The asymptotics of the kernel functions associated to orthogonal polynomials in several variables on the unit ball

Comportamiento asintótico de polinomios ortogonales en varias variables sobre la bola unidad

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Abstract. We consider a family of orthogonal polynomials in several variables with respect to a Sobolev-type inner product, obtained from adding a gradient operator of order j, evaluated in a fixed point to a standard inner product. We study explicit relations between the Sobolev-type polynomials and the standard polynomials, among the kernel functions associated to the Sobolev-type polynomials and the kernel functions associated to the standard polynomials. In addition, an example for a particular choice of a classical measure $\sigma \in \mathbb{R}^d$ is analyzed. Finally, we obtain the asymptotics of the some derivatives of the kernel functions evaluated in some points of the unit ball in d variables.

Keywords: Orthogonal polynomials in several variables, Asymptotics behavior, Sobolev inner products.

Resumen. Consideramos una familia de polinomios ortogonales en varias variables con respecto a un producto interno de tipo Sobolev, el cual se obtiene al adicionar a un producto interno estándar un operador gradiente de orden j, evaluado en un punto fijo. Estudiamos relaciones entre los polinomios de tipo Sobolev y los polinomios estándar, como relaciones entre el núcleo asociado a los polinomios de tipo Sobolev y el núcleo de los polinomios estándar. Adicionalmente, estudiamos un caso particular de una medida $\sigma \in \mathbb{R}^d$. Finalmente, se obtienen los comportamientos asintóticos de las derivadas del núcleo evaluadas en puntos de la bola unidad en d variables.

Palabras claves: Polinomios ortogonales en varias variables, comportamiento asintótico, producto interno de tipo Sobolev.

Mathematics Subject Classification: 33C47, 42C05.

Recibido: noviembre de 2014 Aceptado: marzo de 2015

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1. Introduction

The study of Sobolev orthogonal polynomials on the real line has been evolving in the last fifty years, over all in the last twenty years, when an extensive interest by the researchers has increased because of the uses in some subjects, in particular, in the approximation theory. The study of orthogonal polynomials of several variables has been increasing as a consequence of the results on the real line, as the properties of the zeros, differential equations, and asymptotic behavior. However, one of the motivations of study of asymptotic behavior of orthogonal polynomials on several variables is that this theory has become a tool for numerical solutions of partial differential equations. In [6], we can find an interesting survey of Sobolev orthogonal polynomials.

A special attention has been paid to the families of non-standard orthogonal polynomials, within the Sobolev-type orthogonal polynomials, see [6, 7, 8] and the references therein, which are associated with an inner product defined on the linear space of polynomials with real coefficients. These inner products

$$\langle p,q\rangle = \langle p,q\rangle_\sigma + \sum_{i=0}^{j} \lambda_i p^{(i)}(\xi) q^{(i)}(\xi) = \int_E p(x)q(x)d\sigma(x) + \sum_{i=0}^{j} \lambda_i p^{(i)}(\xi) q^{(i)}(\xi), \tag{1}$$

where $E \subseteq \mathbb{R},\ j \in \mathbb{N},\ \xi \in \mathbb{R},\ \lambda_i \in \mathbb{R}^+$ and σ is a classical measure.

The non-standard features of this kind of inner products is the presence of derivatives and that the operator associated with the multiplication by x is not symmetric; that is, for any pair of polynomials p and q,

$$\langle xp,q\rangle \neq \langle p,xq\rangle.$$

The family of orthogonal polynomials with respect to (1) are called Sobolev-type orthogonal polynomials and the family of orthogonal polynomials with respect to $\langle p,q\rangle_\sigma$ are called standard polynomials.

We will study a particular case of the inner product (1), for orthogonal polynomials in several variables. As in [4], the derivative is replaced by a gradient operator of order j. We analyze the relation among the Sobolev-type polynomials and the standard polynomials, such the relation between the kernel functions associated to the Sobolev-type polynomials and the kernel functions associated to the standard polynomials.

In [4], the authors analyze the asymptotic behavior of the kernel functions associated with the Sobolev-type orthogonal polynomials on the unit ball in d variables on the point $(\mathbf{0},\mathbf{0})$, where $\mathbf{0} = (0,0,\ldots,0)$. They obtain expressions for the kernel functions and the fourth order derivatives (after [3]). We continue with this particular choice of measure, but we analyze the asymptotic behavior not only in $(\mathbf{0},\mathbf{0})$, but also we extend the analysis in any point on the unit sphere.

2. Preliminaries: Orthogonal polynomials in several variables

Let $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_d) \in \mathbb{N}^d$ be a multi-index and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. A monomial in d variables is given by:

$$\mathbf{x}^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \dots x_d^{\kappa_d}.$$

The integer number $|\kappa| = \kappa_1 + \kappa_2 + \cdots + \kappa_d$ is called the *total degree* of $\mathbf{x}^\kappa$.

A polynomial $P(\mathbf{x})$ in d variables is a linear combination of monomials,

$$P(\mathbf{x}) = \sum_\kappa c_\kappa \mathbf{x}^\kappa,$$

where the coefficients c_κ are in a field $\mathbf{C}$, usually the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. The degree of a polynomial is defined as the highest total degree of its monomials.

For a polynomial $P(\mathbf{x})$ in several variables the partial derivative with respect to the $i-th$ component is denoted by

$$\partial_i P(\mathbf{x}) = \frac{\partial P(\mathbf{x})}{\partial x_i}.$$

We will denote,

- Π^d, the set of the polynomials in d variables with real coefficients,

$$\Pi^d = \left\{ \sum_\kappa c_\kappa \mathbf{x}^\kappa : \quad c_\kappa \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^d \right\}.$$

- Π_n^d, the subspace of Π^d consisting on the polynomials of total degree at most n,

$$\Pi_n^d = \left\{ \sum_{|\kappa| \le n} c_\kappa \mathbf{x}^\kappa : \quad c_\kappa \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^d \right\}.$$

A polynomial is called *homogeneous* if its monomials have the same total degree. $\mathcal{H}_n^d$ denote the space of homogeneous polynomials of degree n in d variables, i.e.,

$$\mathcal{H}_n^d = \left\{ \sum_{|\kappa| = n} c_\kappa \mathbf{x}^\kappa : \quad c_\kappa \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^d \right\}.$$

Every polynomial in Π_n^d can be written as a linear combination of homogeneous polynomials. Therefore, for every $P(\mathbf{x}) \in \Pi_n^d$, we have,

$$P(\mathbf{x}) = \sum_{i=0}^{n} \sum_{|\kappa| = i} c_\kappa \mathbf{x}^\kappa.$$

A basis of $\mathcal{H}_n^d$ is $\{\mathbf{x}^\kappa : |\kappa| = n\}$ and the dimension of $\mathcal{H}_n^d$ denoted by r_n^d is given by

$$dim\mathcal{H}_n^d = r_n^d = \begin{pmatrix} n+d-1 \\ n \end{pmatrix},$$

see [5]. An essential difference between polynomials in one variable and in several variables is the fact that in the last ones we do not have an obvious natural order. The natural order among monomials of one variable is the increasing degree, which is: $1,\ x,\ x^2,\ x^3, \ldots$. For polynomials in several variables, there are many choices of well-defined total orders. We present two possibilities.

1. ***The lexicographic order.*** Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_d),\ \eta = (\eta_1, \eta_2, \ldots, \eta_d)$, be two multi-indexes. $\kappa \succ_L \eta$ if the first non-zero entry in the difference $\kappa - \eta = (\kappa_1 - \eta_1, \kappa_2 - \eta_2, \ldots, \kappa_d - \eta_d)$ is positive. Is clear that the lexicographic order does not respect the total degree of the polynomials.

2. ***The graded lexicographic order.*** Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_d)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_d)$, be two multi-indexes. $\kappa \succ_{GL} \eta$ if $|\kappa| > |\eta|$, or in the case that $|\kappa| = |\eta|$ then we have the first non-zero entry in the difference $\kappa - \eta = (\kappa_1 - \eta_1, \kappa_2 - \eta_2, \ldots, \kappa_d - \eta_d)$ is positive. The graded lexicographic order respect the total degree of the polynomials.

We consider the orthogonality only in terms of polynomials of different degrees; that is, equal degree polynomials are orthogonal polynomials of lower degree; but equal degree polynomials are not orthogonal between each other.

Let $d\sigma(\mathbf{x})$ be a positive measure defined on $E \subset \mathbb{R}^d$ a domain with a nonempty interior. Let $\langle \cdot, \cdot \rangle$ denote the inner product defined on Π_n^d by

$$\langle p(\mathbf{x}), q(\mathbf{x}) \rangle_\sigma = \int_E p(\mathbf{x}) q(\mathbf{x}) d\sigma(\mathbf{x}), \tag{2}$$

see [7]. We will say that the polynomial $p(\mathbf{x}) \in \Pi_n^d$ is orthogonal with respect to (2) if $\langle p(\mathbf{x}), q(\mathbf{x}) \rangle = 0,\ \forall q(\mathbf{x}) \in \Pi_{n-1}^d$.

Let V_n^d be linear space of orthogonal polynomials of total degree n with respect to (2). Then

$$dimV_n^d = dim\mathcal{H}_n^d.$$

If $\left\{P_{\kappa_i}^n(\mathbf{x}) : |\kappa_i| = n, 1 \leq i \leq r_n^d\right\}_{n \geq 0}$ is a basis of V_n^d, we will write the (column) polynomial vector as:

$$\mathbb{P}_n(\mathbf{x}) = \begin{pmatrix} P_{\kappa_1}^n(\mathbf{x}) \\ P_{\kappa_2}^n(\mathbf{x}) \\ P_{\kappa_3}^n(\mathbf{x}) \\ \vdots \\ P_{\kappa_{r_n^d}}^n(\mathbf{x}) \end{pmatrix}_{r_n^d \times 1},$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{r_n^d}$ are arranged according to the reverse lexicographic order.

We denote the partial derivative with respect to the $i-th$ component of a polynomial vector

$$\partial_i \mathbb{P}_n(\mathbf{x}) = \begin{pmatrix} \partial_i P^n_{\kappa_1}(\mathbf{x}) \\ \partial_i P^n_{\kappa_2}(\mathbf{x}) \\ \partial_i P^n_{\kappa_3}(\mathbf{x}) \\ \vdots \\ \partial_i P^n_{\kappa_{r^d_n}}(\mathbf{x}) \end{pmatrix}_{r^d_n \times 1} .$$

We will say that $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$ is an *orthogonal polynomial system*, if,

$$\left\langle \mathbb{P}_n(\mathbf{x}), \mathbb{P}^T_m(\mathbf{x}) \right\rangle = \int_S \mathbb{P}_n(\mathbf{x})\mathbb{P}^T_m(\mathbf{x}) d\sigma(\mathbf{x}) = \begin{cases} 0 & \text{if } m \neq n \\ H_n & \text{if } m = n, \end{cases} \tag{3}$$

where H_n is a symmetric and positive $r^d_n \times r^d_n$ matrix. If H_n is the identity matrix $\forall n \geq 0$ then $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$ is called an *orthonormal polynomial system (OPS)*. A step-by-step method of constructing a corresponding orthonormal polynomial sequence is known as the Gram-Schmidt process. It produces real orthonormal polynomials, see [2], generalizing this method it is always possible from an orthogonal polynomials system to obtain a system of orthonormal polynomials.

We define the kernel function of V^d_j by

$$P_j(\mathbf{x}, \mathbf{y}) = \mathbb{P}^T_j(\mathbf{x}) H^{-1}_j \mathbb{P}_j(\mathbf{y}) = P_j(\mathbf{y}, \mathbf{x}), \quad j \geq 0, \tag{4}$$

and the *kernel function* of Π^d_n by

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} P_j(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} \mathbb{P}^T_j(\mathbf{x}) H^{-1}_j \mathbb{P}_j(\mathbf{y}) = K_n(\mathbf{y}, \mathbf{x}), \quad n \geq 0. \tag{5}$$

The definition of $K_n(\mathbf{x}, \mathbf{y})$ does not depend on a particular basis, but it is often more convenient to work with an orthonormal basis, see [5]. Thus, the kernel function adopts a simpler expression

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} \mathbb{P}^T_j(\mathbf{x}) \mathbb{P}_j(\mathbf{y}).$$

Let f be a real valued function on d variables. We will use the *gradient operator* ∇ defined as usual

$$\nabla f(\mathbf{x}) = (\partial_1 f(\mathbf{x}), \partial_2 f(\mathbf{x}), \ldots, \partial_d f(\mathbf{x})) \in M_{1\times d}(\Pi^d).$$

The gradient operator can be extended for vector polynomials. If $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$

is an OPS, for $n \geq 0$, we introduce, see [7]

$$\nabla\mathbb{P}_n(\mathbf{x}) = (\partial_1\mathbb{P}_n(\mathbf{x})|\partial_2\mathbb{P}_n(\mathbf{x})|\dots|\partial_d\mathbb{P}_n(\mathbf{x}))$$
$$= \begin{pmatrix} \partial_1 P_{\kappa_1(\mathbf{x})} & \partial_2 P_{\kappa_1(\mathbf{x})} & \partial_3 P_{\kappa_1(\mathbf{x})} & \cdots & \partial_d P_{\kappa_1(\mathbf{x})} \\ \partial_1 P_{\kappa_2(\mathbf{x})} & \partial_2 P_{\kappa_2(\mathbf{x})} & \partial_3 P_{\kappa_2(\mathbf{x})} & \cdots & \partial_d P_{\kappa_2(\mathbf{x})} \\ \partial_1 P_{\kappa_3(\mathbf{x})} & \partial_2 P_{\kappa_3(\mathbf{x})} & \partial_3 P_{\kappa_3(\mathbf{x})} & \cdots & \partial_d P_{\kappa_3(\mathbf{x})} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial_1 P_{\kappa_{r_n^d}(\mathbf{x})} & \partial_2 P_{\kappa_{r_n^d}(\mathbf{x})} & \partial_3 P_{\kappa_{r_n^d}(\mathbf{x})} & \cdots & \partial_d P_{\kappa_{r_n^d}(\mathbf{x})} \end{pmatrix} \in M_{r_n^d \times d}(\Pi^d),$$

and, for a higher order gradient, we define, see [4]

$$\nabla^{(j)}\mathbb{P}_n = \nabla^{(j)}\mathbb{P}_n(\mathbf{x}) = \left(\partial^j_{\beta_1}\mathbb{P}_n(\mathbf{x})|\partial^j_{\beta_2}\mathbb{P}_n(\mathbf{x})|\dots|\partial^j_{\beta_{d^j}}\mathbb{P}_n(\mathbf{x})\right) \in M_{r_n^d \times d^j}(\Pi^d), \quad (6)$$

where, $\partial^j_{\beta_i} = \frac{\partial^j}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_2}\cdots\partial^j\partial x_d^{\gamma_d}}$ and β_i runs through all d^j combinations of j total derivatives with respect to d different variables (i.e., all different combinations of $\gamma_1, \gamma_2, \dots, \gamma_d \in \mathbb{N}$ such that $\gamma_1 + \gamma_2 + \cdots + \gamma_d = j$, arranged according to the lexicographic order).

Moreover, we define the vectors

$$K_n^{(j,0)}(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{n}\left(\nabla^{(j)}\mathbb{P}_i(\mathbf{x})\right)^T H_i^{-1}\mathbb{P}_i(\mathbf{y}) \in M_{d^j\times 1}(\Pi^d), \quad (7)$$

$$K_n^{(0,j)}(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{n}\mathbb{P}_i^T(\mathbf{x})H_i^{-1}\nabla^{(j)}\mathbb{P}_i(\mathbf{y}) \in M_{1\times d^j}(\Pi^d), \quad (8)$$

which satisfy $K_n^{(j,0)}(\mathbf{x},\mathbf{y}) = (K_n^{(0,j)}(\mathbf{y},\mathbf{x}))^T$, and the matrix

$$K_n^{(j,j)}(\mathbf{x},\mathbf{y}) = \left(\partial^j_{\beta_i}\partial^j_{\eta_k}K_n(\mathbf{x},\mathbf{y})\right)_{i,k=1}^{d^j}, \quad (9)$$

where, as above, β_i (resp. η_k) runs through all d^j combinations of j total derivatives with respect to d different variables in $\mathbf{x}$ (resp. $\mathbf{y}$).

In [4], the authors show the following lemma.

Lemma 2.1. *Let $\lambda \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^d$ a fixed point. For $n \geq 0$, $I_{d^j} + \lambda K_n^{(j,j)}(\xi,\xi)$ is a symmetric and non singular matrix.*

3. A higher order Sobolev-type inner product

We consider the following Sobolev-type inner product, see [4]

$$\langle p(\mathbf{x}), q(\mathbf{x})\rangle_\mu = \langle p(\mathbf{x}), q(\mathbf{x})\rangle_\sigma + \lambda\nabla^{(j)}p(\xi)(\nabla^{(j)}q(\xi))^T \quad (10)$$

where $\xi \in \mathbb{R}^d$, $\lambda \in \mathbb{R}^+$ and $j \in \mathbb{N}$.

We will denote the OPS with respect to (10) by $\{\mathbb{Q}_n(\mathbf{x})\}_{n\geq 0}$ and the OPS with respect to (2) by $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$.

We define the kernel function of V_j^d associated with $\{\mathbb{Q}_n(\mathbf{x})\}_{n\geq 0}$ by

$$Q_i(\mathbf{x},\mathbf{y}) = \mathbb{Q}_i^T(\mathbf{x})G_i^{-1}\mathbb{Q}_j(\mathbf{y}) = Q_i(\mathbf{y},\mathbf{x}), \quad j \geq 0, \quad (11)$$

see [4], and the kernel function of Π_n^d associated with $\{\mathbb{Q}_n(\mathbf{x})\}_{n\geq 0}$ by

$$\tilde{K}_n(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{n} Q_i(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{n} \mathbb{Q}_i^T(\mathbf{x}) G_i^{-1} \mathbb{Q}_i(\mathbf{y}), \qquad n \geq 0. \tag{12}$$

The following two theorems whose proof appears in [4] establish an explicit relation between $\{\mathbb{Q}_n(\mathbf{x})\}_{n\geq 0}$ and $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$ and their respective kernel functions.

Theorem 3.1. *Let $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$ be the OPS associated with the inner product (2). Define a Sobolev-type inner product as in (10). Then if we denote by $\{\mathbb{Q}_n(\mathbf{x})\}_{n\geq 0}$ its corresponding OPS, normalized such that $\mathbb{Q}_n(\mathbf{x})$ and $\mathbb{P}_n(\mathbf{x})$ have the same leading coefficient, we have $\mathbb{Q}_0(\mathbf{x}) = \mathbb{P}_0(\mathbf{x})$, and for $n > 0$,*

$$\mathbb{Q}_n(\mathbf{x}) = \mathbb{P}_n(\mathbf{x}) - \lambda \nabla^{(j)} \mathbb{P}_n(\xi) \left(I_{d^j} + \lambda K_{n-1}^{(j,j)}(\xi,\xi) \right)^{-1} K_{n-1}^{(j,0)}(\xi, x). \tag{13}$$

Conversely, if we define $\{\mathbb{Q}_n(\mathbf{x})\}_{n\geq 0}$ as in (13), then they are an OPS with respect to (10).

Theorem 3.2. *For $i \geq 1$, we have:*

$$\begin{aligned} Q_i(\mathbf{x},\mathbf{y}) = P_i(\mathbf{x},\mathbf{y}) &- \lambda (K_i^{(j,0)}(\xi,\mathbf{x}))^T \left(I_{d^j} + \lambda K_i^{(j,j)}(\xi,\xi) \right)^{-1} K_i^{(j,0)}(\xi,\mathbf{y}) \\ &+ \lambda (K_{i-1}^{(j,0)}(\xi,\mathbf{x}))^T \left(I_{d^j} + \lambda K_{i-1}^{(j,j)}(\xi,\xi) \right)^{-1} K_{i-1}^{(j,0)}(\xi,\mathbf{y}), \end{aligned} \tag{14}$$

where we assume $K_0^{(j,0)}(\mathbf{x},\mathbf{y}) = 0$. Furthermore,

$$\tilde{K}_n(\mathbf{x},\mathbf{y}) = K_n(\mathbf{x},\mathbf{y}) - \lambda (K_n^{(j,0)}(\xi,\mathbf{x}))^T \left(I_{d^j} + \lambda K_n^{(j,j)}(\xi,\xi) \right)^{-1} K_n^{(j,0)}(\xi,\mathbf{y}). \tag{15}$$

4. An example on the unit ball in $\mathbb{R}^d$

Let $B^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\}$ the unit ball in $\mathbb{R}^d$, where $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ is the Euclidean norm. We denote $\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + x_2y_2 + \cdots + x_dy_d$ the standard inner product on $\mathbb{R}^d$.

As in [4], we analyze a particular case of the Sobolev-type inner product defined in (10). We consider the weight function

$$W_\mu(\mathbf{x}) = \left(1 - \|\mathbf{x}\|^2\right)^{\mu - \frac{1}{2}}, \qquad \mu \geq -\frac{1}{2}, \qquad \mathbf{x} \in B^d. \tag{16}$$

Associated with $W_\mu(\mathbf{x})$, we define the inner product on the unit ball

$$\langle f, g\rangle_\sigma = N_\mu \int_{B^d} f(\mathbf{x}) g(\mathbf{x}) W_\mu(\mathbf{x}) dx_1 \ldots dx_d, \tag{17}$$

where N_μ is the normalizing constant in order to have $\langle 1, 1\rangle_\sigma = 1$ and it is given by

$$N_\mu = \left(\int_{B^d} W_\mu(\mathbf{x}) dx \right)^{-1} = \frac{\Gamma\left(\mu + \frac{d+1}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(\mu + \frac{1}{2}\right)}.$$

The family of orthogonal polynomials with respect to (17) is called *the classical orthogonal polynomials on the unit ball*, see [9].

We will denote by $\{P_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ the family of Jacobi polynomials in one variable, which are orthogonal with respect to the weight function $(1-x)^{\alpha}\ (1+x)^{\beta}$ on the interval $[-1,1]$, where $\alpha,\beta > -1$. The family of Jacobi polynomials satisfies

$$P_n^{(\alpha,\beta)}(1) = \left(\begin{array}{c} n+\alpha \\ n \end{array} \right) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}, \tag{18}$$

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n P_n^{(\beta,\alpha)}(1) = (-1)^n \binom{n+\beta}{n} = (-1)^n \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)}, \tag{19}$$

$$\frac{dP_n^{(\alpha,\beta)}(x)}{dx} = C_{n,\alpha,\beta} P_{n-1}^{(\alpha+1,\beta+1)}(x), \qquad C_{n,\alpha,\beta} = \frac{1}{2}(n+\alpha+\beta+1). \tag{20}$$

From the Stirling formula for the asymptotics of the Gamma function, see [1]

$$\frac{\Gamma(n+k)}{\Gamma(n+1)} = n^{k-1}\left(1+\mathcal{O}(n^{-1})\right), \qquad n\to\infty.$$

As a consequence, when $n\to\infty$,

$$P_n^{(\alpha,\beta)}(1) = \frac{1}{\Gamma(\alpha+1)} n^{\alpha}\left(1+\mathcal{O}(n^{-1})\right), \tag{21}$$

$$P_n^{(\alpha,\beta)}(-1) = \frac{1}{\Gamma(\beta+1)}(-1)^n n^{\beta}\left(1+\mathcal{O}(n^{-1})\right), \tag{22}$$

$$C_{n,\alpha,\beta} = n\left(1+\mathcal{O}(n^{-1})\right). \tag{23}$$

In [4] the authors studied a family of orthogonal polynomials associated to the Sobolev-type inner product defined in (10), for a particular case where the measure is $(1-\|\mathbf{x}\|^2)^{\mu-\frac{1}{2}}$. They analyzed the asymptotic behavior of the corresponding kernel function $K_n(\mathbf{x},\mathbf{y})$ and the matrix (9), for $j=2$ and $\mathbf{x}=\mathbf{y}=0\in\mathbb{R}^d$, i.e., $K_n^{(2,2)}(0,0)$. As a continuation, we will first perform an analysis of the asymptotic behavior of the kernel, not only evaluating in the points $\mathbf{x}=\mathbf{y}=0\in\mathbb{R}^d$, but evaluating in the points $\mathbf{x}=0\in\mathbb{R}^d$ and $\mathbf{y}\in\mathbb{R}^d$ with $\|\mathbf{y}\|=1$.

Then we evaluate in the points $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$ such that $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, i.e., we analyze the asymptotic behavior of the kernel function $K_n^{(2,2)}(0,\mathbf{y}_{\|\mathbf{y}\|=1})$ and $K_n^{(2,2)}(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}_{\|\mathbf{y}\|=1})$ respectively, where $\mathbf{y}_{\|\mathbf{y}\|=1}$ means a point $\mathbf{y}\in\mathbb{R}^d$ such that $\|\mathbf{y}\|=1$ (similar to $\mathbf{x}_{\|\mathbf{x}\|=1}$).

4.1. The kernel function $K_n^{(2,2)}(0,\mathbf{y}_{\|\mathbf{y}\|=1})$

We consider the orthogonal polynomials with respect to the Sobolev-type inner product (10), where $\langle f,g\rangle_{\sigma}$ is defined in (17). We are interested in analyzing the asymptotic behavior of the corresponding kernel functions for $j=2$, $\xi_1=0$ and $\|\xi_2\|=1$. At the first time we find an expression for both, $K_n^{(0,2)}(0,\mathbf{y})$ and $K_n^{(2,2)}(0,\mathbf{y}_{\|\mathbf{y}\|=1})$.

Taking into account that

$$K_n(\mathbf{x},\mathbf{y}) = b_{\mu}A_n\int_{-1}^{1} P_n^{(\alpha,\alpha-1)}(w)(1-t^2)^{\mu-1}dt, \tag{24}$$

see [10], where

$$b_\mu = \left[\int_{-1}^{1}(1-t^2)^{\mu-1}dt\right]^{-1} = \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\mu)}, \tag{25}$$

$$\alpha(\mu,d) = \alpha = \mu + \frac{d}{2},$$

$$A_n = \frac{2\Gamma(\alpha+1)\Gamma(n+2\alpha)}{\Gamma(2\alpha+1)\Gamma(n+\alpha)}, \tag{26}$$

$$w(\mathbf{x},\mathbf{y},t) = w = \langle \mathbf{x},\mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}t.$$

Taking partial derivatives with respect to the variables x_r and x_s, integrating by parts and evaluating at $\mathbf{x}=0$,

$$\begin{aligned}\frac{\partial^2}{\partial x_s \partial x_r}K_n(0,\mathbf{y}) = {} & y_r y_s b_\mu A_n C_{n,\alpha,\alpha-1}C_{n-1,\alpha+1,\alpha}\cdot \\ & \int_{-1}^{1} P_{n-2}^{(\alpha+2,\alpha+1)}\left(\sqrt{1-\|\mathbf{y}\|^2}t\right)(1-t^2)^{\mu-1}dt \\ & - \frac{1-\|\mathbf{y}\|^2}{2\mu}\delta_{r,s} b_\mu A_n C_{n,\alpha,\alpha-1}C_{n-1,\alpha+1,\alpha}\cdot \\ & \int_{-1}^{1} P_{n-2}^{(\alpha+2,\alpha+1)}\left(\sqrt{1-\|\mathbf{y}\|^2}t\right)(1-t^2)^{\mu}dt. \end{aligned} \tag{27}$$

Moreover, using,

$$\int_{-1}^{1} P_{n-1}^{(\mu+\frac{d}{2}+1,\mu+\frac{d}{2})}(\sqrt{1-\|\mathbf{y}\|^2}t)(1-t^2)^{\mu-1}dt = h_{n,\mu,d} P_{[\frac{n-1}{2}]}^{(\frac{d}{2}+1,\mu-\frac{1}{2})}(1-2\|\mathbf{y}\|^2), \tag{28}$$

see [7], with

$$h_{n,\mu,d} = \frac{4\Gamma(\mu+\frac{1}{2})\Gamma([\frac{n-1}{2}]+\mu+\frac{d+1}{2}+1)\Gamma(n+2\mu+d)}{\Gamma(\mu+\frac{d+1}{2})\Gamma([\frac{n-1}{2}]+\mu+\frac{1}{2})\Gamma(n+2\mu+d+1)b_\mu A_n}, \tag{29}$$

from (27), (28) and as $\alpha = \mu + \frac{d}{2}$, it follows:

$$\begin{aligned}\frac{\partial^2}{\partial x_s \partial x_r}K_n(0,\mathbf{y}) = {} & y_r y_s B_1 P_{[\frac{n-2}{2}]}^{(\frac{d}{2}+2,\mu-\frac{1}{2})}(1-2\|\mathbf{y}\|^2) - (1-\|\mathbf{y}\|^2)\delta_{r,s}B_2 \\ & P_{[\frac{n-2}{2}]}^{(\frac{d}{2}+1,\mu+\frac{1}{2})}(1-2t\|\mathbf{y}\|^2), \end{aligned} \tag{30}$$

with

$$B_1 = b_\mu A_n C_{n,\mu+\frac{d}{2},\mu+\frac{d}{2}-1}C_{n-1,\mu+\frac{d}{2}+1,\mu+\frac{d}{2}}h_{n-1,\mu,d+2},$$

$$B_2 = (2\mu)^{-1} b_\mu A_n C_{n,\mu+\frac{d}{2},\mu+\frac{d}{2}-1}C_{n-1,\mu+\frac{d}{2}+1,\mu+\frac{d}{2}}h_{n-1,\mu+1,d}.$$

Now, taking partial derivatives of $\frac{\partial^2}{\partial x_s \partial x_r} K_n(0,\mathbf{y})$ with respect to the variables y_i and y_j and evaluating at $\mathbf{y}_{\|\mathbf{y}\|=1}$

$$\begin{aligned}\frac{\partial^4}{\partial y_j \partial y_i \partial x_s \partial x_r} K_n(0,\mathbf{y}_{\|\mathbf{y}\|=1}) = -4\big[&y_i y_r \delta_{s,j} + y_i y_s \delta_{r,j} + y_r y_s \delta_{i,j} + y_j y_r \delta_{s,i} \\ &+ y_j y_s \delta_{r,i}\big]\cdot \\ B_1 C_{[\frac{n-2}{2}],\frac{d}{2}+2,\mu-\frac{1}{2}} P^{(\frac{d}{2}+3,\mu+\frac{1}{2})}_{[\frac{n-2}{2}]-1}(-1) &+ (\delta_{r,j}\delta_{s,i} + \delta_{s,j}\delta_{r,i}) B_1 P^{(\frac{d}{2}+2,\mu-\frac{1}{2})}_{[\frac{n-2}{2}]}(-1) \\ + 16 y_j y_i y_r y_s B_1 C_{[\frac{n-2}{2}],\frac{d}{2}+2,\mu-\frac{1}{2}} &C_{[\frac{n-2}{2}]-1,\frac{d}{2}+3,\mu+\frac{1}{2}} P^{(\frac{d}{2}+4,\mu+\frac{3}{2})}_{[\frac{n-2}{2}]-2}(-1) \\ + 2\delta_{i,j}\delta_{r,s} B_2 P^{(\frac{d}{2}+1,\mu+\frac{1}{2})}_{[\frac{n-2}{2}]}(-1) &- 16 y_i y_j \delta_{r,s} B_2 C_{[\frac{n-2}{2}],\frac{d}{2}+1,\mu+\frac{1}{2}} P^{(\frac{d}{2}+2,\mu+\frac{3}{2})}_{[\frac{n-2}{2}]-1}(-1).\end{aligned} \tag{31}$$

We denote,

$$z_1 = -4B_1 C_{[\frac{n-2}{2}],\frac{d}{2}+2,\mu-\frac{1}{2}} P^{(\frac{d}{2}+3,\mu+\frac{1}{2})}_{[\frac{n-2}{2}]-1}(-1), \tag{32}$$

$$z_2 = B_1 P^{(\frac{d}{2}+2,\mu-\frac{1}{2})}_{[\frac{n-2}{2}]}(-1), \tag{33}$$

$$z_3 = 16 B_1 C_{[\frac{n-2}{2}],\frac{d}{2}+2,\mu-\frac{1}{2}} C_{[\frac{n-2}{2}]-1,\frac{d}{2}+3,\mu+\frac{1}{2}} P^{(\frac{d}{2}+4,\mu+\frac{3}{2})}_{[\frac{n-2}{2}]-2}(-1), \tag{34}$$

$$z_4 = 2B_2 P^{(\frac{d}{2}+1,\mu+\frac{1}{2})}_{[\frac{n-2}{2}]}(-1), \tag{35}$$

$$z_5 = -16 B_2 C_{[\frac{n-2}{2}],\frac{d}{2}+1,\mu+\frac{1}{2}} P^{(\frac{d}{2}+2,\mu+\frac{3}{2})}_{[\frac{n-2}{2}]-1}(-1). \tag{36}$$

As a consequence, the $d^2 \times d^2$ matrix $K_n^{(2,2)}(0,\mathbf{y}_{\|\mathbf{y}\|=1})$, whose entries are arranged according to the orders of their partial derivatives, has elements

$$z_1\left(y_i y_r \delta_{s,j} + y_i y_s \delta_{r,j} + y_r y_s \delta_{i,j} + y_j y_r \delta_{s,i} + y_j y_s \delta_{r,i}\right)$$

$$+ z_2(\delta_{r,j}\delta_{s,i} + \delta_{s,j}\delta_{r,i}) + z_3 y_j y_i y_r y_s + z_4 \delta_{i,j}\delta_{r,s} + z_5 y_i y_j \delta_{r,s},$$

where $1 \leq r,s,i,j \leq d$.

From $Kronecker$'s deltas we can observe,

1. Let i,j,r,s, z_1 survives if at least two are equal.
2. z_2 survives if $(i = s \wedge r = j) \vee (j = s \wedge r = i)$.
3. z_3 survives in all entries of the matrix.
4. If i,j,r,s are different two by two only z_3 survive.
5. z_4 survives if $r = s \wedge i = j$.
6. z_5 survives if $r = s$.

Below we give a first conjecture, regarding the distribution of the d^4 entries of the matrix $K_n^{(2,2)}(0,\mathbf{y}_{\|\mathbf{y}\|=1})$.

Conjecture. The d^4 entries of the matrix $K_n^{(2,2)}(0,\mathbf{y}_{\|\mathbf{y}\|=1})$ are distributed as follows

1. d^2 entries appear only once.
2. $\frac{d(d-1)(d+2)}{2}$ entries appear 2 times each one.

3. $\frac{d(d-1)}{2}$ entries appear 4 times each one.
4. $\frac{d(d-1)(d-2)}{2}$ entries appear 10 times each one.
5. $\frac{d(d-1)(d-2)(d-3)}{24}$ entries appear 24 times each one.

Lemma 4.1. *Let b_μ, A_n and $h_{n,\mu,d}$ defined in (25), (26) and (29) respectively. When $n \longrightarrow \infty$,*

$$b_\mu A_n = \frac{2\Gamma(\mu+\frac{1}{2})\Gamma(\mu+\frac{d}{2}+1)}{\sqrt{\pi}\Gamma(\mu)\Gamma(2\mu+d+1)} n^{\mu+\frac{d}{2}} \left(1+\mathcal{O}(n^{-1})\right), \tag{37}$$

$$h_{n,\mu,d} = \frac{2\sqrt{\pi}\Gamma(\mu)\Gamma(2\mu+d+1)}{\Gamma(\mu+\frac{d+1}{2})\Gamma(\mu+\frac{d}{2}+1)} n^{-\mu} \left(1+\mathcal{O}(n^{-1})\right). \tag{38}$$

Proof. It is obtained using the Stirling formula in the definitions of b_μ, A_n and $h_{n,\mu,d}$. □

Theorem 4.2. *Let z_1, z_2, z_3, z_4 and z_5 defined from (32) to (36). When $n \longrightarrow \infty$,*

$$z_1 = k_1 \cdot n^{\frac{d}{2}+\mu+\frac{7}{2}} \left(1+\mathcal{O}(n^{-1})\right), \tag{39}$$

$$z_2 = k_2 \cdot n^{\frac{d}{2}+\mu+\frac{3}{2}} \left(1+\mathcal{O}(n^{-1})\right), \tag{40}$$

$$z_3 = k_3 \cdot n^{\frac{d}{2}+\mu+\frac{11}{2}} \left(1+\mathcal{O}(n^{-1})\right), \tag{41}$$

$$z_4 = k_4 \cdot n^{\frac{d}{2}+\mu+\frac{3}{2}} \left(1+\mathcal{O}(n^{-1})\right), \tag{42}$$

$$z_5 = k_5 \cdot n^{\frac{d}{2}+\mu+\frac{7}{2}} \left(1+\mathcal{O}(n^{-1})\right), \tag{43}$$

where k_1, k_2, k_3, k_4 and k_5 are constants depending of μ and d.

Proof. It is obtained directly by applying (16), (17), (37) and (38) to the definitions given. □

On the other hand, we have the following result.

Theorem 4.3. *Let $K_n(\mathbf{x},\mathbf{y})$ be the kernel function associated to $\{\mathbb{P}_n(\mathbf{x})\}_{n\geq 0}$, the OPS with respect to (17), then for $n \to \infty$*

(i) $K_n^{(2,0)}(0,\mathbf{y}) = n^{d+2}\left(1+\mathcal{O}(n^{-1})\right)$ *if* $\mathbf{y}=0$.

(ii) $K_n^{(2,0)}(0,\mathbf{y}) = n^{\frac{d+3}{2}}\left(1+\mathcal{O}(n^{-1})\right)$ *uniformly in compacts inside* $B^d - \{0\}$.

(iii) $K_n^{(2,0)}(0,\mathbf{y}) = n^{\mu+\frac{d+3}{2}}\left(1+\mathcal{O}(n^{-1})\right)$ *if* $\|\mathbf{y}\|=1$.

Proof. (i) From (30), evaluating in $\|\mathbf{y}\|=0$ the first term vanishes and considering (21), (23) together with the Lemma 4.1. we get the desired expression.

(ii) We get the result using the fact that $\left|P_n^{(a,b)}(t)\right| \leq Cn^{-1/2}$ uniformly compacts subsets of $(-1,1)$, see [8] and (23) together with the Lemma 4.1.

(iii) From (30), evaluating in $\|\mathbf{y}\|=1$ the second term vanishes and considering (22), (23) together with the Lemma 4.1. we get the desired expression.

□

4.2. The kernel function $K_n^{(2,2)}\left(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}_{\|\mathbf{y}\|=1}\right)$

We consider the orthogonal polynomials with respect to the Sobolev-type inner product (10), where $\langle f,g\rangle_\sigma$ is defined in (17). We are interested in analyzing the asymptotic behavior of the corresponding kernel functions for $j=2$, $\|\xi_1\|=1$ and $\|\xi_2\|=1$. First, we find an expression for both $K_n^{(0,2)}(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y})$ and $K_n^{(2,2)}(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}_{\|\mathbf{y}\|=1})$.

Taking partial derivatives with respect to the variables x_r and x_s in (24), integrating by parts, evaluating at $\|x\|=1$, using (25) and the fact that $2\mu b_{\mu+1}=(2\mu+1)b_\mu$ we get,

$$
\begin{aligned}
\frac{\partial^2}{\partial x_s\partial x_r}K_n(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}) &= \left(y_r y_s-\frac{1-\|\mathbf{y}\|^2}{2\mu+1}\delta_{r,s}\right)A_nC_{n,\alpha,\alpha-1}C_{n-1,\alpha+1,\alpha}\\
&\quad P_{n-2}^{(\alpha+2,\alpha+1)}\left(\left\langle\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}\right\rangle\right)\\
&-\frac{1-\|\mathbf{y}\|^2}{2\mu+1}(x_sy_r+x_ry_s)A_nC_{n,\alpha,\alpha-1}C_{n-1,\alpha+1,\alpha}C_{n-2,\alpha+2,\alpha+1}P_{n-3}^{(\alpha+3,\alpha+2)}\\
&\quad\left(\left\langle\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}\right\rangle\right)\\
&+\frac{\left(1-\|\mathbf{y}\|^2\right)^2}{(2\mu+1)(2\mu+3)}x_rx_sA_nC_{n,\alpha,\alpha-1}C_{n-1,\alpha+1,\alpha}C_{n-2,\alpha+2,\alpha+1}C_{n-3,\alpha+3,\alpha+2}\\
&\quad P_{n-4}^{(\alpha+4,\alpha+3)}\left(\left\langle\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}\right\rangle\right).
\end{aligned}
$$

Now, taking partial derivatives of $\frac{\partial^2}{\partial x_s\partial x_r}K_n(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y})$ with respect to the variables y_i and y_j and evaluating at $\|\mathbf{y}\|=1$, we have

$$
\begin{aligned}
\frac{\partial^4}{\partial y_j\partial y_i\partial x_s\partial x_r}K_n\left(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}_{\|\mathbf{y}\|=1}\right) &= K_n^{(2,2)}\left(\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}_{\|\mathbf{y}\|=1}\right)\\
&=\left(\delta_{r,j}\delta_{s,i}+\delta_{s,j}\delta_{r,i}+\frac{2}{2\mu+1}\delta_{i,j}\delta_{r,s}\right)g_1\\
+\Bigg[\delta_{s,i}\left(x_jy_r+\frac{2}{2\mu+1}x_ry_j\right)&+\delta_{r,i}\left(x_jy_s+\frac{2}{2\mu+1}x_sy_j\right)\\
&+\delta_{s,j}\left(x_iy_r+\frac{2}{2\mu+1}x_ry_i\right)\Bigg]g_2\\
+\Bigg[\delta_{r,j}\left(x_iy_s+\frac{2}{2\mu+1}x_sy_i\right)&+\delta_{r,s}\left(x_jy_i+\frac{2}{2\mu+1}x_iy_j\right)\\
&+\delta_{i,j}\left(x_sy_r+\frac{2}{2\mu+1}x_ry_s\right)\Bigg]g_2\\
+\Bigg[\left(x_jx_iy_ry_s+\frac{4}{2\mu+3}x_rx_sy_jy_i\right)&+\frac{2}{2\mu+1}\left(x_ix_sy_jy_r+x_ix_ry_jy_s\right)\\
&+\frac{2}{2\mu+1}\left(x_jx_sy_iy_r+x_jx_ry_iy_s\right)\Bigg]g_3,
\end{aligned}
$$

where $1\le r,s,i,j\le d$ and

$$
g_1=A_nC_{n,\alpha,\alpha-1}C_{n-1,\alpha+1,\alpha}P_{n-2}^{(\alpha+2,\alpha+1)}\left(\left\langle\mathbf{x}_{\|\mathbf{x}\|=1},\mathbf{y}_{\|\mathbf{y}\|=1}\right\rangle\right), \tag{44}
$$

$$g_2 = A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} P_{n-3}^{(\alpha+3,\alpha+2)} \left(\left\langle \mathbf{x}_{\|\mathbf{x}\|=1}, \mathbf{y}_{\|\mathbf{y}\|=1} \right\rangle \right), \qquad (45)$$

$$g_3 = A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} C_{n-3,\alpha+3,\alpha+2} \cdot P_{n-4}^{(\alpha+4,\alpha+3)} \left(\left\langle \mathbf{x}_{\|\mathbf{x}\|=1}, \mathbf{y}_{\|\mathbf{y}\|=1} \right\rangle \right). \qquad (46)$$

From *Kronecker's* deltas we can observe,

1. j, i, r, s, g_1 survive only in two cases: when all of them are equal and when two of them are equal to each other and the other two are equal to each other.
2. If j, i, r, s are different two by two, g_2 does not survive.
3. g_3 survives in all entries of the matrix.

Below we give a second conjecture, regarding the distribution of the d^4 entries of the matrix $K_n^{(2,2)}(\mathbf{x}_{\|\mathbf{x}\|=1}, \mathbf{y}_{\|\mathbf{y}\|=1})$.

Conjecture. The d^4 entries of the matrix $K_n^{(2,2)}\left(\mathbf{x}_{\|\mathbf{x}\|=1}, \mathbf{y}_{\|\mathbf{y}\|=1}\right)$ are distributed as follows

1. d^2 entries appear only once.
2. $d^2(d-1)$ entries appear 2 times each one.
3. $\left[\frac{d(d-1)}{2}\right]^2$ entries appear 4 times each one.

Lemma 4.4. *Let A_n defined as in (26), then for $n \longrightarrow \infty$,*

$$A_n = \frac{2\Gamma(\mu + \frac{d}{2} + 1)}{\Gamma(2\mu + d + 1)} n^{\mu + \frac{d}{2}} (1 + \mathcal{O}(n^{-1})), \qquad (47)$$

Proof. It is obtained using the Stirling formula in the definitions in (26). □

The behavior of g_1, g_2 and g_3 is analyzed in two situations: when $\mathbf{x} = \mathbf{y}$ and when $\mathbf{x} = -\mathbf{y}$,

Theorem 4.5. *Let g_1, g_2 and g_3 defined in (44), (45) and (46), when $n \longrightarrow \infty$.*

(i) If $\mathbf{x} = \mathbf{y}$

$$g_1 = k_6 \cdot n^{2\mu+d+4} \left(1 + \mathcal{O}(n^{-1})\right), \qquad (48)$$

$$g_2 = k_7 \cdot n^{2\mu+d+6} \left(1 + \mathcal{O}(n^{-1})\right), \qquad (49)$$

$$g_3 = k_8 \cdot n^{2\mu+d+8} \left(1 + \mathcal{O}(n^{-1})\right), \qquad (50)$$

where k_6, k_7 and k_8 are constants depending of μ and d.

(ii) $\mathbf{x} = -\mathbf{y}$

$$g_1 = k_9 \cdot n^{2\mu+d+3} \left(1 + \mathcal{O}(n^{-1})\right), \qquad (51)$$

$$g_2 = k_{10} \cdot n^{2\mu+d+5} \left(1 + \mathcal{O}(n^{-1})\right), \qquad (52)$$

$$g_3 = k_{11} \cdot n^{2\mu+d+7} \left(1 + \mathcal{O}(n^{-1})\right), \qquad (53)$$

where k_9, k_{10} and k_{11} are constants depending on μ and d.

Proof. (i) When $\mathbf{x}=\mathbf{y}$, g_1, g_2 and g_3 are given by

$$
\begin{aligned}
g_1 &= A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} P_{n-2}^{(\alpha+2,\alpha+1)}(1),\\
g_2 &= A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} P_{n-3}^{(\alpha+3,\alpha+2)}(1),\\
g_3 &= A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} C_{n-3,\alpha+3,\alpha+2} P_{n-4}^{(\alpha+4,\alpha+3)}(1).
\end{aligned}
$$

using (21), (23) and the Lemma 4.4., when $n \longrightarrow \infty$,

$$
\begin{aligned}
g_1 &= k_6 \cdot n^{2\mu+d+4}\left(1+\mathcal{O}(n^{-1})\right),\\
g_2 &= k_7 \cdot n^{2\mu+d+6}\left(1+\mathcal{O}(n^{-1})\right),\\
g_3 &= k_8 \cdot n^{2\mu+d+8}\left(1+\mathcal{O}(n^{-1})\right).
\end{aligned}
$$

(ii) When $\mathbf{x}=-\mathbf{y}$, g_1, g_2 and g_3 are given by

$$
\begin{aligned}
g_1 &= A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} P_{n-2}^{(\alpha+2,\alpha+1)}(-1),\\
g_2 &= A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} P_{n-3}^{(\alpha+3,\alpha+2)}(-1),\\
g_3 &= A_n C_{n,\alpha,\alpha-1} C_{n-1,\alpha+1,\alpha} C_{n-2,\alpha+2,\alpha+1} C_{n-3,\alpha+3,\alpha+2} P_{n-4}^{(\alpha+4,\alpha+3)}(-1),
\end{aligned}
$$

using (22), (23) and the lemma 4.4., when $n \longrightarrow \infty$,

$$
\begin{aligned}
g_1 &= k_9 \cdot n^{2\mu+d+3}\left(1+\mathcal{O}(n^{-1})\right),\\
g_2 &= k_{10} \cdot n^{2\mu+d+5}\left(1+\mathcal{O}(n^{-1})\right),\\
g_3 &= k_{11} \cdot n^{2\mu+d+7}\left(1+\mathcal{O}(n^{-1})\right).
\end{aligned}
$$

□

Acknowledgements

The authors thank the referees for their comments. They greatly improved the manuscript.

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