A new proof of the Benedetti's inequality and some applications to perturbation to real eigenvalues and singular values

Una nueva prueba de desigualdad de Benedetti y algunas aplicaciones a la perturbación de valores propios reales y valores singulares

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Abstract. Using the standard deviation of the real samples $\mu_n \geq \cdots \geq \mu_1$ and $\lambda_n \geq \cdots \geq \lambda_1$, we refine the Chebyshev's inequality (refer to [5]),

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right) \leq \frac{1}{n}\sum_{i=1}^{n}\mu_i\lambda_i. \tag{1}$$

As a consequence, we obtain a new proof of the Benedetti's inequality (refer to [1], [2] and [4])

$$\frac{1}{n-1} \leq \frac{Cov(\mu,\lambda)}{s(\mu)s(\lambda)}, \tag{2}$$

where $Cov[\mu,\lambda]$, $s(\mu)$ and $s(\lambda)$ denote the covariance, and the standard deviations ($\neq 0$) of the sample vectors $\mu = (\mu_1,\ldots,\mu_n)$ and $\lambda = (\lambda_1,\ldots,\lambda_n)$, respectively.

We can also get very interesting applications to eigenvalues and singular values perturbation theory. For some kinds of matrices, the result that we present improves the well known Hoffmand-Weiland's inequality.

Keywords: Chebyshev's inequality, Hoffmand-Weiland's inequality, eigenvalues perturbation, singular value perturbation.

Resumen. Usando la desviación estándar de las muestras reales $\mu_n \geq \cdots \geq \mu_1$ y $\lambda_n \geq \cdots \geq \lambda_1$, refinamos la desigualdad de Chebyshev, ver [5],

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right) \leq \frac{1}{n}\sum_{i=1}^{n}\mu_i\lambda_i. \tag{1}$$

Como consecuencia, obtenemos una nueva prueba de desigualdad de Benedetti, ver [1], [2] y [4]

$$\frac{1}{n-1} \leq \frac{Cov(\mu,\lambda)}{s(\mu)s(\lambda)}, \tag{2}$$

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dónde $Cov[\mu, \lambda]$, $s(\mu)$ y $s(\lambda)$ denotan la covarianza, y las desviaciones estándar ($\neq 0$) de los vectores de muestra $\mu = (\mu_1, \dots, \mu_n)$ y $\lambda = (\lambda_1, \dots, \lambda_n)$, respectivamente.

También podemos obtener aplicaciones muy interesantes para los valores propios y la teoría de perturbaciones de valores singulares. Para algunos tipos de matrices, el resultado que presentamos mejora la conocida desigualdad de Hoffmand-Weiland.

Palabras claves: Desigualdad de Hoffmand-Weiland, Desigualdad de Chebyshev, perturbación de valores propios, peturbación de valores singulares.

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1. Introduction

Carlo Benedetti presented in [1] (1957), one of the most interesting inequalities (2) of the applied statistic, used in measures of correlation. N. Georgescu [2] (1959) got a simpler proof of the Benedetti's theorem, using methods concerning the extrema of symmetrical, homogeneous algebraic forms. In 1994, T. Y Hwang and Ch. Y. Hu [4] rediscover the Benedetti's result, studying bounds of sample correlation coefficient, and in 2001 H. Sarria [6] discovered the inequality (3) in context of the matrix perturbation eigenvalues, that is equivalent to the Benedetti's inequality and is clearly a refinement of the Chebyshev's inequality (1). The inequality (3) was initially obtained using mathematical programming techniques. Once found, however, we remarked that the inequality admitted a shorter proof, using an elementary algebraic process. In this paper, we present this scheme.

For any real vector $\lambda = (\lambda_1, \dots, \lambda_n)$, we define the mean and the standard deviation by

$$m(\lambda) = \frac{1}{n}\sum_{i=1}^{n} \lambda_i \quad \text{and} \quad s(\lambda)^2 = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} \lambda_i\right)^2,$$

respectively. The covariance of the vectors μ and λ are given by

$$Cov(\mu, \lambda) = \frac{1}{n}\mu^{tr}\lambda - m(\mu)m(\lambda).$$

The main result in this paper is:

Theorem 1.1. *If $\mu_n \geq \cdots \geq \mu_1$ and $\lambda_n \geq \cdots \geq \lambda_1$, with $n \geq 2$, then*

$$m(\lambda)m(\mu) + \frac{1}{n-1}s(\lambda)s(\mu) \leq \frac{1}{n}\mu^{tr}\lambda. \tag{3}$$

Equality holds, if and only if, any of the following cases is given: **(a)** $n = 2$, **(b)** $\mu_1 \leq \mu_2 = \cdots = \mu_n$ *and* $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \leq \lambda_n$, **(c)** $\mu_1 = \mu_2 = \cdots = \mu_{n-1} \leq \mu_n$ *and* $\lambda_1 \leq \lambda_2 = \cdots = \lambda_n$.

In order to obtain the proof of the theorem above, we will use the following result:

Lemma 1.2. *Let* $a_1, a_2, \ldots, a_n \geq 0$ *and* $b_1, b_2, \ldots, b_n \geq 0$*, with* $n \geq 1$ *then*

$$(n-1)^2 \left(\sum_{i,j=1}^{n} c_{ij} a_i b_j \right)^2 \geq \sum_{i,j=1}^{n} c_{ij} a_i a_j \sum_{i,j=1}^{n} c_{ij} b_i b_j \tag{4}$$

where $c_{ij} = (n - \max\{i,j\} + 1)(\min\{i,j\} - 1)$*. Equality holds if, and only if, one of the following holds:* **(a)** $n = 1$*,* **(b)** $n = 2$*,* **(c)** $a_3 = \cdots = a_n = 0$ *and* $b_2 = \cdots = b_{n-1} = 0$ **(d)** $a_2 = \cdots = a_{n-1} = 0$ *and* $b_3 = \cdots = b_n = 0$*.*

Proof. If $n = 1$ or $n = 2$, the statement (4) is evident. Now we suppose $n > 2$. Note that $c_{ij} = c_{ji}$ for $i, j = 1, \ldots, n$. Then

$$\left(\sum_{i,j=1}^{n} c_{ij} a_i b_j \right)^2 = \sum_{i,j=1}^{n} c_{ij} a_i b_j \sum_{k,t=1}^{n} c_{kt} a_k b_t = \sum_{i,j,k,t=1}^{n} c_{ij} c_{kt} a_i a_k b_j b_t.$$

We can also write,

$$\sum_{i,j=1}^{n} c_{ij} a_i a_j \sum_{i,j=1}^{n} c_{ij} b_i b_j = \sum_{i,j,k,t=1}^{n} c_{ik} c_{jt} a_i a_k b_j b_t.$$

Therefore,

$$\begin{aligned} (n-1)^2 \left(\sum_{i,j=1}^{n} c_{ij} a_i b_j \right)^2 - \sum_{i,j=1}^{n} c_{ij} a_i a_j \sum_{i,j=1}^{n} c_{ij} b_i b_j &= \\ (n-1)^2 \sum_{i,j,k,t=1}^{n} c_{ij} c_{kt} a_i a_k b_j b_t &- \sum_{i,j,k,t=1}^{n} c_{ik} c_{jt} a_i a_k b_j b_t \\ = \sum_{i,j,k,t=1}^{n} \left[(n-1)^2 c_{ij} c_{kt} - c_{ik} c_{jt} \right] a_i a_k b_j b_t. & \end{aligned} \tag{5}$$

Note that $c_{i1} = c_{1i} = 0$, so we can assume $2 \leq i \leq n$ and $2 \leq j \leq n$. In this case, $1 \leq n - i + 1 \leq n - 1$ and $1 \leq i - 1 \leq n - 1$, for any i, $2 \leq i \leq n$. As a consequence, we have $1 \leq (n - j + 1)(i - 1) \leq (n-1)^2$, for any $2 \leq i, j \leq n$. On the other hand, if $a = \min\{i,j,k,t\}$ and $A = \max\{i,j,k,t\}$, then

$$c_{aA} \leq c_{ij} c_{kt} \leq c_{aA} (n-1)^2, \tag{6}$$

for $i, j, k, t \in \{2, \ldots, n\}$. Therefore

$$(n-1)^2 c_{ij} c_{kt} - c_{ik} c_{jt} \geq 0, \tag{7}$$

for $2 \leq i, j, k, t \leq n$. Using (5), and (7)

$$(n-1)^2 \left(\sum_{i,j=2}^{n} c_{ij} a_i b_j \right)^2 - \sum_{i,j=2}^{n} c_{ij} a_i a_j \sum_{i,j=2}^{n} c_{ij} b_i b_j \geq 0,$$

and this proves our inequality.

We are going to get the equality conditions. Note that $(n-1)^2 c_{2n}^2 - c_{22} c_{nn} = (n-1)^2 c_{n2}^2 - c_{22} c_{nn} = 0$, so from (5), clearly equality holds if, both $a_3 = \cdots = a_n = 0$ and $b_2 = \cdots = b_{n-1} = 0$, or if both $a_2 = \cdots = a_{n-1} = 0$ and $b_3 = \cdots = b_n = 0$. Now, suppose that the equality holds, and there exist $p, q \in \{2, \ldots, n\}$, $p \leq q$ such that $a_p, b_q > 0$, from here, (5), and (7)

$$(n-1)^2 c_{pq}^2 - c_{pp} c_{qq} = 0. \tag{8}$$

But from (6), (8) is true if, and only if, both $c_{pq}^2 = c_{aA}$, and $c_{pp} c_{qq} = c_{aA}(n-1)^2$. This implies that $p = 2$ and $q = n$, i.e., if $a_3 = \cdots = a_n = 0$ and $b_2 = \cdots = b_{n-1} = 0$. The symmetry case $q \leq p$ implies, $a_2 = \cdots = a_{n-1} = 0$ and $b_3 = \cdots = b_n = 0$. □

Proof of theorem 1.1. First, observe that (3) is equivalent to the inequality

$$(n-1)^2 \left(n \sum_{i=1}^{n} \lambda_i \mu_i - \sum_{i=1}^{n} \lambda_i \sum_{i=1}^{n} \mu_i \right)^2 \geq \left(n \sum_{i=1}^{n} \lambda_i^2 - \left(\sum_{i=1}^{n} \lambda_i \right)^2 \right) \left(n \sum_{i=1}^{n} \mu_i^2 - \left(\sum_{i=1}^{n} \mu_i \right)^2 \right).$$

But

$$n \sum_{i=1}^{n} \lambda_i \mu_i - \sum_{i=1}^{n} \lambda_i \sum_{i=1}^{n} \mu_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_j - \lambda_i)(\mu_j - \mu_i),$$

so the inequality (3) is equivalent to

$$(n-1)^2 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_j - \lambda_i)(\mu_j - \mu_i) \right)^2 \geq \sum_{i=1}^{n} \sum_{j=1}^{n} (\lambda_j - \lambda_i)^2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} (\mu_j - \mu_i)^2,$$

or to

$$(n-1)^2 \left(\sum_{i<j}^{n} (\lambda_j - \lambda_i)(\mu_j - \mu_i) \right)^2 \geq \sum_{i<j}^{n} (\lambda_j - \lambda_i)^2 \cdot \sum_{i<j}^{n} (\mu_j - \mu_i)^2. \tag{9}$$

If $i < j$, we can write,

$$(\lambda_j - \lambda_i)(\mu_j - \mu_i) = \Big[(\lambda_j - \lambda_{j-1}) + (\lambda_{j-1} - \lambda_{j-2}) + \cdots + (\lambda_{i+1} - \lambda_i) \Big] \cdot \Big[(\mu_j - \mu_{j-1}) + (\mu_{j-1} - \mu_{j-2}) + \cdots + (\mu_{i+1} - \mu_i) \Big]. \tag{10}$$

Expanding (10), we get a sum of terms of the form $(\lambda_r - \lambda_{r-1})(\mu_k - \mu_{k-1})$ where $i < r, k \leq j$. Substituting (10) in the sum

$$\sum_{i<j}^{n}(\lambda_j - \lambda_i)(\mu_j - \mu_i)$$

of the left side sum of (9), we can observe that if $2 \leq r, k \leq n$, the term $(\lambda_r - \lambda_{r-1})(\mu_k - \mu_{k-1})$ appears in this sum, once, each time that the term $(\lambda_j - \lambda_i)(\mu_j - \mu_i)$ appears for both $1 < i \leq \min\{k, r\}$ and $\max\{k, r\} \leq j \leq r$; that is $c_{rk} = (n - \max\{r, k\} + 1)(\min\{r, k\} - 1)$ times. Making a similar analysis in every sum in the right side of (9), we can see that (9) is equivalent to the inequality

$$(n-1)^2 \left(\sum_{i=2}^{n}\sum_{j=2}^{n} c_{ij}(\lambda_i - \lambda_{i-1})(\mu_j - \mu_{j-1}) \right)^2 \geq \sum_{i=2}^{n}\sum_{j=2}^{n} c_{ij}(\lambda_i - \lambda_{i-1})(\lambda_j - \lambda_{j-1}) \cdot \sum_{i=2}^{n}\sum_{j=2}^{n} c_{ij}(\mu_i - \mu_{i-1})(\mu_j - \mu_{j-1}).$$

Finally, we make $a_i = \lambda_i - \lambda_{i-1}$ and $b_j = \mu_j - \mu_{j-1}$, for $i, j = 2, \ldots, n$ and apply the Lemma 1.2, in order to get (3). □

Observation: Benedetti's inequality (2), is clearly equivalent to inequality (3).

2. Some applications to matrix theory

We will use $\mathcal{C}$ to denote the set of complex numbers, and $M_{n,m}(\mathcal{C})$ for the space of $n \times m$ complex matrices. If $n = m$, we will use $M_n(\mathcal{C})$ for the square complex matrix space. If $X \in M_n(\mathcal{C})$, X^* and X^{tr} will denote the conjugate transpose, and the transpose of X, respectively, and $Tr(X)$ its trace.

2.1. Real eigenvalues

One of the problems in classical perturbation theory for matrix eigenvalues consists in providing bounds to the Euclidean distance $\sqrt{\sum_{i=1}^{n} |\mu_i - \lambda_i|^2}$, where $\mu_1, \mu_2, \ldots, \mu_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $n \times n$ complex matrices A and B, respectively. The most interesting result in this area is the Hoffman-Weilandt theorem (e.g., refer to [3]):

Theorem 2.1. *(Hoffman-Weilandt) If A and B are normal matrices, then there exists a suitable numbering of the eigenvalues such that*

$$\sum_{i=1}^{n} |\mu_i - \lambda_i|^2 \leq \|A - B\|_F^2$$

where $\|\cdot\|_F$ denoted the Frobenius' norm.

In this section, we suppose that the matrices A and B have real eigenvalues. Also, we will define the mean and the standard deviation of a matrix X with real eigenvalues respectively by

$$m(X) = \frac{Tr(X)}{n} \quad \text{and} \quad s(X)^2 = \frac{Tr(X^2)}{n} - \frac{Tr(X)^2}{n^2}.$$

The inequalities given here are tightest when all the eigenvalues are real; this happens for example when the matrices are Hermitian. Our main purpose is to give bounds for the Euclidean distance (considering eigenvalues in increasing order) using traces of the matrices A, B, A^2 and B^2.

Theorem 2.2. *If $\mu_n \geq \cdots \geq \mu_1$ and $\lambda_n \geq \cdots \geq \lambda_1$, with $n \geq 2$, then*

(i) $nm(A)m(B) + \frac{n}{n-1}s(A)s(B) \leq \mu^{tr}\lambda$,

(ii) $\sum_{i=1}^{n}(\lambda_i - \mu_i)^2 \leq Tr(A^2) + Tr(B^2) - 2nm(A)m(B) - \frac{2n}{n-1}s(A)s(B)$,

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{tr}$ *and* $\mu = (\mu_1, \mu_2, \cdots, \mu_n)^{tr}$.

Equality holds in (i) and (ii) if, and only if, any the following cases is given: **(a)** $n = 2$, **(b)** $\mu_1 \leq \mu_2 = \cdots = \mu_n$ *and* $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} \leq \lambda_n$, **(c)** $\mu_1 = \mu_2 = \cdots \mu_{n-1} \leq \mu_n$ *and* $\lambda_1 \leq \lambda_2 = \cdots = \lambda_n$.

Proof. *(i)* follows from Theorem 1.1. *(ii)* is equivalent to *(i)*. □

Observation: Inequalities in Theorem 2.2, are equivalent to the inequality just given in terms of the mean and the standard deviation

$$\sum_{i=1}^{n}(\lambda_i - \mu_i)^2 \leq (\alpha_1 - \alpha_2)^2 + (n-2)(\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_4)^2, \tag{11}$$

where

$$\alpha_1 = \frac{Tr(A)}{n} - \sqrt{n-1}\, s(A)$$
$$\alpha_2 = \frac{Tr(B)}{n} - \sqrt{\frac{1}{n-1}}\, s(B)$$
$$\alpha_3 = \frac{Tr(A)}{n} + \sqrt{\frac{1}{n-1}}\, s(A)$$
$$\alpha_4 = \frac{Tr(B)}{n} + \sqrt{n-1}\, s(B).$$

The proof is long but straightforward.

2.2. Singular values

We can extend the bounds in Theorem 2.2 to the case of perturbation of singular values. Let be $n + m > 2$, and A, $B \in M_{n,m}(\mathcal{C})$, with singular eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_q$ and $\beta_1, \beta_2, \dots, \beta_q$ respectively, where $q = \min\{n, m\}$. The proof is straightforward using Theorem 2.2 and the following result (refer to [3] Theorem 7.3.7).

Theorem 2.3. *Let $\tilde{A} \in M_{m,n}$ defined by*

$$\tilde{A} = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

Let be $\iota_1, \iota_2, \dots, \iota_q$ nonnegative real numbers. The singular values of A are $\iota_1, \iota_2, \dots, \iota_q$ if, and only if, the $m + n$ eigenvalues of $\tilde{A}$ are $\iota_1, \iota_2, \dots, \iota_q$, $-\iota_1, -\iota_2, \dots, -\iota_q$, and $|m - n|$ additional 0's.

Theorem 2.4. *If $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_q$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_q$. Then,*

(i) $\frac{n}{n+m-1}\sqrt{m(AA^*)m(BB^*)} \leq \alpha^{tr}\beta$.

(ii) $\sum_{i=1}^{n}(\alpha_i - \beta_i)^2 \leq Tr(AA^*) + Tr(BB^*) - \frac{2n}{n+m-1}\sqrt{m(AA^*)m(BB^*)}$.

Equality holds in (i) and (ii) if, and only if, either $A = 0$ or $B = 0$.

3. Examples

To illustrate the bounds given here, we present some numerical examples.

Example 3.1. Let

$$A = \begin{pmatrix} 6 & 0 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{pmatrix},$$

These matrices have real eigenvalues $(-1, 4, 6)$ and $\left(0, \frac{1}{2}(7 - \sqrt{65}), \frac{1}{2}(7 + \sqrt{65})\right)$ respectively. Note that Hoffmand - Weiland's theorem is not applicable here, because A and B are not normal.

Considering order of the eigenvalues, by the Theorem 2.2*(i)*, we get the lower bound:

$$37.2583 \leq \mu^{tr}\lambda \approx 45.7179.$$

Considering order of the eigenvalues, by the Theorem 2.2*(ii)*, we get the upper bound:

$$4.3086 \approx \sqrt{\sum_{i=1}^{3}(\mu_i - \lambda_i)^2} \leq 5.9568$$

The following four examples compare Hoffman-Weiland's bound and the bound *(ii)* in Theorem 2.2:

Example 3.2. (a) Let

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 5 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}.$$

be. We generate B, through the MATLAB's instructions

$$B = rand(4); \qquad B = B' * B;$$

in order to get the following

$\sqrt{\sum_{i=1}^{4}(\mu_i - \lambda_i)^2}$	Theorem 2.2	Hoffman-Weiland
4.69	7.23	8.05
5.01	6.91	7.66
5.58	7.15	7.68
4.92	7.01	7.98
3.74	6.74	7.50

Table 1

(b) Now, we make

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 5 & 0 & 0 \\ -3 & 0 & 1 & 2 \\ 1 & 0 & 2 & 3 \end{pmatrix}$$

$\sqrt{\sum_{i=1}^{4}(\mu_i - \lambda_i)^2}$	Theorem 2.2	Hoffman-Weiland
4.46	7.14	8.01
5.41	7.26	8.20
5.64	7.39	8.13
5.17	7.38	7.41
5.24	7.10	7.89

Table 2

(c)

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 5 & 7 & 4 \\ -3 & 7 & 1 & 2 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$\sqrt{\sum_{i=1}^{4}(\mu_i - \lambda_i)^2}$	Theorem 2.2	Hoffman-Weiland
10.21	12.86	11.25
9.72	12.73	12.03
10.75	13.12	11.34
9.56	12.83	12.80
9.77	12.90	12.14

Table 3

(d) For small perturbation, Hoffman-Weiland's bound is better than bound *(ii)* in Theorem 2.2. In the following table we illustrate this aspect. The matrices A and B are generated via the MATLAB's instructions

$$A = rand(4); A = 10 * A * A';$$

$$B = rand(4); B = A + 0.1 * B * B';$$

$\sqrt{\sum_{i=1}^{4}(\mu_i - \lambda_i)^2}$	Theorem 2.2	Hoffman-Weiland
2.93	50.38	3.20
3.48	39.56	4.08
6.23	51.35	6.62
4.81	37.56	4.93
2.72	59.64	3.62

Table 4

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