U, b \cdot U) = D_\infty([b + (a-b)] \cdot U, b \cdot U) = D_\infty(b \cdot U + (a-b) \cdot U, b \cdot U).$$

Since D_∞ is invariant to translations we get

$$D_\infty(a \cdot U, b \cdot U) = D_\infty((a-b) \cdot U, 0) = |b-a| \parallel V \parallel_{_F} .$$

(vi) Given $U, V \in \mathcal{F}_C(\mathbb{R})$, Let $W = U \ominus_{gH} V$ be their $gH-$difference, then

$$U = V \dotplus W$$

and

$$\begin{aligned} D_\infty(U,V) &= D_\infty(V \dotplus W, V) \\ &\underset{\textbf{Theorem 4.30}}{=} D_\infty(W,0) \\ &= \parallel W \parallel_{_F} \\ &= \parallel U \ominus_{gH} V \parallel_{_F} \end{aligned}$$

□

5. Conclusions

Fuzzy sets theory tries to formalize the capability to perform a wide variety of physical and mental tasks and it is suitable for dealing with imprecision and aproximate reasoning and has become in a very important field of investigation, as much their mathematical implications as their practical applications, which have been successfully applied in the world. Here, some of the basics notions of this branch of mathematics were given and it can be inferred that the notion of fuzzy number is an important concept for fuzzy calculus and fuzzy modeling since it allows us to manage different kind of problems like decision making problems and to deal with certain differential equations [15, 19].

Acknowledgment

The author expresses his thanks to everyone who supported and encouraged him in the preparation of this work. Also, the author expresses his gratitude to his family, classmates and teachers because their comments helped or contributed in the development of this work, especially to the referee and Carlos Isaac Zainea because their suggestions given were very useful.

References

[1] B. Bede, *Studies in fuzziness and soft computing 295 - Mathematics of Fuzzy Sets and Fuzzy Logic*, Springer, 2013.

[2] J. J. Buckley and E. Eslami, *An introduction to fuzzy logic and fuzzy sets*, vol. 13, Springer Science & Business Media, 2002.

[3] A. Celikyilmaz and I. B. Turksen, *Modeling uncertainty with fuzzy logic*, Studies in Fuzziness and Soft Computing **240** (2009).

[4] R. Coppi, M. A. Gil, and H. A. Kiers, *The fuzzy approach to statistical analysis*, Computational statistics & data analysis **51** (2006), no. 1, 1–14.

[5] E. Cox, M. O'Hagan, R. Taber, and M. O'Hagen, *The fuzzy systems handkbook with cdrom*, Academic Press, Inc., 1998.

[6] P. Diamond and P. Kloeden, *Metric topology of fuzzy numbers and fuzzy analysis*, Fundamentals of Fuzzy Sets, Springer, 2000, pp. 583–641.

[7] P. Diamond, P. E. Kloeden, P. E. Kloeden, Australia Mathematician, and P. E. Kloeden, *Metric spaces of fuzzy sets: theory and applications*, World Scientific, 1994.

[8] V. Dimitrov and V. Korotkich, *Fuzzy logic: a framework for the new millennium*, vol. 81, Springer Science & Business Media, 2002.

[9] D. Dubois, H. M. Prade, and H. Prade, *Fundamentals of fuzzy sets*, vol. 7, Springer Science & Business Media, 2000.

[10] D. J. Dubois, *Fuzzy sets and systems: Theory and applications*, vol. 144, Academic press, 1980.

[11] B. A. Faybishenko, *Introduction to modeling of hydrogeologic systems using fuzzy differential equations*, Fuzzy Partial Differential Equations and Relational Equations, Springer, 2004, pp. 267–284.

[12] M. A. Gil, M. López-Díaz, and D. A. Ralescu, *Overview on the development of fuzzy random variables*, Fuzzy sets and systems **157** (2006), no. 19, 2546–2557.

[13] M. Hanss, *Applied fuzzy arithmetic*, Springer, 2005.

[14] L. C. Jain and N. M. Martin, *Fusion of neural networks, fuzzy systems and genetic algorithms: Industrial applications*, vol. 4, CRC press, 1998.

[15] V. Lakshmikantham and R. N. Mohapatra, *Theory of fuzzy differential equations and inclusions*, CRC Press, 2004.

[16] K. H. Lee, *First course on fuzzy theory and applications*, vol. 27, Springer Science & Business Media, 2006.

[17] J. N. Mordeson, D. S. Malik, and T. D. Clark, *Application of fuzzy logic to social choice theory*, CRC Press, 2015.

[18] H. T. Nguyen and M. Sugeno, *Fuzzy systems: modeling and control*, vol. 2, Springer Science & Business Media, 1998.

[19] J. R. Ni no Quevedo, *Ecuaciones diferenciales difusas: Una forma de modelar la incertidumbre y la ambigüedad.*, 2015.

[20] W. Pedrycz and F. Gomide, *Fuzzy systems engineering: toward human-centric computing*, John Wiley & Sons, 2007.

[21] T. J. Ross, *Fuzzy logic with engineering applications*, John Wiley & Sons, 2009.

[22] M. Spivak, *Cálculo infinitesimal*, Reverté, 1996.

[23] L. H. Tsoukalas and R. E. Uhrig, *Fuzzy and neural approaches in engineering*, John Wiley & Sons, Inc., 1996.

[24] M. J. Wierman, *An introduction to the mathematics of uncertainty*, (2010).

[25] L. A. Zadeh, *Fuzzy sets*, Information and control **8** (1965), no. 3, 338–353.

[26] ———, *The concept of a linguistic variable and its application to approximate reasoning - i*, Information sciences **8** (1975), no. 3, 199–249.

[27] ———, *The concept of a linguistic variable and its application to approximate reasoning - ii*, Information sciences **8** (1975), no. 4, 301–357.

[28] ———, *The concept of a linguistic variable and its application to approximate reasoning-iii*, Information sciences **9** (1975), no. 1, 43–80.

[29] ———, *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers*, 1996.

[30] H. Zarei, A. Kamyad, and A. A. Heydari, *Fuzzy modeling and control of hiv infection*, Computational and mathematical methods in medicine **2012** (2012).

[31] H. Zhang and D. Liu, *Fuzzy modeling and fuzzy control*, Springer Science & Business Media, 2006.

[32] H. J. Zimmermann, *Fuzzy sets, decision making, and expert systems*, vol. 10, Springer Science & Business Media, 1987.

[33] ———, *Fuzzy set theory and its applications*, Springer Science & Business Media, 2001.