Using weak forms to derive asymptotic expansions of elliptic equations with high-contrast coefficients

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Abstract. In this work we review some recent results on approximation of solutions of elliptic problems with high-contrast coefficients. In particular, we detail the derivation of asymptotic expansions for the solution in terms of the high-contrast of the coefficients. We consider the case of high-contrast elliptic equations and we present the case of only one high-contrast inclusion. The case of more inclusions follows similarly. In order to simplify the presentation we consider first the one dimensional case so no further complicated requirements regarding the computation of two or higher dimensional integrals are needed. We review the case of two-dimensional problems and give some numerical examples.

Keywords: Elliptic equations, high-contrast coefficients, asymptotic expansions.

Resumen. En este trabajo revisamos algunos resultados recientes sobre la aproximación de soluciones de problemas elípticos con coeficientes de alto contraste. En particular, detallamos la derivación de expansiones asintóticas para la solución en términos del alto contraste de los coeficientes. Consideramos el caso de ecuaciones elípticas de alto contraste y presentamos únicamente el caso de una inclusión. El caso de más inclusiones se sigue de manera similar. Con el fin de simplificar la presentación consideramos primero el caso unidimensional, para evitar complicaciones con respecto al cálculo de integrales de dos o más dimensiones. Revisamos el caso de problemas bi-dimensionales y se dan algunos ejemplos numéricos.

Palabras claves: Ecuaciones elípticas, coeficientes de alto contraste, expansiones asintóticas.

Mathematics Subject Classification: 35Q35, 65M60.

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1. Introduction

The mathematical and numerical analysis of partial differential equations in multi-scale and high-contrast media are important in many practical applications. In fact, in many applications related to fluid flows in porous media the

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coefficient represents the permeability (which is how easy the porous media let the fluid flow). The values of the permeability usually, and specially for complicated porous media, vary in several orders of magnitude. We refer to this case as the high-contrast case and we say that the corresponding elliptic equation that models (e.g. the pressure) has high-contrast coefficients, see for instance [4, 7, 8].

A fundamental purpose is to understand the effects on the solution related to the variations of high-contrast in the properties of the porous media. In terms of the model, these variations appear in the coefficients of the differential equations. In particular, this interest gives an importance for the computation of numerical solutions.

In order to devise efficient numerical strategies it is important to understand the behavior of solutions of these equations. Deriving, and manipulating expansions representing solutions certainly help in this task.

In this work we detail the derivations of asymptotic expansions for high-contrast elliptic problems. We consider expansions of the form

$$u_\eta(x) = \sum_{j=0}^{\infty} \eta^{-j} u_j(x), \tag{1}$$

and detail procedures to define and compute each of the terms in the series. This work is preliminary and complements current work in the use of similar expansions to design and analyze efficient numerical approximations of high-contrast elliptic equations, see [8].

This manuscript is organized as follows. In Section 2 we detailedly show how to work out the expansion for one dimensional problems. We give some explicit examples. In Section 3 we summarize the procedure for two dimensional problems and give some numerical examples. In Section 4 we present some conclusions and work perspective.

2. Asymptotic expansion in one dimension

In this section we detail the procedure to derive (by using weak formulations) the asymptotic expansion for the solution of a high-contrast elliptic problem. In order to simplify the presentation we have chosen a one dimensional problem with only one high-contrast inclusion. In Section 3 we summarize the procedure for higher dimensional problems.

Let us consider the following one dimensional problem (in its strong form)

$$\begin{cases} -\left(\kappa(x)\, u'(x)\right)' = f(x), & \text{for all } x \in (-1,1), \\ u(-1) = u(1) = 0, \end{cases} \tag{2}$$

where the high-contrast coefficient is given by,

$$\kappa(x) = \begin{cases} 1, & -1 \le x < -\delta \text{ or } \delta < x \le 1, \\ \eta, & -\delta \le x \le \delta. \end{cases} \tag{3}$$

In this case we assume $\eta >> 1$. This differential equation models the stationary temperature of a bar represented by the one dimensional domain $[-1,1]$. In this case, the coefficient κ models the conductivity of the bar which depends on the material the bar is made of. For this particular coefficient, the part of the domain represented by the interval $(-\delta,\delta)$ is highly-conducting when compared with the rest of the domain and we say that this medium (bar) has high-contrast conductivity properties.

We first write the weak form of this problem. We follow the usual procedure, that is, we select a space of test functions, multiply both sides of the equations by this test functions, then, we use integration by parts formula (in one dimension) and obtain the weak form. In order to fix ideas and concentrate on the derivation of the asymptotic expansion we use the usual test function and solutions spaces, in this case, that would be subspaces of $H^1(-1,1)$ for both, solutions and test functions.

Let $v \in H_0^1(-1,1)$ be a test function and $u \in H^1(-1,1)$ is the sought solution, for more details of the spaces H^1 and H_0^1 see for instance [2, 6]. Then after integration by parts we write

$$\int_{-1}^{1} \kappa u' v' = \int_{-1}^{1} f v, \qquad \text{for all } v \in H_0^1(-1,1).$$

Here u is the weak solution of problem (2). Existence and uniqueness of the weak solution follows from usual arguments (Lax-Milgram, in e.g., [2]). To emphasis the dependence of u on the contrast η, from now on, we write u_η for the solution of (2). Note that in order to simplify the notation we have omitted the integration variable x and the integration measure dx. We can split this integral in sub-domains integrals and recalling the definitions of the high-contrast conductivity coefficient $\kappa = \kappa(x)$ in (3), we obtain

$$\int_{-1}^{-\delta} u'v' + \eta \int_{-\delta}^{\delta} u'v' + \int_{\delta}^{1} u'v' = \int_{-1}^{1} fv, \tag{4}$$

for all $v \in H_0^1(-1,1)$. We observe that each integral on the left side is finite (since all the factors are in $L^2(-1,1)$).

Our goal is to write an expansion of the form

$$u_\eta(x) = \sum_{j=0}^{\infty} \eta^{-j} u_j(x), \tag{5}$$

with individual terms in $H^1(-1,1)$ such that they satisfy the Dirichlet boundary condition $u_j(-1) = u_j(1) = 0$ for $j \geq 1$. Other boundary conditions for (2) can be handled similarly. Each term will solve (weakly) boundary value problems in the sub-domains $(-1,-\delta)$, $(-\delta,\delta)$ and $(\delta,1)$. The different data on the boundary of sub-domains are revealed by the corresponding local weak formulation derived from the power series above (5). We discuss this in detail below.

We first assume that (5) is a valid solution of problem (4), see [5], thus we can substitute (5) into (4). We obtain that for all $v \in H_0^1(-1,1)$ the following holds

$$\int_{1}^{-\delta} \sum_{j=0}^{\infty} \eta^{-j} u_j' v' + \eta \int_{-\delta}^{\delta} \sum_{j=0}^{\infty} \eta^{-j} u_j' v' + \int_{\delta}^{1} \sum_{j=0}^{\infty} \eta^{-j} u_j' v' = \int_{-1}^{1} fv,$$

or, after formally interchanging integration and summation signs,

$$\sum_{j=0}^{\infty} \eta^{-j} \left[\int_{1}^{-\delta} u_j' v' + \int_{\delta}^{1} \eta^{-j} u_j' v' \right] + \sum_{j=0}^{\infty} \eta^{-j+1} \int_{-\delta}^{\delta} u_j' v' = \int_{-1}^{1} fv.$$

Rearrange these to obtain,

$$\sum_{j=0}^{\infty} \eta^{-j} \left[\int_{1}^{-\delta} u_j' v' + \int_{\delta}^{1} \eta^{-j} u_j' v' \right] + \eta \int_{-\delta}^{\delta} u_0' v' + \sum_{j=0}^{\infty} \eta^{-j} \int_{-\delta}^{\delta} u_{j+1}' v' = \int_{-1}^{1} fv,$$

which after collecting terms can be written as

$$\sum_{j=0}^{\infty} \eta^{-j} \left[\int_{1}^{-\delta} u_j' v' + \int_{-\delta}^{\delta} u_{j+1}' v' + \int_{\delta}^{1} \eta^{-j} u_j' v' \right] + \eta \int_{-\delta}^{\delta} u_0' v' = \int_{-1}^{1} fv, \quad (6)$$

which holds for all test functions $v \in H_0^1(-1,1)$. Now we match up the coefficients corresponding to equal powers on the both sides of equation (6).

2.1. Terms corresponding to η

In the equation (6) above, there is only one term with η so that we obtain

$$\int_{-\delta}^{\delta} u_0' v' = 0, \quad \text{for all } v \in H_0^1(-1,1).$$

Thus $u_0' = 0$ (which can be readily seen if we take a test function v such that $v = u_0$) and therefore u_0 is a constant in $(-\delta, \delta)$.

2.2. Terms corresponding to $\eta^0 = 1$

The next coefficients to match up are those of $\eta^0 = 1$, the coefficients in (6) with $j = 0$.

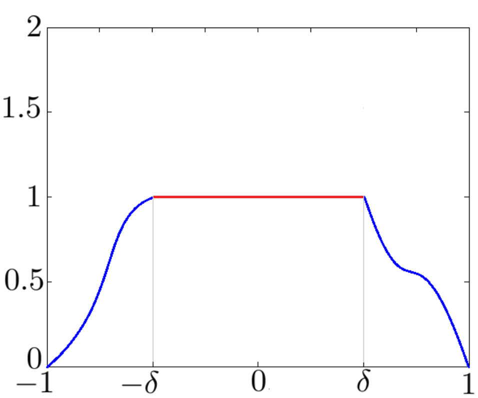


Figure 1: Function v in V_{const}.

Let

$$V_{\text{const}} = \left\{ v \in H_0^1(-1,1): \ v^{(1)} = v|_{(-\delta,\delta)} \text{ is constant} \right\}.$$

For an illustration of the function v in V_{const}, see Figure 1. We have

$$\int_{-1}^{-\delta} u_0' v' + \int_{\delta}^{1} u_0' v' = \int_{-1}^{1} f v, \quad \text{for all } v \in V_{\text{const}}. \tag{7}$$

To study further this problem we introduce the following decomposition for functions in $v \in V_{\text{const}}$. For any $v \in V_{\text{const}}$, we write

$$v = c_0 \chi + v^{(1)} + v^{(2)},$$

where $v^{(1)} \in H_0^1(-1,-\delta)$, $v^{(2)} \in H_0^1(\delta,1)$ and χ is a continuous function defined by

$$\chi(x) = \begin{cases} 1, & x \in (-\delta,\delta), \\ 0, & x = -1,\, x = 1, \\ \text{harmonic}, & \text{otherwise}. \end{cases} \tag{8}$$

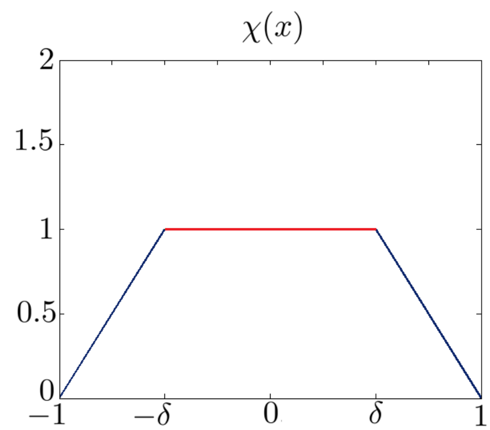


Figure 2: χ function.

See Figure 2 for an illustration of χ. Note also that

$$\chi'(x) = \begin{cases} 0, & x \in (-\delta,\delta), \\ 1/(1-\delta), & x \in (-1,-\delta), \\ -1/(1-\delta), & x \in (\delta,1), \end{cases} \tag{9}$$

and observe that $\chi \in H_0^1(-1,1)$.

The same decomposition holds for u_0, that is, $u_0 = c_0\chi + u^{(1)} + u^{(2)}$, since $u_0 \in V_{\text{const}}$. Note that this is an orthogonal decomposition, which can be verified by direct integration. Next we split (7) into three equations using the decomposition introduced above. This is done by testing again subset of test functions determined by the decomposition introduced above.

1. Test against χ, that is, let $v = \chi \in H_0^1(-1,1)$ in (7) and using that $u_0 = d\chi + u^{(1)} + u^{(2)}$ we get,

$$\int_{-1}^{-\delta} \left(c_0\chi + u^{(1)} + u^{(2)}\right)' \chi' + \int_{\delta}^{1} \left(c_0\chi + u^{(1)} + u^{(2)}\right)' \chi' = \int_{-1}^{1} f\chi, \tag{10}$$

which after simplification (using the definition of χ and the fundamental theorem of calculus) gives

$$\int_{-1}^{-\delta} c_0 \left(\chi'\right)^2 + \int_{\delta}^{1} c_0 \left(\chi'\right)^2 = \int_{-1}^{1} f\chi,$$

from which we get

$$c_0 = \frac{\int_{-1}^{1} f\chi}{\int_{-1}^{-\delta} \left(\chi'\right)^2 + \int_{\delta}^{1} \left(\chi'\right)^2} = \frac{1-\delta}{2} \int_{-1}^{1} f\chi, \tag{11}$$

where we have used the derivative of χ defined in (9) for $(-1,-\delta)$ and $(\delta,1)$ respectively.

2. We now test (7) against $v^{(1)} \in H_0^1(-1,-\delta)$ (extended by zero), that is, we take $v = v^{(1)}$ in $v^{(1)} \in H_0^1(-1,-\delta)$ to get

$$\int_{-1}^{-\delta} \left(c_0\chi + u^{(1)} + u^{(2)}\right)' \left(v^{(1)}\right)' + \int_{\delta}^{1} \left(c_0\chi + u^{(1)} + u^{(2)}\right)' \left(v^{(1)}\right)' = \int_{-1}^{1} f v^{(1)}.$$

Note that $\int_{\delta}^{1} \left(c_0\chi + u^{(1)} + u^{(2)}\right)' \left(v^{(1)}\right)' = 0$ (since $v^{(1)}$ is supported in $(-1,-\delta)$). Using this, we have that

$$\int_{-1}^{-\delta} \left(c_0\chi + u^{(1)} + u^{(2)}\right)' \left(v^{(1)}\right)' = \int_{-1}^{-\delta} f v^{(1)}.$$

After simplifying (using the definition of χ and the fundamental theorem of calculus) we have

$$\int_{-1}^{-\delta} \left(u^{(1)}\right)' \left(v^{(1)}\right)' = \int_{-1}^{-\delta} f v^{(1)}, \quad \text{for all } v^{(1)} \in H_0^1(-1,-\delta). \tag{12}$$

Recalling the boundary values of $u^{(1)}$ we see that (12) is the weak formulation of Dirichlet problem

$$\begin{cases} -\left(u^{(1)}\right)'' = f, & \text{in } (-1,-\delta), \\ u^{(1)}(-1) = 0,\ u^{(1)}(-\delta) = 0. \end{cases} \tag{13}$$

3. Testing against $v^{(2)} \in H_0^1(\delta,1)$ we get (in a similar fashion) that

$$\int_{\delta}^{1} \left(u^{(2)}\right)' \left(v^{(2)}\right)' = \int_{\delta}^{1} f v^{(2)}, \quad \text{for all } v^{(2)} \in H_0^1(\delta,1). \tag{14}$$

Then $u^{(2)}$ is the weak formulation to the Dirichlet problem

$$\begin{cases} -\left(u^{(2)}\right)'' = f, & \text{in } (\delta,1), \\ u^{(2)}(\delta) = 0,\ u^{(2)}(1) = 0. \end{cases}$$

2.3. A remark about Dirichlet and Neumann sub-domain problems

Now we make the following remark that will be central for the upcoming arguments. Looking back to (7) we make the following observation about u_0. Taking test functions $v \in H_0^1(-1,-\delta)$ we see that u_0 solve the following Dirichlet problem

$$\int_{-1}^{-\delta} u_0' v' = \int_{-1}^{-\delta} f v, \quad \text{for all } v \in H_0^1(-1,-\delta),$$

with the corresponding boundary data. The strong form for this Dirichlet problem is given by

$$\begin{cases} -z'' = f, & \text{in } (-1,-\delta) \\ z(-1) = 0, \; z(-\delta) = u_0(-\delta), \end{cases}$$

with only solution $z = u_0$ in $(-1,-\delta)$. From here we conclude that $z = u_0$ is also the solution of the following mixed Dirichlet and Neumann boundary condition problem

$$\begin{cases} -z'' = f, & \text{in } (-1,-\delta), \\ z(-1) = 0, \; z'(-\delta^-) = u_0'(-\delta^-), \end{cases}$$

with weak formulation given by

$$\int_{-1}^{-\delta} z' v' = u_0'\left(-\delta^-\right) v\left(-\delta^-\right) + \int_{-1}^{-\delta} f v, \quad \text{for all } v \in H^1(-1,-\delta).$$

Since $z = u_0 \in H^1(-1,-\delta)$ is solution of this problem we can write

$$\int_{-1}^{-\delta} u_0' v' = u_0'\left(-\delta^-\right) v\left(-\delta^-\right) + \int_{-1}^{-\delta} f v, \quad \text{for all } v \in H^1(-1,-\delta). \tag{15}$$

Analogously we can write

$$\int_{\delta}^{1} u_0' v' = u_0'\left(\delta^+\right) v\left(\delta^+\right) + \int_{\delta}^{1} f v, \quad \text{for all } v \in H^1(\delta,1). \tag{16}$$

By substituting the equations (15) and (16) into (7) we have

$$\left(u_0'\left(-\delta^-\right) v\left(-\delta^-\right) + \int_{-1}^{-\delta} f v\right) + \int_{-\delta}^{\delta} u_1' v' + \left(\int_{\delta}^{1} f v - u_0'\left(\delta^+\right) v\left(\delta^+\right)\right) = \int_{-1}^{1} f v,$$

so that, after simplifying it gives, for all $v \in H^1(-1,1)$, that

$$\int_{-\delta}^{\delta} u_1' v' = \int_{-\delta}^{\delta} f v + u_0'\left(\delta^+\right) v\left(\delta^+\right) - u_0'\left(-\delta^-\right) v\left(-\delta^-\right). \tag{17}$$

This last equation (17) is the weak formulation of the Neumann problem for u_1 defined by

$$\begin{cases} -u_1'' = f, & \text{in } (-\delta,\delta), \\ u_1'(-\delta^+) = u_0'(-\delta^-), \; u_1'(\delta^-) = u_0'(\delta^+). \end{cases} \tag{18}$$

This classical Neumann problem has solution only if the compatibility condition is satisfied. The compatibility condition comes from the fact that if we integrate directly the first equation in (18), we have

$$-\int_{-\delta}^{\delta} u_1'' = \int_{-\delta}^{\delta} f, \text{ and then we need } u_1'(-\delta^+) - u_1'(\delta^-) = \int_{-\delta}^{\delta} f.$$

Here, the derivatives are defined using side limits for the function u_1. Using the fact that $u_1'(-\delta^+) = u_0'(-\delta^-)$ and $u_1'(\delta^-) = u_0'(\delta^+)$ then, the compatibility conditions becomes

$$u_0'(-\delta^-) - u_0'(\delta^+) = \int_{-\delta}^{\delta} f. \tag{19}$$

In order to verify this compatibility condition we first observe that if we take $v = \chi$ in (15) and (16) and recalling the definition of χ we obtain

$$\int_{-1}^{-\delta} u_0'\chi' = u_0'(-\delta^-)(1) + \int_{-1}^{-\delta} f\chi \quad \text{and} \quad \int_{\delta}^{1} u_0'\chi' = u_0'(\delta^+)(-1) + \int_{\delta}^{1} f\chi.$$

On the other hand, if we take $v = \chi$ in (7) we have

$$\int_{-1}^{-\delta} u_0'\chi' + 0 + \int_{\delta}^{1} u_0'\chi' = \int_{-1}^{1} f\chi.$$

Combining these three equations we conclude that (19) holds true.

Now, observe that (18) has unique solution up to a constant so, the solution takes the form $u_1 = \widetilde{u}_1 + c_1$, with c_1 an integration constant. In addition, $\widetilde{u}_1$ is a function with the property that its means measure is 0, i.e.,

$$\int_{-\delta}^{\delta} \widetilde{u}_1 = 0.$$

In this way, we need to determine the value of constant c_1, for this, we substitute the function u_1 with a total function $\widetilde{u}_1 + c_1$, but the constant c_1 cannot be computed in this part, so it will be specified later.

Therefore $\widetilde{u}_1$ solves the Neumann problem in $(-\delta, \delta)$

$$\int_{-\delta}^{\delta} \widetilde{u}_1' v' = \int_{-\delta}^{\delta} fv - \left[u_0'(\delta^+) - u_0'(-\delta^-)\right], \text{ for all } v \in H^1(-\delta, \delta). \tag{20}$$

2.4. Terms corresponding to η^{-1}

For the other parts of u_1 in the interval, we need the term of η with $j = 1$, which is given from the equation (6), we have

$$\int_{-1}^{-\delta} u_1'v' + \int_{-\delta}^{\delta} u_2'v' + \int_{\delta}^{1} u_1'v' = 0, \quad \text{for all } v \in H_0^1(-1, 1). \tag{21}$$

Note that if we restrict this equation to test functions $v \in H_0^1(-1, -\delta)$ and $v \in H_0^1(\delta, 1)$ such as in (7), i.e., $\int_{-\delta}^{\delta} u_2'v' = 0$, we have

$$\int_{-1}^{-\delta} u_1'v' = 0, \text{ for all } v \in H_0^1(-1, -\delta),$$

and

$$\int_{\delta}^{1} u_1' v' = 0, \text{ for all } v \in H_0^1(\delta, 1),$$

where each integral is a weak formulation to problems with Dirichlet conditions

$$\begin{cases} -u_1'' = 0, & \text{in } (-1, -\delta), \\ u_1(-1) = 0, \\ u_1(-\delta^-) = u_1(-\delta^+) = \widetilde{u}_1(-\delta^+) + c_1, \end{cases} \tag{22}$$

and

$$\begin{cases} -u_1'' = 0, & \text{in } (\delta, 1), \\ u_1(\delta^+) = u_1(\delta^-) = \widetilde{u}_1(\delta^-) + c_1, \\ u_1(1) = 0, \end{cases} \tag{23}$$

respectively.

Back to the problem (21) above, we compute u_2 with given solutions in (22) and (23) we get

$$\int_{-\delta}^{\delta} u_2' v' = u_1'(-\delta^-) v(-\delta^-) - u_1'(\delta^+) v(\delta^+).$$

This equation is the weak formulation to the Neumann problem

$$\begin{cases} -u_2'' = 0, & \text{in } (-\delta, \delta), \\ u_2'(-\delta^+) = u_1'(-\delta^-), \; u_2'(\delta^-) = u_1'(\delta^+). \end{cases} \tag{24}$$

Note that, since u_2 depends only on the (normal) derivative of u_1, then, it does not depend on the value of c_1. But the value c_1 is chosen such that compatibility condition holds.

2.5. Terms corresponding to η^{-j}, $j \geq 2$

In order to determine the other parts of u_2 we need the term of η but this procedure is similar for the case of u_j with $j \geq 2$, so we present the deduction for general u_j with $j \geq 2$. Thus we have that

$$\int_{-1}^{-\delta} u_j' v' + \int_{-\delta}^{\delta} u_{j+1}' v' + \int_{\delta}^{1} u_j' v' = 0, \quad \text{for all } v \in H_0^1(-1, 1). \tag{25}$$

Again, if we restrict this last equation to $v \in H_0^1(-1, -\delta)$ and $v \in H_0^1(\delta, 1)$ to be defined in (7), we have

$$\int_{-1}^{-\delta} u_j' v' = 0, \text{ for all } v \in H_0^1(-1, -\delta),$$

and

$$\int_{\delta}^{1} u_j' v' = 0, \text{ for all } v \in H_0^1(\delta, 1),$$

respectively. Where each integral is a weak formulation to the Dirichlet problem

$$\begin{cases} -u_j'' = 0, & \text{in } (-1, \delta), \\ u_j(-1) = 0, \\ u_j(-\delta^-) = u_j(-\delta^+) = \widetilde{u}_j(-\delta^+) + c_j, \end{cases} \tag{26}$$

and

$$\begin{cases} -u_j'' = 0, & \text{in } (\delta,1),\\ u_j(\delta^+) = u_j(\delta^-) = \widetilde{u}_j(\delta^-) + c_j, & \\ u_j(1) = 0. & \end{cases} \tag{27}$$

Following a similar argument to the one given above, we conclude that u_j is harmonic in the intervals $(-1,-\delta)$ and $(\delta,1)$ for all $j \geq 1$ and u_{j-1} is harmonic in $(-\delta,\delta)$ for $j \geq 2$. As before, we have

$$u_j'(\delta^-) - u_j'(-\delta^+) = -\left[u_{j-1}'(\delta^+) - u_{j-1}'(-\delta^-)\right], \quad \text{for all } j \geq 2.$$

Note that u_j is given by the solution of a Neumann problem in $(-\delta,\delta)$. Thus, the function takes the form $u_j = \widetilde{u}_j + c_j$, with c_j being a integration constant, though, $\widetilde{u}_j$ is a function with the property that its integral is 0, i.e.,

$$\int_{-\delta}^{\delta} \widetilde{u}_j = 0, \quad \text{for all } j \geq 2.$$

In this way, we need to determine the value of constant c_j, for this, we substitute the function u_j with a total function $\widetilde{u}_j + c_j$. Note that $\widetilde{u}_j$ solves the Neumann problem in $(-\delta,\delta)$

$$\int_{-\delta}^{\delta} \widetilde{u}_j' v' = -\left[u_{j-1}'(\delta^+) - u_{j-1}'(-\delta^-)\right], \quad \text{for all } v \in H^1(-\delta,\delta). \tag{28}$$

Since c_j, with $j = 2,\ldots$, are constants, their harmonic extensions are given by $c_j\chi$ in $(-1,1)$, (see Remark 3.1 below for more details). We have

$$u_j = \widetilde{u}_j + c_j\chi, \quad \text{in } (-\delta,\delta).$$

This complete the construction of u_j.

From the equation (25), and solutions of Dirichlet problems (26) and (27) we have

$$\int_{-\delta}^{\delta} u_{j+1}' v' = u_j'(-\delta^-)v(-\delta^-) - u_j'(\delta^+)v(\delta^+). \tag{29}$$

This equation is the weak formulation to the Neumann problem

$$\begin{cases} -u_{j+1}'' = 0, & \text{in } (-\delta,\delta),\\ u_{j+1}'(-\delta^+) = u_j'(-\delta^-),\ u_{j+1}'(\delta^-) = u_j'(\delta^+). & \end{cases} \tag{30}$$

As before, we have

$$u_{j+1}'(\delta^-) - u_{j+1}'(-\delta^+) = -\left[u_j'(\delta^+) - u_j'(-\delta^-)\right].$$

The compatibility conditions need to be satisfied. Observe that

$$\begin{aligned} u_{j+1}'(\delta^-) - u_{j+1}'(-\delta^+) &= -\left[u_j'(\delta^+) - u_j'(-\delta^-)\right]\\ &= -\widetilde{u}_j'(\delta^+) - c_j\chi'(\delta^+) + \widetilde{u}_j'(-\delta^-) + c_j\chi'(-\delta^-)\\ &= \widetilde{u}_j'(-\delta^-) - \widetilde{u}_j'(\delta^+) + c_j\left[\chi'(-\delta^-) - \chi'(\delta^+)\right]\\ &= 0. \end{aligned}$$

By the latter we conclude that, in order to have the compatibility condition of (30) it is enough to set,

$$c_j = -\frac{\widetilde{u}_j'(-\delta^-) - \widetilde{u}_j'(\delta^+)}{\chi'(-\delta^-) - \chi'(\delta^+)}.$$

We can choose u_{j+1} in $(-\delta, \delta)$ such that

$$u_{j+1} = \widetilde{u}_{j+1} + c_{j+1}, \qquad \text{where } \int_{-\delta}^{\delta} \widetilde{u}_{j+1} = 0,$$

and $\widetilde{u}_{j+1}$ solves the Neumann problem

$$\int_{-\delta}^{\delta} \widetilde{u}_{j+1}' v' = -\left[u_j'(\delta^+) - u_j'(-\delta^-)\right], \quad \text{for all } v \in H^1(-\delta, \delta). \tag{31}$$

and as before

$$c_{j+1} = -\frac{\widetilde{u}_{j+1}'(-\delta^-) - \widetilde{u}_{j+1}'(\delta^+)}{\chi'(-\delta^-) - \chi'(\delta^+)}.$$

2.6. Illustrative example in one dimension

In this part we show a simple example of the weak formulation with the purpose of illustrating of the development presented above. First take the next (strong) problem

$$\begin{cases} (\kappa(x)\, u'(x))' = 0, & \text{in } (-2, 2), \\ u(-2) = 0,\ u(2) = 4, \end{cases} \tag{32}$$

with the function $\kappa(x)$ defined by

$$\kappa(x) = \begin{cases} 1, & -2 \leq x < -1, \\ \eta, & -1 \leq x < 1, \\ 1, & 1 \leq x \leq 2. \end{cases}$$

The weak formulation for problem (32) is to find a function $u \in H^1(-2, 2)$ such that

$$\begin{cases} \int_{-2}^{2} \kappa(x) u'(x) v'(x) dx = 0, \\ u(-2) = 0,\ u(2) = 4, \end{cases} \tag{33}$$

for all $v \in H_0^1(-2, 2)$.

Note that the boundary condition is not homogeneous, but this case is similar, and only the term u_0 inherit a non-homogeneous boundary condition.

As before, for $j = 0$ we have the equation (7), then u_0 is constant in $(-1, 1)$ and we write the decomposition $u_0 = c_0\chi + u^{(1)} + u^{(2)}$, and if $v = \chi \in H_0^1(-2, 2)$ from equation (10) we have that $c_0 = 2$.

Similarly, we can take $v = v^{(1)} \in H_0^1(-2, -1)$ and we obtain the weak formulation of Dirichlet problem

$$\begin{cases} -\left(u^{(1)}\right)'' = 0, & \text{in } (-2, -1), \\ u^{(1)}(-2) = 0,\ u^{(1)}(-1) = 0, \end{cases}$$

that after integrating directly twice gives a linear function $u^{(1)} = \alpha_1(x+2)$ in $(-2,-1)$ and α_1 is an integration constant. Using the boundary data we have $u^{(1)} = 0$.

If $v = v^{(2)} \in H_0^1(1,2)$ we have the weak formulation to the Dirichlet problem (13) that in this case becomes,

$$\begin{cases} -\left(u^{(2)}\right)'' = 0, & \text{in } (1,2), \\ u^{(2)}(1) = 0,\ u^{(2)}(2) = 4. \end{cases}$$

Integrating twice in the interval $(1,2)$ we obtain the solution, $u^{(2)} = \alpha_2(x-1)$, which becomes $u^{(2)} = 4(x-1)$. We use the boundary condition to determine the integration constant α_2. Then, we find the decomposition for $u_0 = c_0\chi + u^{(1)} + u^{(2)}$ for each part of interval which is given by

$$u_0 = \begin{cases} 2(x+2), & x \in (-2,-1), \\ 2, & x \in (-1,1), \\ 2x, & x \in (1,2). \end{cases}$$

With u_0 already computed we can to obtain the boundary data for the Neumann problem that determines u_1 in the equation (18). We have

$$\begin{cases} -u_1'' = 0, & \text{in } (-1,1), \\ u_1'(-1) = 2,\ u_1'(1) = 2. \end{cases}$$

As before, we see that the compatibility condition holds and therefore it has solution in the interval $(-1,1)$. Easy calculation gives $u_1 = 2x$. For computing the constant c_1, we consider the Neumann problem (20), which has the solution $\widetilde{u}_1 = 2x$. By the definition of $u_1 = \widetilde{u}_1 + c_1$ we conclude that $c_1 = 0$. Now, in order to compute u_1 in the $(-2,-1)$ and $(1,2)$ we apply the condition in (21). It follows from the Dirichlet problems (22) and (23), that the solutions are $u_1 = -2(x+2)$ and $u_1 = -2x+4$ respectively. We summarize the expression for u_1 as

$$u_1 = \begin{cases} -2(x+2), & x \in (-2,-1), \\ 2x, & x \in (-1,1), \\ -2(x-2), & x \in (1,2). \end{cases}$$

Returning to equation (21) we can obtain u_2 in the interval $(-1,1)$ by solving the Neumann problem

$$\begin{cases} -u_2'' = 0 \text{ in } (-1,1), \\ u_2'(-1) = -2,\ u_2'(1) = 2. \end{cases}$$

Note again that the compatibility condition holds. The solution is given by $u_2 = -2x$. Computing the constant c_2, we consider the Neumann problem (28) for $j = 2$, has the solution $\widetilde{u}_2 = -2x$. Note that we assume the boundary conditions of Dirichlet problems (26) and (27). Then $c_2 = 0$. So, in order to find u_2 in the intervals $(-2,-1)$ and $(1,2)$, we apply the condition in (25). It follows from (26) and (27) that the solutions are $u_2 = 2(x+2)$ and $u_2 = 2(x-2)$, respectively.

For the case of terms u_{j+1} with $j \geq 2$, we consider the Neumann problem (30) with solution $u_{j+1} = \pm 2x$ in $(-1,1)$. Note that the compatibility condition holds. Again we recall the Neumann problem (31) with solution $\widetilde{u}_{j+1} = \pm 2x$ in this case. We conclude that $c_j = 0$ for each $j = 2, 3 \ldots$. Again in order to find u_{j+1} in the intervals $(-2, -1)$ and $(1, 2)$. It follows of analogous form above that the solutions $u_{j+1} = \mp 2(x+2)$ in $(-2, -1)$ and $u_{j+1} = \mp 2(x-2)$ in $(1, 2)$, for $j = 2, 3, \ldots$. So, we can be calculated following terms of the power series.

We note that above we computed an approximation of the solution by solving local problems to the inclusion and the background. In this example, we can directly compute the solution for the problem (32) and verify that the expansion is correct. We have

$$u(x) = \alpha \int_{-2}^{x} \frac{1}{\kappa(t)} dt.$$

Then with boundary condition of problem (32) we calculate a constant $\alpha = \frac{2\eta}{1+\eta}$, so we have

$$u\left(x\right) = \begin{cases} \frac{2\eta}{1+\eta}\left(x+2\right), & x \in [-2, -1), \\ \frac{2\eta}{1+\eta}\left[1 + \frac{1}{\eta}\left(x+1\right)\right], & x \in [-1, 1), \\ \frac{2\eta}{1+\eta}\left[1 + \frac{2}{\eta} + \left(x-1\right)\right], & x \in [1, 2]. \end{cases} \tag{34}$$

Recall that the term $\frac{\eta}{1+\eta}$ can be written as a power series given by $\frac{\eta}{1+\eta} = \sum_{j=0}^{\infty} \frac{(-1)^j}{\eta^j}$.

By inserting this expression into (34) and after some manipulations, we have

$$u(x) = \begin{cases} 2(x+2)\sum_{j=0}^{\infty} \frac{(-1)^j}{\eta^j}, & x \in [-2, -1), \\ 2 - 2x\sum_{j=1}^{\infty} \frac{(-1)^j}{\eta^j}, & x \in [-1, 1), \\ 2x\sum_{j=0}^{\infty} \frac{(-1)^j}{\eta^j} - 4x\sum_{j=1}^{\infty} \frac{(-1)^j}{\eta^j}, & x \in [1, 2]. \end{cases} \tag{35}$$

We can rewrite this expression to get

$$u\left(x\right) = \underbrace{\begin{Bmatrix} 2\left(x+2\right) \\ 2 \\ 2x \end{Bmatrix}}_{u_0} + \frac{1}{\eta}\underbrace{\begin{Bmatrix} -2\left(x+2\right) \\ 2x \\ -\left(2x-4\right) \end{Bmatrix}}_{u_1} + \frac{1}{\eta^2}\underbrace{\begin{Bmatrix} 2\left(x+2\right) \\ -2x \\ \left(2x-4\right) \end{Bmatrix}}_{u_2} + \cdots, \tag{36}$$

Observe that those were the same terms computed before by solving local problems in the inclusions and the background.

3. Asymptotic expansion in two and three dimensions

In this section we use the notation introduced in [5] and we detail the derivation of asymptotic expansions in higher dimensions for high-contrast elliptic problems of the form,

$$-\text{div}(\kappa(x)\nabla u(x)) = f(x), \quad \text{in } D, \tag{37}$$

with Dirichlet data defined by $u = g$ on ∂D. We assume that D is the disjoint union of a background domain and inclusions, $D = D_0 \cup (\bigcup_{m=1}^{M} \overline{D}_m)$. We assume that $D_0, D_1, \dots, D_M$, are polygonal domains (or domains with smooth boundaries). We also assume that each D_m is a connected domain, $m = 0, 1, \dots, M$. Additionally, we assume that D_m is compactly included in the open set $D \setminus \bigcup_{\ell=1,\ell\neq m}^{M} \overline{D}_\ell$, i.e., $\overline{D}_m \subset D \setminus \bigcup_{\ell=1,\ell\neq m}^{M} \overline{D}_\ell$, and we define $D_0 := D \setminus \bigcup_{m=1}^{M} \overline{D}_m$. Let D_0 represent the background domain and the subdomains $\{D_m\}_{m=1}^{M}$ represent the inclusions. For simplicity of the presentation we consider only interior inclusions. Other cases can be studied similarly.

We consider a coefficient with multiple high-conductivity inclusions. Let κ be defined by

$$\kappa(x) = \begin{cases} \eta, & x \in D_m, \quad m = 1, \dots, M, \\ 1, & x \in D_0 = D \setminus \bigcup_{m=1}^{M} \overline{D}_m. \end{cases} \tag{38}$$

We seek to determine $\{u_j\}_{j=0}^{\infty} \subset H^1(D)$ such that

$$u_\eta = u_0 + \frac{1}{\eta} u_1 + \frac{1}{\eta^2} u_2 + \cdots = \sum_{j=0}^{\infty} \eta^{-j} u_j, \tag{39}$$

and such that they satisfy the following Dirichlet boundary conditions,

$$u_0 = g \text{ on } \partial D \quad \text{ and } \quad u_j = 0 \text{ on } \partial D \text{ for } j \geq 1. \tag{40}$$

This work complements current work in the use of similar expansions to design and analyze efficient numerical approximations of high-contrast elliptic equations, see [8, 9].

We have the following weak formulation of problem (37): to find $u \in H^1(D)$ such that

$$\begin{cases} \mathcal{A}(u, v) = \mathcal{F}(v), & \text{for all } v \in H_0^1(D), \\ \qquad u = g, & \text{on } \partial D. \end{cases} \tag{41}$$

The bilinear form $\mathcal{A}$ and the linear functional $\mathcal{F}$ are defined by

$$\begin{aligned} \mathcal{A}(u, v) &= \int_D \kappa(x) \nabla u(x) \cdot \nabla v(x) dx, && \text{for all } u, v \in H_0^1(D), \\ \mathcal{F}(v) &= \int_D f(x) v(x) dx, && \text{for all } v \in H_0^1(D). \end{aligned} \tag{42}$$

We denote by u_η the solution of problem (37) with Dirichlet boundary condition (40). From now on, we use the notation $w^{(m)}$, which means that the function w is restricted on domain D_m, that is $w^{(m)} = w|_{D_m}$, $m = 0, 1$.

3.1. Derivation for one high-conductivity inclusion

Let us denote by u_η the solution of (37) with the coefficient $\kappa(x)$ defined in (38). We express the expansion as in (39) with the functions u_j, $j = 0, 1, \dots$, satisfying the conditions on the boundary of D given in (40). For the case of the one inclusion with $m = 0, 1$, we consider D as the disjoint union of a background domain D_0 and one inclusion D_1, such that $D = D_0 \cup \overline{D}_1$. We assume that D_1 is compactly included in D ($\overline{D}_1 \subset D$).

We obtain the following development for each term of this asymptotic expansion in the problem (37). First we replace $u(x)$ for the expansion (39) in the bilinear form of the (42), we have

$$\begin{aligned}\int_D \kappa(x)\sum_{j=0}^{\infty}\eta^{-j}\nabla u_j\cdot\nabla v &= \int_{D_0}\sum_{j=0}^{\infty}\eta^{-j}\nabla u_j\cdot\nabla v+\eta\int_{D_1}\sum_{j=0}^{\infty}\eta^{-j}\nabla u_j\cdot\nabla v\\ &=\sum_{j=0}^{\infty}\eta^{-j}\int_{D_0}\nabla u_j\cdot\nabla v+\eta\sum_{j=0}^{\infty}\eta^{-j}\int_{D_1}\nabla u_j\cdot\nabla v\\ &=\sum_{j=0}^{\infty}\eta^{-j}\int_{D_0}\nabla u_j\cdot\nabla v+\sum_{j=0}^{\infty}\eta^{-j+1}\int_{D_1}\nabla u_j\cdot\nabla v,\end{aligned}$$

we change the index in the last sum to obtain,

$$\sum_{j=0}^{\infty}\eta^{-j}\int_{D_0}\nabla u_j\cdot\nabla v+\sum_{j=-1}^{\infty}\eta^{-j}\int_{D_1}\nabla u_{j+1}\cdot\nabla v,$$

and then, we have

$$\sum_{j=0}^{\infty}\eta^{-j}\int_{D_0}\nabla u_j\cdot\nabla v+\eta\int_{D_1}\nabla u_0\cdot\nabla v+\sum_{j=0}^{\infty}\eta^{-j}\int_{D_1}\nabla u_{j+1}\cdot\nabla v.$$

We obtain,

$$\eta\int_{D_1}\nabla u_0\cdot\nabla v+\sum_{j=0}^{\infty}\eta^{-j}\left(\int_{D_0}\nabla u_j\cdot\nabla v+\int_{D_1}\nabla u_{j+1}\cdot\nabla v\right)=\int_D fv. \quad (43)$$

In brief, we obtain the following equations after matching equal powers,

$$\int_{D_1}\nabla u_0\cdot\nabla v=0, \quad (44)$$

$$\int_{D_0}\nabla u_0\cdot\nabla v+\int_{D_1}\nabla u_1\cdot\nabla v=\int_D fv, \quad (45)$$

and for $j\geq 1$,

$$\int_{D_0}\nabla u_j\cdot\nabla v+\int_{D_1}\nabla u_{j+1}\cdot\nabla v=0, \quad (46)$$

for all $v\in H_0^1(D)$.

The equation (44) tells us that the function u_0 restricted to D_1 is constant, that is $u_0^{(1)}$ is a constant function. We introduce the following subspace,

$$V_{\text{const}}=\{v\in H_0^1(D),\ \text{such that } v^{(1)}=v|_{D_1}\ \text{is constant}\}.$$

If in equation (45) we choose test function $z\in V_{\text{const}}$, then, we see that u_0 satisfies the problem

$$\begin{aligned}\textstyle\int_D\nabla u_0\cdot\nabla z=\int_D fz, &\quad \text{for all } z\in V_{\text{const}}\\ u_0=g, &\quad \text{in } \partial D.\end{aligned} \quad (47)$$

The problem (47) is elliptic and it has a unique solution. This follows from the ellipticity of bilinear form $\mathcal{A}$.

We introduce the *harmonic characteristic function* $\chi_{D_1} \in H_0^1(D)$ with the condition

$$\chi_{D_1}^{(1)} = 1, \text{ in } D_1,$$

and which is equal to the harmonic extension of its boundary data in D_0 (see [5]). We then have,

$$\begin{aligned} \int_{D_0} \nabla \chi_{D_1}^{(0)} \cdot \nabla z &= 0, && \text{for all } z \in H_0^1(D_0), \\ \chi_{D_1}^{(0)} &= 1, && \text{on } \partial D_1, \\ \chi_{D_1}^{(0)} &= 0, && \text{on } \partial D. \end{aligned} \tag{48}$$

To obtain an explicit formula for u_0 we use the facts that the problem (47) is elliptic and has unique solution, and a property of the harmonic characteristic functions described in the next remark (see [5]).

Remark 3.1. Let w be a harmonic extension to D_0 of its Neumann data on ∂D_0. That is, w satisfies the following problem

$$\int_{D_0} \nabla w \cdot \nabla z = \int_{\partial D_0} \nabla w \cdot n_0 z, \quad \text{for all } z \in H^1(D_0).$$

Since $\chi_{D_1} = 0$ on ∂D and $\chi_{D_1} = 1$ on ∂D_1, we have that

$$\int_{D_0} \nabla \chi_{D_1} \cdot \nabla w = \int_{\partial D_0} \nabla w \cdot n_0 \chi_{D_1} = 0 \left(\int_{\partial D} \nabla w \cdot n \right) + 1 \left(\int_{\partial D_1} \nabla w \cdot n_0 \right),$$

and we conclude that for every harmonic function on D_0,

$$\int_{D_0} \nabla \chi_{D_1} \cdot \nabla w = \int_{\partial D_1} \nabla w \cdot n_0.$$

Note that if $\xi \in H^1(D)$ is such that $\xi^{(1)} = \xi|_{D_1} = c$ is a constant in D_1 and $\xi^{(0)} = \xi|_{D_0}$ is harmonic in D_0, then $\xi = c\chi_{D_1}$.

An explicit formula for u_0 is obtained as

$$u_0 = u_{0,0} + c_0 \chi_{D_1}, \tag{49}$$

where $u_{0,0} \in H^1(D)$ is defined by $u_{0,0}^{(1)} = 0$ in D_1 and $u_{0,0}^{(0)}$ solves the Dirichlet problem

$$\begin{aligned} \int_{D_0} \nabla u_{0,0}^{(0)} \cdot \nabla z &= \int_{D_0} f z, && \text{for all } z \in H_0^1(D_0), \\ u_{0,0}^{(0)} &= 0, && \text{on } \partial D_1, \\ u_{0,0}^{(0)} &= g, && \text{on } \partial D. \end{aligned} \tag{50}$$

From equation (47), (49) and the facts in the Remark 3.1 we have

$$\int_{D_0} \nabla u_0 \cdot \nabla \chi_{D_1} = \int_D f \chi_{D_1},$$

or

$$\int_{D_0} \nabla(u_{0,0} + c_0\chi_{D_1}) \cdot \nabla\chi_{D_1} = \int_D f\chi_{D_1},$$

from which we can obtain the constant c_0 given by

$$c_0 = \frac{\int_D f\chi_{D_1} - \int_{D_0} \nabla u_{0,0} \cdot \nabla\chi_{D_1}}{\int_{D_0} |\nabla\chi_{D_1}|^2}, \tag{51}$$

or, using the Remark above we also have

$$c_0 = \frac{\int_{D_1} f - \int_{\partial D_1} \nabla u_{0,0} \cdot n_0}{\int_{\partial D_1} \nabla\chi_{D_1} \cdot n_0}. \tag{52}$$

Thus, by (52) c_0 balances the fluxes across ∂D_1, see [5].

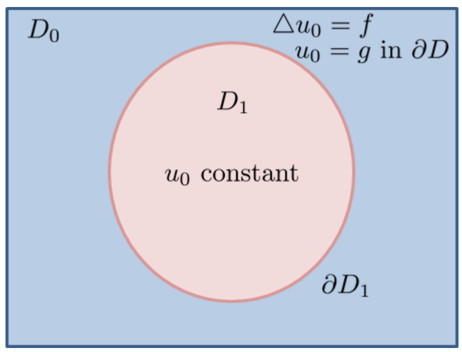


Figure 3: Illustration of local problems related to u_0 in the inclusion and the background.

In Figure 3 we illustrate the properties of u_0, that is, the local problem u_0 is solved in the inclusions and the background.

To get the function u_1 we proceed as follows. We first write

$$u_1^{(1)} = \widetilde{u}_1^{(1)} + c_1, \quad \text{where } \int_{D_1} \widetilde{u}_1^{(1)} = 0$$

and $\widetilde{u}_1^{(1)}$ solves the Neumann problem

$$\int_{D_1} \nabla\widetilde{u}_1^{(1)} \cdot \nabla z = \int_{D_1} fz - \int_{\partial D_1} \nabla u_0^{(0)} \cdot n_1 z \quad \text{for all } z \in H^1(D_1). \tag{53}$$

The Problem (53) satisfies the compatibility condition so it is solvable. The constant c_1 is given by

$$c_1 = -\frac{\int_{\partial D_1} \nabla\widetilde{u}_1^{(0)} \cdot n_0}{\int_{\partial D_1} \nabla\chi_{D_1}^{(0)} \cdot n_0} = -\frac{\int_{D_0} \nabla\widetilde{u}_1 \cdot \nabla\chi_{D_1}}{\int_{D_0} |\nabla\chi_{D_1}|^2}. \tag{54}$$

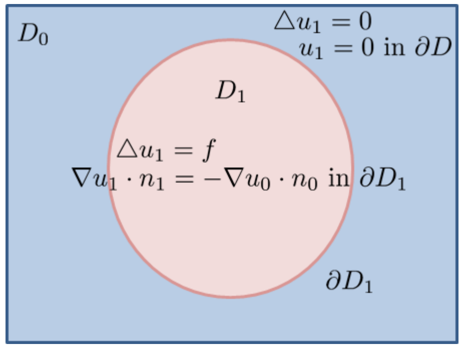


Figure 4: Illustration of local problems related to u_1 in the inclusions and the background. First a problem in D_1 is solved and then a problem in D_0 is solved using the boundary data in ∂D_1.

In Figure 4 we illustrate the properties of u_1, that is, the local problem u_1 is solved in the inclusions and the background. Now, we complete the construction of u_1 and show how to construct u_j, $j = 2, 3, \ldots$. For a given $j = 1, 2, \ldots$ and given $u_j^{(1)}$ we show how to construct $u_j^{(0)}$ and $u_{j+1}^{(1)}$.

Assume that we already have constructed $u_j^{(1)}$ in D_1 as the solution of a Neumann problem in D_1 and can be written as

$$u_j^{(1)} = \widetilde{u}_j^{(1)} + c_j, \qquad \text{where } \int_{D_1} \widetilde{u}_j = 0. \tag{55}$$

We find $u_j^{(0)}$ in D_0 by solving a Dirichlet problem with known Dirichlet data, that is,

$$\begin{aligned} \int_{D_0} \nabla u_j^{(0)} \cdot \nabla z = 0, \text{ for all } z \in H_0^1(D_0), \\ u_j^{(0)} = \widetilde{u}_j^{(1)} \text{ on } \partial D_1 \quad \text{and} \quad u_j = 0 \text{ on } \partial D. \end{aligned} \tag{56}$$

Since c_j, $j = 1, \ldots,$ are constants, their harmonic extensions are given by $c_j \chi_{D_1}$, $j = 1, \ldots$. Then, we conclude that

$$u_j = \widetilde{u}_j + c_j \chi_{D_1}, \tag{57}$$

where $\widetilde{u}_j^{(0)}$ is defined by (56). The balancing constant c_j is given as

$$c_j = -\frac{\int_{\partial D_1} \nabla \widetilde{u}_j^{(0)} \cdot n_0}{\int_{\partial D_1} \nabla \chi_{D_1}^{(0)} \cdot n_0} = -\frac{\int_{D_0} \nabla \widetilde{u}_j \cdot \nabla \chi_{D_1}}{\int_{D_0} |\nabla \chi_{D_1}|^2}, \tag{58}$$

so we have $\int_{\partial D_1} \nabla u_j^{(0)} \cdot n_0$. This completes the construction of u_j. The compatibility conditions are satisfied, if we use the definition of c_j in (58). In the Figure 5 we illustrate the properties of u_j, that is, the local problem u_j is solved in the inclusions and the background.

We recall the convergence result obtained in [5]. There it is proven that there is a constant $C > 0$ such that for every $\eta > C$, the expansion (39) converges (absolutely) in $H^1(D)$. The asymptotic limit u_0 satisfies (49). Additionally, for every $\eta > C$, we have

$$\left\| u - \sum_{j=0}^{J} \eta^{-j} u_j \right\|_{H^1(D)} \leq C_1 \left(\|f\|_{H^{-1}(D)} + \|g\|_{H^{1/2}(\partial D)} \right) \sum_{j=J+1}^{\infty} \left(\frac{C}{\eta} \right)^j,$$

for $J \geq 0$. For the proof we refer to [5].

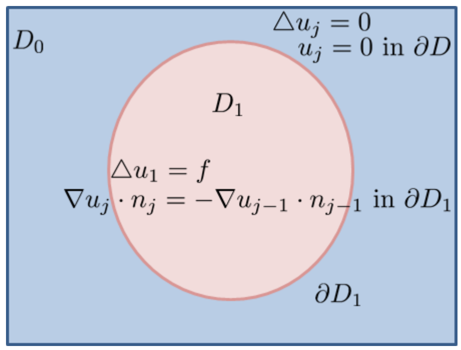


Figure 5: Illustration of local problems related to u_j , $j \geq 2$, in the inclusion and the background. First a problem in D_1 is solved and then a problem in D_0 is solved using the boundary data in ∂D_1.

3.2. Derivation for multiply high-conductivity inclusions

In this section we express the expansion as in (39) with the coefficient

$$\kappa(x) = \begin{cases} \eta, & x \in D_m, \quad m = 1, \ldots, M, \\ 1, & x \in D_0 = D \setminus \bigcup_{m=1}^{M} \overline{D}_m. \end{cases}$$

First, we describe the asymptotic problem.

We recall the space of constant functions inside the inclusions

$$V_{\text{const}} = \left\{ v \in H_0^1(D), \text{ such that } v|_{D_m} \text{ is a constant for all } m = 1, \ldots, M \right\}.$$

By analogy with the case of one high-conductivity inclusion, if we choose test function $z \in V_{\text{const}}$, then, we see that u_0 satisfies the problem

$$\begin{array}{rl} \int_D \nabla u_0 \cdot \nabla z = \int_D fz, & \text{for all } z \in V_{\text{const}} \\ u_0 = g, & \text{in } \partial D. \end{array} \tag{59}$$

The problem (47) is elliptic and it has a unique solution. For each $m = 1, \ldots, M$ we introduce the harmonic characteristic function $\chi_{D_m} \in H_0^1(D)$ with the condition

$$\chi_{D_m}^{(1)} \equiv \delta_{m\ell} \text{ in } D_\ell, \text{ for } \ell = 1, \ldots, M, \tag{60}$$

and which is equal to the harmonic extension of its boundary data in D_0, χ_{D_m} (see [5]).

We then have,

$$\int_{D_0} \nabla\chi_{D_m} \cdot \nabla z = 0, \qquad \text{for all } z \in H_0^1(D_0). \tag{61}$$
$$\chi_{D_m} = \delta_{m\ell}, \qquad \text{on } \partial D_\ell, \text{ for } \ell = 1,\dots,M,$$
$$\chi_{D_m} = 0, \qquad \text{on } \partial D.$$

Where $\delta_{m\ell}$ represent the Kronecker delta, which is equal to 1 when $m = \ell$ and 0 otherwise. To obtain an explicit formula for u_0 we use the facts that the problem (47) is elliptic and has unique solution, and a similar property of the harmonic characteristic functions described in the Remark 3.1, we replace χ_{D_1} by χ_{D_m} for this case.

We decompose u_0 into the harmonic extension to D_0 of a function in V_{const} plus a function $u_{0,0}$ with value g on the boundary and zero boundary condition on ∂D_m, $m = 1,\dots,M$. We write

$$u_0 = u_{0,0} + \sum_{m=1}^{M} c_m(u_0)\chi_{D_m},$$

where $u_{0,0} \in H^1(D)$ with $u_{0,0} = 0$ in D_m for $m = 1,\dots,M$, and $u_{0,0}$ solves the problem in D_0

$$\int_{D_0} \nabla u_{0,0} \cdot \nabla z = \int_{D_0} fz, \quad \text{for all } z \in H_0^1(D_0), \tag{62}$$

with $u_{0,0} = 0$ on ∂D_m, $m = 1,\dots,M$ and $u_{0,0} = g$ on ∂D. We compute the constants c_m using the same procedure as before. We have

$$\sum_{m=1}^{M} c_m(u_0) \int_{D_0} \nabla\chi_{D_m} \cdot \chi_{D_\ell} = \int_D f\chi_{D_\ell} - \int_{D_0} \nabla u_{0,0} \cdot \nabla\chi_{D_\ell}, \quad \text{for } \ell = 1,\dots,M,$$

and this last problem is equivalent to the linear system,

$$\mathbf{A}_{\text{geom}}\mathbf{X} = \mathbf{b},$$

where $\mathbf{A}_{\text{geom}} = [a_{m\ell}]$ and $\mathbf{b} = (b_1,\dots,b_M) \in \mathbb{R}^M$ are defined by

$$a_{m\ell} = \int_D \nabla\chi_{D_m} \cdot \nabla\chi_{D_\ell} = \int_{D_0} \nabla\chi_{D_m} \cdot \chi_{D_\ell}, \tag{63}$$
$$b_\ell = \int_D f\chi_{D_\ell} - \int_{D_0} \nabla u_{0,0} \cdot \nabla\chi_{D_\ell}, \tag{64}$$

and $\mathbf{X} = (c_1(u_0),\dots,c_M(u_0)) \in \mathbb{R}^M$. Then we have

$$\mathbf{X} = \mathbf{A}_{\text{geom}}^{-1}\mathbf{b}. \tag{65}$$

Now using the conditions given for χ_{D_m} in the Remark 3.1 we have

$$a_{m\ell} = \int_D \nabla\chi_{D_m} \cdot \nabla\chi_{D_\ell} = \int_{\partial D_m} \chi_{D_\ell} \cdot n_m = \int_{\partial D_\ell} \nabla\chi_{D_\ell} \cdot n_\ell. \tag{66}$$

Note that $\sum_{m=1}^{M} c_m\chi_{D_m}$ is the solution of a Galerkin projection in the space $\text{Span}\,\{\chi_{D_m}\}_{m=1}^{M}$. For more details see [5], and references therein.

Now, we describe the next individual terms of the asymptotic expansion. As before, we have the restriction of u_1 to the sub-domain D_m, that is

$$u_1^{(m)} = \widetilde{u}_1^{(m)} + c_{1,m}, \quad \text{with} \int_{D_m} \widetilde{u}_1^{(m)} = 0,$$

and $\widetilde{u}_1^{(m)}$ satisfies the Neumann problem

$$\int_{D_m} \nabla\widetilde{u}_1^{(m)}\cdot\nabla z = \int_{D_m} fz - \int_{\partial D_m} \nabla u_0^{(0)}\cdot n_m z, \text{ for all } z\in H^1(D_m),$$

for $m = 1,\dots,M$. The constants $c_{1,m}$ will be chosen later.

Now, for $j = 1, 2, \dots$ we have that $u_j^{(m)}$ in D_m, $m = 1,\dots,M$, then we find $u_j^{(0)}$ in D_0 by solving the Dirichlet problem

$$\begin{aligned} \int_{D_0} \nabla u_j^{(0)}\cdot\nabla z &= 0, && \text{for all } z\in H_0^1(D_0), \\ u_j^{(0)} &= u_j^{(m)}\,(= \widetilde{u}_j^{(m)} + c_{j,m}), && \text{on } \partial D_m,\ m = 1,\dots,M, \\ u_j^{(0)} &= 0, && \text{on } \partial D. \end{aligned} \tag{67}$$

Since $c_{j,m}$ are constants, we define their corresponding harmonic extension by $\sum_{m=1}^{M} c_{j,m}\chi_{D_m}$. So we rewrite

$$u_j = \widetilde{u}_j + \sum_{m=1}^{M} c_{j,m}\chi_{D_m}. \tag{68}$$

The $u_{j+1}^{(m)}$ in D_m satisfy the following Neumann problem

$$\int_{D_m} \nabla u_{j+1}^{(m)}\cdot\nabla z = -\int_{\partial D_m} \nabla u_j^{(0)}\cdot n_0 z, \quad \text{for all } z\in H^1(D).$$

For the compatibility condition we need that for $\ell = 1,\dots,M$

$$\begin{aligned} 0 = \int_{\partial D_\ell} \nabla u_{j+1}^{(\ell)}\cdot n_\ell &= -\int_{\partial D_\ell} \nabla u_j^{(0)}\cdot n_0 \\ &= -\int_{D_\ell} \nabla\left(\widetilde{u}_j^{(0)} + \sum_{m=1}^{M} c_{j,m}\chi_{D_m}^{(0)}\right)\cdot n_0 \\ &= -\int_{\partial D_\ell} \nabla\widetilde{u}_j^{(0)}\cdot n_0 - \sum_{m=1}^{M} c_{j,m}\int_{\partial D_m} \nabla\chi_{D_m}^{(0)}\cdot n_0. \end{aligned}$$

From (63) and (66) we have that $\mathbf{X}_j = (c_{j,1},\dots,c_{j,M})$ is the solution of the system

$$\mathbf{A}_{\text{geom}}\mathbf{X}_j = \mathbf{Y}_j,$$

where

$$\mathbf{Y}_j = \left(-\int_{\partial D_1} \nabla u_j^{(0)}\cdot n_0,\dots,-\int_{\partial D_m} \nabla\widetilde{u}_j^{(0)}\cdot n_0\right),$$

or

$$\mathbf{Y}_j = \left(- \int_{\partial D_0} \nabla u_j^{(0)} \cdot \nabla \chi_{D_1}, \ldots, - \int_{\partial D_0} \nabla \widetilde{u}_j^{(0)} \cdot \nabla \chi_{D_M} \right).$$

For the convergence we have the result obtained in [5]. There it is proven that there are constants $C, C_1 > 0$ such that $\eta > C$, the expansion (39) converges absolutely in $H^1(D)$ for η sufficiently large. We recall the following result.

Theorem 3.2. *Consider the problem* (37) *with coefficient* (38). *The corresponding expansion* (39) *with boundary condition* (40) *converges absolutely in* $H^1(D)$ *for* η *sufficiently large. Moreover, there exist positive constants* C *and* C_1 *such that for every* $\eta > C$, *we have*

$$\left\| u - \sum_{j=0}^{J} \eta^{-j} u_j \right\|_{H^1(D)} \leq C_1 \left(\|f\|_{H^{-1}(D)} + \|g\|_{H^{1/2}(\partial D)} \right) \sum_{j=J+1}^{\infty} \left(\frac{C}{\eta} \right)^j,$$

for $J \geq 0$.

3.3. Examples in two dimensions

In this section we show some examples of the expansion terms in two dimensions. In particular, few terms are computed numerically using a Finite Element method. For details on the Finite Element Method, see for instance [1, 3]. We recall that, apart for verifying the derived asymptotic expansions numerically, these numerical studies are a first step to understand the approximation u_0 and with this understanding try to devise cheap numerical approximations for u_0 (and then for u_η). We make some comments in the next section.

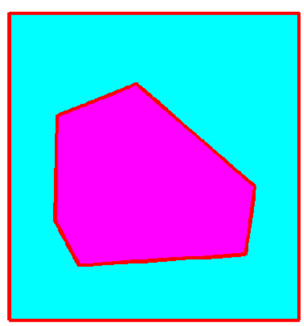
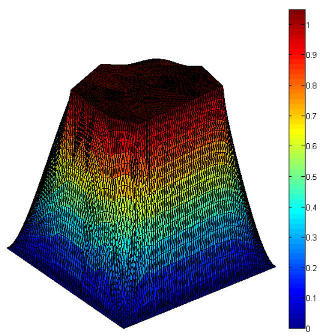


Figure 6: Example of geometry configurations with interior inclusion to solve the problem (37) with $u(x) = 0$ on ∂D and the coefficient κ in (38).

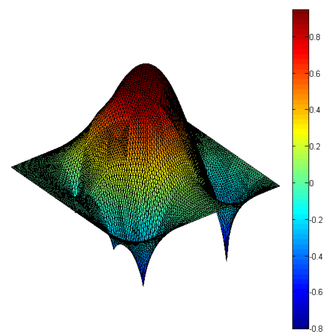
We numerically solve the problem (37) with $u(x) = 0$ on ∂D and the coefficient κ in (38) where the domain configuration is illustrated in Figure 6. This configuration contains one polygonal interior inclusion. We applied the conditions above develop through an numerical implementation in MatLab. We set $\eta = 10$ for this example. We show the computed first four terms of the asymptotic expansions in Figure 7. In particular we show the asymptotic limit u_0. For this case $\eta = 10$ and the errors form the truncated series and the whole domain solution u_η is reported in Figure 8. We see a linear decay of the logarithm of the error with respect to the number of terms that corresponds to the decay of the power series tail.

As a second example we consider several inclusions. The domain is illustrated in Figure 9 as well as the corresponding u_0 term. In Figure 10 we show the behavior of the first three terms but inside the inclusions regions.

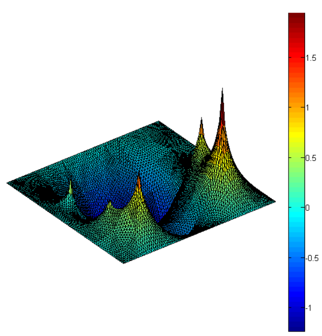
In the Figure 10 we have a plot of the asymptotic solutions and show its behaviour on inclusions.



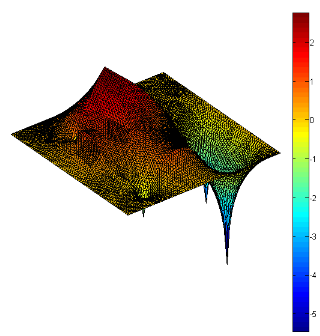
(A) u_0



(B) u_1



(C) u_2



(D) u_3

Figure 7: Few terms of the asymptotic expansion (39) for the solution of (37) with $u(x) = 0$ on ∂D and the coefficient κ in (38) where the domain configuration is illustrated in Figure 6. Here we consider $\eta = 10$.

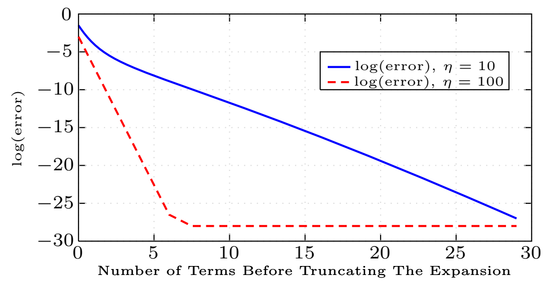


Figure 8: Error of the difference of the solution u_η computed directly and the addition of the truncation of the expansion (39). We consider $\eta = 10$ (solid line) and $\eta = 100$ (dashed line).

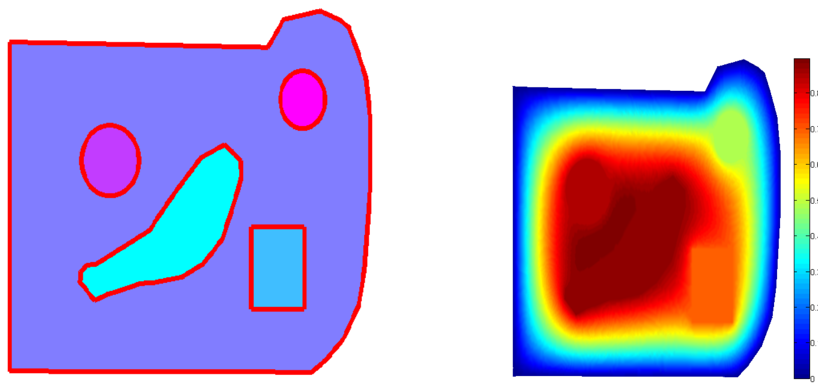
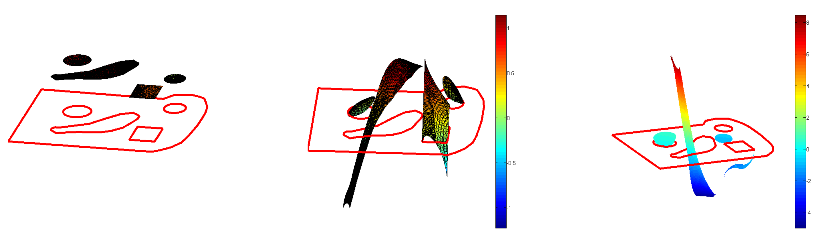


Figure 9: Example of geometry configurations with interior inclusion (left picture). Asymptotic limit u_0 for the solution of (37) with $u(x) = 0$ on ∂D and the coefficient κ in (38) where the domain configuration is illustrated in the left picture. Here we consider $\eta = 10$ (right picture).



(A) u_0 (B) u_1 (C) u_2

Figure 10: Illustration, inside inclusions, of first few terms of the asymptotic expansion (39) for the solution of (37) with $u(x) = 0$ on ∂D and the coefficient κ in (38) where the domain configuration is illustrated in Figure 9. Here we consider $\eta = 10$.

4. Final comments and ongoing work

We review some results and examples concerning asymptotic expansions for high-contrast coefficient elliptic equations. In particular we gave some explicit examples for the computations of the few terms in one dimension that several numerical examples in two dimensions. We mention and a main application in mind is to find ways to quickly is an approximation compute a few terms, in particular the term u_0, which, as seen in the paper, is approximation of order η^{-1} to the solution. A main difficulty is that the computation of the *harmonic characteristic functions* is computationally expensive. One option is to approximate these functions by solving a local problem (instead of a whole background problem). For instance the domain where harmonic characteristic functions can computed is illustrated in Figure 11 where the approximated harmonic characteristic function will be zero on the a boundary of a neighborhood of the inclusions. See Figure 11. This approximation will be consider for the case of many highly, dense high-contrast inclusions. This is under study and results will be presented elsewhere.

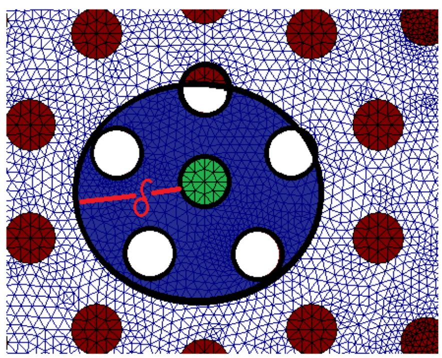


Figure 11: Illustration of $\delta-$ neighborhood of an inclusion (blue region).

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