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On Chern's conjecture about the Euler charateristic of affine manifolds

Daniela MARTINEZ

Advisor: Prof. Dr. Camilo Arias Abad
National University of Colombia
at Medellín

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Daniela Martínez Madrid

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Chapter 1

Introduction

The development the theory of characteristic classes allowed Shiing-Shen Chern to generalize the Gauss Bonnet theorem to Riemannian manifolds of arbitrary dimension. The Chern Gauss Bonnet theorem expresses the Euler characteristic as an integral of a polynomial evaluated on the curvature tensor, i.e if K is the curvature form of the Levi-Civita connection, the Chern Gauss Bonnet formula is

$$\chi(M) = \left(\frac{1}{2\pi}\right)^n \int_M \mathsf{Pf}(K),$$

In particular, the theorem implies that if the Levi Civita connection is flat, the Euler characteristic is zero.
An affine structure on a manifold is an atlas whose transition functions are affine transformations. The existence of such a structure is equivalent to the existence of a flat torsion free connection on the tangent bundle. Around 1955 Chern conjectured the following:

Conjecture. The Euler characteristic of a closed affine manifold is zero.

Not all flat torsion free connections on TM admit a compatible metric, and therefore, Chern-Weil theory cannot be used in general to write down the Euler class in terms of the curvature.

In 1955, Benzécri [1] proved that a closed affine surface has zero Euler characteristic. Later, in 1958, Milnor [11] proved inequalities which completely characterise those oriented rank two bundles over a surface that admit a flat connection. These inequalities prove that in case of a surface the condition "be torsion free" in Chern's conjecture is not necessary. In 1975, Kostant and Sullivan [9] proved Chern's conjecture in the case where the manifold is complete. In 1977, Smillie [15] proved that the condition that the connection is torsion free matters. For each even dimension greater than 2, Smillie constructed closed manifolds with non-zero Euler characteristic that admit a flat connection on their tangent bundle.

In 2015, Klingler [14] proved the conjecture for special affine manifolds. That is, affine manifolds that admit a parallel volume form.

The goal of this thesis is to provide an exposition of the previous results concerning the Euler characteristic of affine manifolds and study the Chern Weil theory.

Chapter 2

Preliminaries on bundles

2.1 Definitions

Definition 2.1.1 (Vector bundles)**.** *A vector bundle is a tuple* (M, E, π) *where* M *and* E *are smooth manifolds and* $\pi : E \to M$ *is a surjective submersion that satisfies:*

- *For every* $p \in M$*, the fiber* $\pi^{-1}(p)$ *is a vector space.*
- *Every* $p \in M$ *has an open neighborhood* $U \subseteq M$ *and a diffeomorphism* $\varphi : \pi^{-1}(U) \to U \times \mathbb{F}^k$ *(where* $\mathbb{F}$ *is the field* $\mathbb{C}$ *or* $\mathbb{R}$*) in such way that the following diagram commutes:*

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times \mathbb{F}^k \arrow[dl, "\pi"] \\
& U. &
\end{tikzcd}$$

Furthermore, for any $p \in M$ *the map* $v \mapsto \varphi^{-1}(p, v)$ *is linear isomorphism between the vector spaces* $\mathbb{F}^k$ *and* $\pi^{-1}(p)$*.*

Example 2.1.2. The map $\pi : M \times \mathbb{F}^k \to M$ is a vector bundle called the trivial bundle.

The main example is the tangent bundle. If M is a manifold then the tangent bundle $TM = \bigsqcup_{p \in M} T_pM$, where T_pM denotes the tangent sapce to M at p. The vector bundle structure is given by the natural projection $\pi : TM \to M$.

Definition 2.1.3 (Principal bundles)**.** *An action of a group* G *on a manifold* M *is a smooth function*

$$\begin{aligned} \mu : M \times G &\longrightarrow M \\ (m, g) &\longmapsto m \cdot g, \end{aligned}$$

such that: $m \cdot e = m$ *and* $(m \cdot g)h = m \cdot (gh)$*. In particular* μ *is called free action if for all* $m \in M$*,* $mg = m$ *implies* $g = e$*. If* G *is a Lie group, a principal* G*-bundle is a surjective submersion* $\pi : P \to M$ *with a smooth action of* G *on* P *that satisfies:*

(i) *The action is free.*

(ii) $\pi(mg) = \pi(m)$.

(iii) *If* $\pi(m) = \pi(m')$ *then* $m = m'g$.

Since π is a submersion then for all $q \in M$, $\pi^{-1}(q)$ is a manifold of dimension $d = \mathsf{dim}(P) - \mathsf{dim}(M) = \mathsf{dim}(G)$.

Lemma 2.1.4. *Every fiber of the principal G-bundle is diffeomorphic to G.*

Proof. Let $p \in M$ then if $q = \pi(p)$ the result of theorem is given by the diffeomorphism:

$$\begin{aligned} \psi : G &\longrightarrow \pi^{-1}(q) \\ g &\longmapsto pg. \end{aligned}$$

□

A priori P/G has structure of topological space. Then we will give to P/G differential structure induced via a isomorphism as bellow:

Theorem 2.1.5. *If* $\pi : P \to M$ *is a principal G-bundle. Then there is a homeomorphism* $P/G \xrightarrow{\tilde{\pi}} M$. *In particular* P/G *is a smooth manifold.*

Proof. We will prove that $\tilde{\pi}$ is a homeomorphism. This function is continuous by quotient topology. It remains to prove that it is bijective and its inverse map is continuous.

If $\tilde{\pi}[p] = \tilde{\pi}[p']$ then $\pi(p) = \pi(p')$ and $p' = pg$. This implies $[p] = [p']$ and $\tilde{\pi}$ is injective.
The surjectivity holds bye hypothesis. Since that π is a submersion, the implicit function theorem assures the existence of a local section $\sigma : U \to P$ of a open $U \subset M$, define $\tilde{\sigma}$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[dr, "\tilde{\sigma}"'] & P \arrow[d] \\ & P/G \end{tikzcd}$$

Note that $\tilde{\sigma}$ is continous map and $\tilde{\sigma} = \tilde{\pi}^{-1}\big|_U$. Therefore $\tilde{\pi}^{-1}$ is continuous. □

Theorem 2.1.6. *Every principal bundle is locally isomorphic to the trivial bundle.*

Proof. Let $P \to M$ be a principal bundle, then for every $q \in M$ there exists a open $U \subseteq M$ neighborhood of q and section $\sigma : U \to P$ such that $\pi \circ \sigma = \mathsf{id}$. Then the following function is a diffeomorphism:

$$\begin{array}{rccc} \tau & : U \times G & \longrightarrow & \pi^{-1}(U) \\ & (x, g) & \mapsto & \sigma(x) \cdot g. \end{array}$$

□

2.2 Examples of principal bundles

Frame bundles and associated bundles

If $\pi : E \to M$ is a vector bundle over M of rank k. Then the geometric structure that contains information about the set of all ordered bases (or frames) for a fiber of E is called frame bundle. A frame at $p \in M$ is an ordered basis for the vector space E_p. Equivalently, it can be viewed as a linear isomorphism $\varphi : \mathbb{F}^k \to E_p$ (Here $\mathbb{F}$ is $\mathbb{R}$ when we consider a real vector bundle or it is $\mathbb{C}$ when we consider a complex vector bundle).

Define the set of all frames at p as $ISO(\mathbb{F}^k, E_p) = \{\varphi : \mathbb{F}^k \to E_p | \varphi \quad \text{is an isomorphism}\}$. This set has the right action by the group $GL(k, \mathbb{F})$ via the composition of functions, i.e a element $g \in GL(k, \mathbb{F})$ acts on $ISO(\mathbb{F}^k, E_p)$ by $\varphi \circ g$.

A frame bundle is a principal bundle with structural group $GL(\mathbb{F}^k)$ defined by :

$$\begin{array}{rcl} \rho \ : Fr(E) & \longrightarrow & M \\ \varphi & \mapsto & p, \end{array}$$

where $Fr(E) := \coprod_{p \in M} ISO(\mathbb{F}^k, E_p)$.

If E is real vector bundle equipped with a Riemannian metric its orthogonal frame bundle of E is the principal bundle $Fr^O(E) := \coprod_{p \in M} ISO^O(\mathbb{R}^k, E_p)$ (the set of all orthogonal frames at each fiber), with structural group $O(k)$. Similarly, if a complex vector bundle E is equipped with a hermitian metric, then the hermitian frame bundle is $Fr^U(E) := \coprod_{p \in M} ISO^U(\mathbb{C}^k, E_p)$ with structural group $U(k)$.

Conversely, let us consider a principal G-bundle $\pi : P \to M$ and a representation $\rho : G \to GL(V)$ of group G. Then it is possible to construct a vector bundle over M, called the associated bundle. The group G acts on $P \times V$: for a element $g \in G$ and $(p, v) \in P \times V$ the action is given by $(p, v) \cdot g = (p \cdot g, \rho(g^{-1})(v))$. Then we can define $P \times_G V := P \times V/G$, where $(p \cdot g, v) \sim (p, \rho(g)(v))$. The following submersion $\tilde{\pi} : P \times_G V \to M$ determines the associated vector bundle:

$$\begin{array}{rcl} \tilde{\pi} \ : P \times_G V & \longrightarrow & M \\ (p, v) & \mapsto & \pi(p). \end{array}$$

When the structural group is $GL(n, \mathbb{F})$ and the representation is id the construction of the associated bundle is inverse to the frame bundle construction.

Pullback bundles

Definition 2.2.1. *Let $S \subseteq M$ be a submanifold of M and $h : N \to M$ be a function between two manifolds, we say that h is transversal to S (denoted by $h \pitchfork S$) if for every $g \in h^{-1}(S)$*

the following holds:

$$T_{h(g)}M = d_y h(T_y N) + T_{h(g)}S.$$

Theorem 2.2.2. *If* $h \pitchfork S$ *then* $h^{-1}(S)$ *is a submanifold of* N*, Moreover:*

$$\mathsf{dim}(N) - \mathsf{dim}(h^{-1}(S)) = \mathsf{dim}(M) - \mathsf{dim}(S).$$

Proof. The proof of this theorem can be found in the book of the Victor Guillemin and Alan Pollack [16]. ☐

Let $\pi : P \to M$ be a principal G-bundle and $f : N \to M$ be a function between two manifolds. Define the set $f^*(P) = \{(x,p) \in N \times P / f(x) = \pi(p)\}$. If we consider the function

$$\begin{array}{rccc} \lambda : & N \times P & \longrightarrow & M \times M \\ & (n,p) & \mapsto & (f(x), \pi(p)), \end{array}$$

It is easy to verify that $f^*(P) = \lambda(M \times M)$ and $\lambda \pitchfork M \times M$, then $f^*(P)$ is a submanifold of $N \times P$.

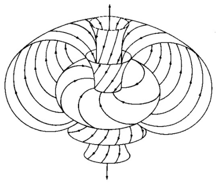
The right action of G on $f^*(P)$ is given by: for a element $g \in G$ and $(x,p) \in f^*(P)$, $(x,p) \cdot g = (x, p \cdot g)$.

Therefore $f^*(P)$ has structure of principal G-bundle, called called the pullback of E by f, so that the following diagram commutes:

$$\begin{tikzcd} f^*(P) \arrow[r] \arrow[d, "\bar{\pi}"'] & P \arrow[d, "\pi"] \\ N \arrow[r, "f"'] & M. \end{tikzcd}$$

Hopf fibration

The Lie group $\mathfrak{u}(1) = \mathbb{S}^1$ acts freely on $\mathbb{S}^3$ given by $(z_1, z_2) \cdot w = (z_1 w, z_2 w)$. The Hopf fibration is a principal $\mathfrak{u}(1)$-bundle with submersion given by $\pi : \mathbb{S}^3 \to \mathbb{C}P^1$, $(z_1, z_2) \mapsto [z_1, z_2]$.



The Hopf fibraton allows one to think of $\mathbb{S}^3$ as a decomposed as a family of circles parametrized by the complex projective plane.

Chapter 3

Preliminaries on connections

3.1 Connections on vector bundles

If $\alpha \in \Gamma(E)$ is a section of a vector bundle E, there is no natural way to differentiate α in the direction of a vector field. A connection on a vector bundle E is the solution to this problem.

Definition 3.1.1. *Let $\pi : E \to M$ be a vector bundle. A connection ∇ on E is a linear map:*

$$\begin{array}{rcl} \nabla \ : \mathfrak{X}(M) \otimes \Gamma(E) & \longrightarrow & \Gamma(E) \\ (X, \alpha) & \mapsto & \nabla_X \alpha \end{array}$$

such that for any smooth function $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$ and $\alpha \in \Gamma(E)$ the following two conditions are satisfied:

1. $$\nabla_{fX}\alpha = f\nabla_X\alpha,$$

2. $$\nabla_X (f\alpha) = (X(f))\,\alpha + f\nabla_X\alpha.$$

Definition 3.1.2. *If E is a vector bundle with connection, we will say that a section $\alpha \in \Gamma(E)$ is covariantly constant if $\nabla_X(\alpha) = 0$ for any vector field $X \in \mathfrak{X}(M)$.*

Let us now consider the case $E = TM$ and local coordinates $\varphi = (x^1, \dots, x^m)$. The Christoffel symbols $\Gamma_{ij}^k : M \to \mathbb{R}$ are smooth functions determined by the condition

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k.$$

The connection ∇ is determined by the Christoffel symbols. Given vector fields $X = \sum_i a^i\partial_i$,

$Y = \sum_j b^j \partial_j$ one computes:

$$\begin{aligned}
\nabla_X Y &= \sum_i a^i \nabla_{\partial_i} \left(\sum_j b^j \partial_j \right) = \sum_{i,j} a^i \nabla_{\partial_i} \left(b^j \partial_j \right) \\
&= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial x^i} \partial_j + b^j \nabla_{\partial_i} \partial_j \right) \\
&= \sum_{i,j} a^i \left(\frac{\partial b^j}{\partial x^i} \partial_j + b^j \sum_k \Gamma^k_{ij} \partial_k \right) \\
&= \sum_{i,j} a^i \frac{\partial b^j}{\partial x^i} \partial_j + \sum_{i,j} a^i b^j \sum_k \Gamma^k_{ij} \partial_k \\
&= \sum_k \left(\sum_i a^i \frac{\partial b^k}{\partial x^i} \partial_k + \sum_{i,j} \Gamma^k_{ij} b^j a^i \right) \partial_k .
\end{aligned}$$

A Riemannian metric g on a manifold M induces a connection, called the *Levi-Civita Connection*, on the tangent bundle TM.

Definition 3.1.3. *Let ∇ be a connection on TM. The torsion of ∇ is the function*

$$\begin{array}{rccl}
T & : \mathfrak{X}(M) \otimes \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\
 & (X,Y) & \mapsto & \nabla_X Y - \nabla_Y X - [X,Y]
\end{array}$$

Given vector fields $X, Y, Z \in \mathfrak{X}(M)$, the torsion satisfies:

- Linearity with respect to functions:

$$T(fX,Y) = fT(X,Y); \quad T(X,fY) = fT(X,Y).$$

- Skewsymmetry:

$$T\left(X,Y\right) + T\left(Y,X\right) = 0.$$

Then we can view the torsion as a tensor:

$$T \in \Omega^2(M, TM) = \Gamma(\Lambda^2(T^*M) \otimes TM),$$

defined by:

$$T(p)(v,w) = \nabla_X Y(p) - \nabla_Y X(p) - [X,Y](p),$$

for any choice of vector fields X, Y such that $X(p) = v$ and $Y(p) = w$.

Definition 3.1.4. *A connection on TM is called symmetric if its torsion is zero.*

Remark 3.1.5. A connection ∇ is symmetric if and only if for any choice of coordinates, the Christoffel symbols satisfy $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Definition 3.1.6. *A connection on a Riemannian manifold* (M, g) *is compatible with the metric if:*

$$X(g(Y, Z)) = g\left(\nabla_X Y,\ Z\right) + g\left(Y,\ \nabla_X Z\right),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Theorem 3.1.7 (Levi-Civita)**.** *Let* (M, g) *be a riemannian manifold. There exists a unique symmetric connection* ∇ *which is compatible with the metric. Moreover, this connection satisfies:*

$$\begin{aligned} g\left(Z, \nabla_Y X\right) = \frac{1}{2} &\left(X g\left(Y, Z\right) + Y g\left(Z, X\right) - Z g\left(X, Y\right)\right. \\ &\left. - g\left(\left[X, Z\right], Y\right) - g\left(\left[Y, Z\right], X\right) - g\left(\left[X, Y\right], Z\right)\right). \end{aligned} \tag{3.1}$$

Proof. Any connection compatible with the metric satisfies:

$$X g\left(Y, Z\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right),$$

$$Y g\left(Z, X\right) = g\left(\nabla_Y Z, X\right) + g\left(Z, \nabla_Y X\right)$$

$$Z g\left(X, Y\right) = g\left(\nabla_Z X, Y\right) + g\left(X, \nabla_Z Y\right),$$

Adding the first two equations, subtracting the third and using the symmetry one obtains:

$$\begin{aligned} &X g\left(Y, Z\right) + Y g\left(Z, X\right) - Z g\left(X, Y\right) \\ &= g\left(\left[X, Z\right], Y\right) + g\left(\left[Y, Z\right], X\right) + g\left(\left[X, Y\right], Z\right) + 2 g\left(Z, \nabla_Y X\right), \end{aligned}$$

which implies:

$$\begin{aligned} g\left(Z, \nabla_Y X\right) = \frac{1}{2} &\left(X g\left(Y, Z\right) + Y g\left(Z, X\right) - Z g\left(X, Y\right)\right. \\ &\left. - g\left(\left[X, Z\right], Y\right) - g\left(\left[Y, Z\right], X\right) - g\left(\left[X, Y\right], Z\right)\right). \end{aligned}$$

Since the metric is nondegenerate, this implies uniqueness. In order to prove existence we define $\nabla_Y X$ to be the unique vector field that satisfies Equation (3.1). In order to prove that ∇ defined in this way is a connection, the only nontrivial statement is:

$$\nabla_X (fY) = f \nabla_X Y + X(f) Y.$$

For this we compute:

$$\begin{aligned} g\left(Z, \nabla_Y \left(fX\right)\right) = \frac{1}{2} &\left(f X g\left(Y, Z\right) + Y g\left(Z, fX\right) - Z g\left(fX, Y\right)\right. \\ &\left. - g\left(\left[fX, Z\right], Y\right) - g\left(\left[Y, Z\right], fX\right) - g\left(\left[fX, Y\right], Z\right)\right). \end{aligned}$$

Using the equations

$$Y g\left(Z, fX\right) = \left(Y f\right) g\left(Z, X\right) + f Y g\left(Z, X\right),$$

$$Z g\left(fX, Y\right) = \left(Z f\right) g\left(X, Y\right) + f Z g\left(X, Y\right),$$

$$g\left(\left[fX, Z\right], Y\right) = f g\left(\left[X, Z\right], Y\right) - \left(Z f\right) g\left(X, Y\right)$$

$$g\left(\left[fX,Y\right],Z\right)=fg\left(\left[X,Y\right],Z\right)-\left(Yf\right)g\left(X,Z\right)$$

we obtain:

$$\begin{aligned} g\left(Z,\nabla_Y\left(fX\right)\right) &= fg\left(Z,\nabla_Y X\right)+\frac{1}{2}\left(2\left(Yf\right)g\left(Z,X\right)\right)\\ &= g\left(Z,f\nabla_Y X+\left(Yf\right)X\right). \end{aligned}$$

It is easy to check that ∇ is symmetric and compatible with the metric. ☐

The connection described above is called the Levi-Civita connection on (M,g).

3.2 Geodesics and the exponential map

Here we will explain the notions of parallel transport and geodesics. It will be convenient to first discuss some natural operations on vector bundles and connections.

Let $f:N\to M$ a smooth function and $\pi:E\to M$ a vector bundle then the set $f^*(E)=\coprod_{p\in N}E_{f(p)}$, admits a unique structure of a vector bundle over N such that:

1. The map $\tilde{f}:f^*(E)\to E;\quad v\in E_{f(p)}\mapsto v\in E_{f(p)}$ is smooth.
2. The projection $\pi:f^*(E)\to M$ is given by $v\in E_{f(p)}\mapsto p$.
3. The diagram:

$$\begin{tikzcd} f^*(E) \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes and is a linear isomorphism on each fiber.

Let ∇ be a connection on $\pi:E\to M$ and $f:N\to M$ a smooth function. Then there exists a unique connection $f^*(\nabla)$ on $f^*(E)$ such that for any $\alpha\in\Gamma(E)$, $X\in\mathfrak{X}(N)$ and $Y\in\mathfrak{X}(M)$ with $Df(p)(X(p))=Y(f(p))$ the following holds:

$$f^*(\nabla)_X(f^*(\alpha))(p)=\nabla_Y(\alpha)(f(p)). \tag{3.2}$$

Recall that we say that a section $\alpha\in\Gamma(E)$ of a vector bundle with connection is covariantly constant if $\nabla_X(\alpha)=0$, for any vector field $X\in\mathfrak{X}(M)$. By imposing this conditions on vector bundles over an interval one obtains the notion of parallel transport along a path.

Proposition 3.2.1. *Let ∇ be a connection on a vector bundle $\pi:E\to I$, where $I=[a,b]$ is an interval. Given a vector $v\in E_a$ there exists a unique covariantly constant section $\alpha\in\Gamma(E)$ such that $\alpha(a)=v$. Moreover, the function $P_a^b:E_a\to E_b$ given by $P_a^b(v)=\alpha(b)$ is a linear isomorphism. The function P_a^b is called the parallel transport of the connection ∇.*

Proof. Since all vector bundles over an interval are trivializable, we may choose a frame $\{\alpha_1, \dots, \alpha_k\}$ for E. There exists a one form $\theta \in \Omega^1(I, \mathsf{End}(E))$ such that:

$$\nabla_X(\alpha_i) = \theta(X, \alpha_i).$$

Let us fix $v = \sum_i \lambda_i \alpha_i(a) \in E_a$. A section $\alpha = \sum_i f_i \alpha_i$ is covariantly constant if it satisfies the differential equation:

$$\sum_i \nabla_{\partial_t}(f_i \alpha_i) = 0,$$

which is equivalent to:

$$\sum_i \frac{\partial f_i}{\partial t}\alpha_i + f_i \theta(\partial_t, \alpha_i) = 0.$$

The Picard-Lindelöf theorem guarantees the existence and uniqueness of a solution of this equation. In order to show that P_a^b is linear it is enough to observe that if α and β are covariantly constant, so is $\alpha + \beta$. It remains to show that P_a^b is an isomorphism. Suppose that $v \in E_a$ is such that $P_a^b(v) = 0$. By symmetry we know that there exists a unique section $\alpha \in \Gamma(E)$ such that $\alpha(b) = 0$. This section is the zero section and we conclude that $v = 0$. □

Definition 3.2.2. *Let ∇ be a connection on $\pi : E \to M$ and $\gamma : [a, b] \to M$ a smooth curve. The parallel transport along γ with respect to ∇ is the linear isomorphism:*

$$P_\nabla(\gamma) : E_{\gamma(a)} \to E_{\gamma(b)}; \quad P_\nabla(\gamma)(v) = P_a^b(v),$$

where P_a^b denotes the parallel transport associated with the vector bundle $\gamma^*(E)$ over the interval $I = [a, b]$ with respect to the connection $\gamma^*(\nabla)$.

Lemma 3.2.3. *Let $\gamma : [a, c] \to M$ be a curve and $b \in (a, c)$. Set $\mu = \gamma|_{[a,b]}; \quad \sigma = \gamma|_{[b,c]}$. Then $P_\nabla(\gamma) = P_\nabla(\sigma) \circ P_\nabla(\mu)$.*

Proof. It is enough to observe that if $\alpha \in \Gamma(\gamma^*(E))$ is covariantly constant then $\alpha|_{[a,b]}$ and $\alpha|_{[b,c]}$ are also covariantly constant. □

Remark 3.2.4. The parallel transport is a parametrization invariant. That is, if ∇ is a connection on $\pi : E \to M$, $\gamma : [a, b] \to M$ is a curve and $\varphi : [c, d] \to [a, b]$ is an orientation preserving diffeomorphism then $P_\nabla(\gamma) = P_\nabla(\gamma \circ \varphi)$.

Definition 3.2.5. *Let ∇ be a connection on TM. A curve $\gamma : [a, b] \to M$ is called a geodesic if the section $\gamma' \in \Gamma(\gamma^*(TM))$ is covariantly constant with respect to the connection $\gamma^*(\nabla)$.*

In local coordinates $\varphi = (x^1, \dots, x^m)$ where $\gamma = (u_1, \dots, u_m)$ and ∇ has Christoffel symbols Γ_{ij}^k one has $\gamma'(t) = \sum_i u_i'(t)\partial_i$, and the geodesic equation takes the form:

$$\begin{aligned}
\gamma^*(\nabla)_{\partial_t}(\gamma'(t)) &= \sum_i \gamma^*(\nabla)_{\partial_t}(u_i'(t)\partial_i) \\
&= \sum_i \Big(u_i''(t)\partial_i + u_i'(t)\gamma^*(\nabla)_{\partial_t}\partial_i\Big) \\
&= \sum_i \Big(u_i''(t)\partial_i + u_i'(t)\sum_j u_j'(t)\nabla_{\partial_j}\partial_i\Big) \\
&= \sum_i \Big(u_i''(t)\partial_i + u_i'(t)\sum_{j,k} u_j'(t)\Gamma_{ij}^k\partial_k\Big).
\end{aligned}$$

We conclude that γ is a geodesic precisely when it satisfies the system of differential equations:

$$u_i''(t) + \sum_{j,k} u_j'(t)u_k'(t)\Gamma^i_{kj} = 0, \ \forall i. \tag{3.3}$$

Example 3.2.6. On Euclidian space $\mathbb{R}^m$ the Christoffel symbols are $\Gamma^k_{ij} = 0$, and therefore the differential equation for a geodesic is just $u_i''(t) = 0$. We conclude that geodesics in euclidean space are straight lines.

Theorem 3.2.7. *Let ∇ be a connection on TM. Given $v \in T_pM$, there exists an interval $(-\epsilon, \epsilon)$ for which there is a unique geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Proof. Let $\varphi = (x^1, \ldots, x^m)$ be local coordinates such that $\varphi(p) = 0$. We write $\gamma(t) = (u_1(t), \ldots, u_m(t))$ and want to solve the system of equations:

$$u_i''(t) + \sum_{j,k} u_j'(t)u_k'(t)\Gamma^i_{kj} = 0.$$

This is a second order ordinary differential equation. The existence and uniqueness of solutions is guaranteed by the Pickard-Lindelöf theorem. ☐

Definition 3.2.8. *Let ∇ be a connection on TM and $p \in M$. We define $A_p \subseteq T_pM$ as follows:*

$$A_p := \{v \in T_pM : \textit{there exists a geodesic } \gamma_v : [-1, 1] \to M, \ \textit{with } \gamma_v(0) = p \textit{ and } \gamma_v'(0) = v\}.$$

The exponential map is defined by:

$$\mathsf{Exp_p} : A_p \to M;\ v \mapsto \gamma_v(1).$$

The proof of the following theorem can be found in any text on riemannian geometry, for example [10].

Theorem 3.2.9. *Let ∇ be a connection on TM and $p \in M$. The domain A_p of the exponential map contains an open neighborhood around $0 \in T_pM$. Moreover, the derivative of the exponential map at 0 is the identity and therefore the exponential map is a local diffeomorphism.*

Proof. The prove of this theorem is in the book "Introduction to smooth manifolds" by John M. Lee [18]. ☐

3.3 Connections on principal bundles

Definition 3.3.1. *A distribution E on M is a subbundle of the tangent bundle. If $\Gamma(E) \subseteq \Gamma(TM)$ is a Lie subalgebra then E is called foliation.*

Definition 3.3.2. *Let $\phi : M \to N$ a smooth function between two manifolds. Given two vector fields $X \in \mathfrak{X}(M)$, $X' \in \mathfrak{X}(N)$, we say that X is ϕ-related to X' (denoted by $\sim_\phi$) if the equation $D_p\phi(X(p)) = X'(\phi(p))$ holds.*

Lemma 3.3.3. *If $X \sim_\phi X'$ and $Y \sim_\phi Y'$ then $[X,Y] \sim_\phi [X',Y']$.*

Proof. In this prove we will use a alternative definition to the tangent space at a point $p \in M$.

$$T_pM = \{X \in Hom(C^\infty(M), \mathbb{R})/X(fg) = X(f)g + X(g)f\}$$

Note that if $X \sim_\phi X'$ by this definition $X(f \circ \phi) = (D_p\phi(X))(f) = X'(f) \circ \phi$, for every $f \in C^\infty(N)$. Then $X \sim_\phi X'$ if only if $X(f \circ \phi) = X'(f) \circ \phi$ for every $f \in C^\infty(N)$.

Let $f \in C^\infty(N)$. Suppose that $X \sim_\phi X'$ and $Y \sim_\phi Y'$ then $X(f \circ \phi) = X'(f) \circ \phi$ and $Y(f \circ \phi) = Y'(f) \circ \phi$. Then compute:

$$(X \circ Y)(f \circ \phi) = X(Y'(f) \circ \phi) = (X' \circ Y')(f) \circ \phi$$

and $(Y \circ X)(f \circ \phi) = (Y' \circ X')(f) \circ \phi$.

Therefore

$$[X,Y](f \circ \phi) = (X \circ Y)(f \circ \phi) - (Y \circ X)(f \circ \phi) = (X' \circ Y')(f) \circ \phi - (Y' \circ X')(f) \circ \phi = [X',Y'](f) \circ \phi.$$

□

Theorem 3.3.4. *If $\pi : E \to M$ is a submersion then $Ker(d\pi)$ is a foliation.*

Proof. First, we will prove that $ker(d\pi)$ is a distribution.
Since $\pi : E \to M$ is a submersion for every $q \in M$, $F = \pi^{-1}(q)$ is a submanifold of dimension $d = \mathsf{dim}(E) - \mathsf{dim}(M)$. In particular, we will show that for $p \in E$, $T_p(F) = Ker(d_p(\pi))$.

If $v \in T_p(F)$ then there exist a curve γ so that $v = [\gamma]$, hence that $D_p(\pi)(v) = [\pi \circ \gamma] = [q] = 0$ and $T_pF \subseteq Ker(d\pi)$.

Note that $\mathsf{dim}(T_pF) = \mathsf{dim}(E) - \mathsf{dim}(M) = \mathsf{dim}(Ker(d\pi))$ then $T_pF = Ker(d(\pi))$.

Now, we will prove $Ker(d(\pi))$ is a foliation. Let $v, v' \in \Gamma(Ker(d\pi))$ then $v \sim_\pi 0$ and $v' \sim_\pi 0$ therefore by the lemma 3.3.3 $[v, v'] \sim_\pi 0$ and $[v, v'] \in \Gamma(Ker(d\pi))$.

□

Definition 3.3.5. *Let $\pi : X \to Y$ be a submersion, then an Ehresmann connection is a distribution $\mathcal{H} \subseteq TX$ such that every $p \in X$, $\mathcal{H}(p) \oplus Ker(d\pi(p)) = T_pX$.*

Definition 3.3.6. *Let $\pi : P \to M$ be a principal G-bundle over a manifold M. A connection on P is an Eheresmann connection which is G-equivariant, in other words, for $q = pg$, $D(R_g)(p)(H_p) = H_q$.*

Theorem 3.3.7. *If $\pi : P \to M$ is a principal G-bundle, then there exists an isomorphism between the bundles $P \times \mathfrak{g}$ and the vector bundle which has fiber $Ker(d\pi(p))$.*

Proof. Since that $P \to M$ is a principal G-bundle, there exists a Lie algebra homomorphism:

$$\begin{array}{rccc} \mu: & \mathfrak{g} & \longrightarrow & \mathfrak{X}(P) \\ & v & \mapsto & v^* \end{array}$$

where $v^*(p) = \frac{d}{dt}\big|_{t=0} p \cdot exp(tv)$.

In fact, for a point fix $p \in P$ the following function is linear a isomorphism.

$$\begin{array}{rccc} \mu_p: & \mathfrak{g} & \longrightarrow & Ker(d\pi(p)) \\ & v & \mapsto & v^*(p). \end{array}$$

Note that $\mathsf{dim}(\mathfrak{g}) = \mathsf{dim}(Ker(d\pi(p)))$, it is enough to prove injectivity. Let $\mu_p(v) = 0 = v^*(p) = \frac{d}{dt}\big|_{t=0} p \cdot exp(tv)$.

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=t_0} p \cdot exp(tv) &= \frac{d}{dz}\bigg|_{z=0} p \cdot exp((z+t_0)v) \\ &= \frac{d}{dz}\bigg|_{z=0} p \cdot exp(zv) \cdot exp(t_0 v) \\ &= D\left(R_{exp(t_0 v)}\right)\left(\frac{d}{dz}\bigg|_{z=0} p \cdot exp(zv)\right) \\ &= D(R_{exp(t_0 v)})(0) = 0 \end{aligned}$$

then $v = 0$. □

Next, we identify a connection $\mathcal{H}$ on a principal bundle P with a Lie algebra-valued form on P.This identification will simplify computations.

Lemma 3.3.8. *Let $\mathcal{H} \subseteq TP$ be a connection on P, then for each $p \in P$ there is an isomorphism $T_pP = \mathfrak{g} \oplus T_{\pi(p)}M$.*

Proof. Previous results have demonstrated that:

$$T_pP = Ker(d\pi(p)) \oplus \mathcal{H}_p \quad y \quad Ker(d\pi(p)) = \mathfrak{g},$$

furthermore the following function is a linear isomorphism

$$d\pi_p : \mathcal{H}_p \xrightarrow{\sim} T_{\pi(p)}M.$$

□

Lemma 3.3.9. *If $\mathcal{H} \subseteq TP$ is a connection on P then there exits a 1-form $\theta \in \Omega^1(P, \mathfrak{g})$ that satisfies:*

- *$\theta(v^*(p)) = v$.*
- *θ is equivariant, i.e for $X \in T_pP$, $\theta(q)D(R_g)(X) = Ad(g^{-1})(\theta(p)(X))$.*

Proof. The 1-form θ is defined as $\theta(p)(X) = v$, where $v \in \mathfrak{g}$ is the unique vector such that $v^*(p) = X^v$ in the decomposition $T_pP = \mathfrak{g} \oplus T_{\pi(p)}M$, for $X \in T_pP$, $X = X^v + X^h$. Then by construction $\theta(v^*(p)) = v$.

Show that θ is equivariant is equivalent show that $\mu_q(\theta(q)(DR_g)(X)) = \mu_q(Ad(g^{-1})(\theta(p)(X)))$, i.e $(D(R_g)(X))^v = \mu_q(Ad(g^{-1}))(\theta(p)(X))$.

One computes

$$\begin{aligned}
\mu_q(Ad(g^{-1}))(\theta(p)(X)) &= \frac{d}{dt}\bigg|_{t=0} q(exp(Ad(g^{-1})(t\theta(p)(X)) \\
&= \frac{d}{dt}\bigg|_{t=0} q \cdot g^{-1} exp(t\theta(p)(X))g \\
&= \frac{d}{dt}\bigg|_{t=0} p(exp(t\theta(p))(X^v))g \\
&= D(R_g)\left(\frac{d}{dt}\bigg|_{t=0} p(exp(t\theta(p)(X^v))\right) \\
&= D(R_g)(\theta(p)(X^v)^*(p)) \\
&= D(R_g)(X^v) = (D(R_g)(X))^v.
\end{aligned}$$

□

Lemma 3.3.10. *If $\theta \in \Omega^1(P, \mathfrak{g})$ is an equivariant 1-form such that $\theta(v^*(p)) = p$ then there exists a connection in P given by $Ker(\theta)$.*

Proof. First we will show that for a every point $p \in P$ one has $Ker(d\pi(p)) \oplus Ker(\theta(p)) \cong T_pP$. The map $\theta(p) : T_pP \to \mathfrak{g}$ is surjective, then $\mathsf{dim}(Ker(\theta)) = \mathsf{dim}(P) - \mathsf{dim}(\mathfrak{g})$ i.e $\mathsf{dim}(Ker(\theta)) + \mathsf{dim}(Ker(d\pi(p))) = \mathsf{dim}P$. Also, if $X \in Ker(d\pi(p)) \bigcap Ker(\theta(p))$ then $X = v^*(p)$ but $\theta(p)(v^*(p)) = v = 0$ and this implies that $X = 0$. It only remains to prove that $Ker(\theta)$ is equivariant with respect to the action of the group G. For $v \in Ker(\theta(p))$ we obtain that $\theta(q)(D(R_g)(p)(V)) = Ad(g^{-1})\theta(p)(v) = 0$ i.e $D(R_g)(p)(Ker(\theta(p))) \subseteq Ker(\theta(q))$. □

Definition 3.3.11. *A horizontal form on a principal bundle $\pi : P \to M$ is a form $\theta \in \Omega(P)$ such that $\theta(p)(X_1, X_2, ..., X_n) = 0$ if any $X_i \in Ker(d\pi(p))$.*

Definition 3.3.12. *A differential graded algebra $A = \oplus_{i \geq 0} A_i$ is a graded algebra equipped with a map $d : A_i \to A_{i+1}$ that satisfies:*

i $d \circ d = 0$.

ii $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$.

Definition 3.3.13. *A commutative differential graded algebra is a differential graded algebra so that the product in the algebra satisfies:*

$$a \cdot b = (-1)^{|a||b|} b \cdot a.$$

Definition 3.3.14. *A differential graded Lie algebra (dgla) is a graded vector space* $\mathfrak{g} = \oplus_{k\geqslant 0}\mathfrak{g}^k$ *together with a bilinear map* $[\cdot,\cdot] : \mathfrak{g}^i \otimes \mathfrak{g}^j \to \mathfrak{g}^{i+j}$ *and a differential map* $\delta : \mathfrak{g}^k \to \mathfrak{g}^{k+1}$ *satisfying:*

i $[v, w] + (-1)^{|v||w|}[w, v] = 0$ *(graded skew-symmetric).*

ii $[v, [w, z]] = [[v, w], z] + (-1)^{|v||w|}[w, [v, z]]$ *(Jacobi Identity).*

iii $\delta([v, w]) = [\delta v, w] + (-1)^{|v|}[v, \delta w]$ *(graded Leibniz rule).*

iv $\delta \circ \delta = 0$

Lemma 3.3.15. *If* A *is a commutative differential graded algebra and* $\mathfrak{g}$ *is a Lie algebra then* $A \otimes \mathfrak{g}$ *together with the maps:* $\delta(a \otimes v) = da \otimes v$ *and* $[a \otimes v, b \otimes w] = ab \otimes [v, w]$ *is a dgla.*

Proof. $A \otimes \mathfrak{g}$ satisfies the graded skew symmetric property i.e $[a \otimes v, b \otimes w] + (-1)^{|a||b|}[b \otimes w, a \otimes v] = 0$.
One computes

$$[a \otimes v, b \otimes w] = ab \otimes [v, w]$$

and

$$[b \otimes w, a \otimes v] = ba \otimes [w, v] = -(-1)^{|a||b|}ab \otimes [v, w]$$

then

$$[a \otimes v, b \otimes w] = ab \otimes [v, w] + (-1)^{|a||b|}[b \otimes w, a \otimes v] = ab \otimes [v, w] - ab \otimes [v, w] = 0.$$

$A \otimes \mathfrak{g}$ satisfies the Jacobi identity property i.e $[a \otimes v, [b \otimes w, c \otimes z]] = [[a \otimes v, b \otimes w]c \otimes z] + (-1)^{|a||b|}[b \otimes w, [a \otimes v, c \otimes z]]$.
One computes

$$[a \otimes v, [b \otimes w, c \otimes z]] = [a \otimes v, bc \otimes [w, z]] = abc \otimes [v, [w, z]]$$

using the Jacobi identity in $\mathfrak{g}$.

$$abc \otimes [v, [w, z]] = abc \otimes [[v, w], z] + [w, [v, z]] = abc \otimes [[v, w], z] + abc \otimes [w, [v, z]]$$

where $[[a\otimes v, b\otimes w], c\otimes z] = abc\otimes [[v, w], z]$ and $[b\otimes w, [a\otimes v, c\otimes z]] = (-1)^{|a||b|}abc\otimes [w, [v, z]]$ therefore

$$[a \otimes v, [b \otimes w, c \otimes z]] = [[a \otimes v, b \otimes w]c \otimes z] + (-1)^{|a||b|}[b \otimes w, [a \otimes v, c \otimes z]].$$

$A \otimes \mathfrak{g}$ satisfies the graded Leibniz rule i.e $\delta([a \otimes v, b \otimes w]) = [\delta(a \otimes v), b \otimes w] + (-1)^{|a|}[a \otimes v, \delta(b \otimes w)]$.
One computes

$$\delta([a \otimes v, b \otimes w]) = \delta(ab \otimes [v, w]) = d(ab) \otimes [v, w]$$

where $d(ab) = (da)b + (-1)^{|a|}a(db)$, then

$$\begin{aligned} d(ab) \otimes [v, w] &= (da)b \otimes [v, w] + (-1)^{|a|}a(db) \otimes [v, w] \\ &= [da \otimes v, b \otimes w] + (-1)^{|a|}[a \otimes v, db \otimes w] \\ &= [\delta(a \otimes v), b \otimes w] + (-1)^{|a|}[a \otimes v, \delta(b \otimes w)]. \end{aligned}$$

□

Note that if $\pi : P \to M$ is a principal G-bundle, then $\Omega(P)$ is a commutative differential graded algebra and $\Omega(P) \otimes \mathfrak{g}$ has structure of dgla, therefore the following definition makes sense.

Definition 3.3.16. *Let $\theta \in \Omega^1(P, \mathfrak{g}) := \Omega(P) \otimes \mathfrak{g}$ be a connection 1-form on a principal bundle P. Then the* ***curvature*** *$F_\theta \in \Omega^2(P, \mathfrak{g})$ of θ is defined by:*

$$F_\theta = d\theta + \frac{1}{2}[\theta, \theta].$$

A connection is said to be flat if its curvature is zero.

Lemma 3.3.17. *If F_θ is the curvature of a connection on a principal G-bundle then it satisfies the formula $d(F_\theta) + [\theta, F_\theta] = 0$. This is called the Bianchi Identity.*

Proof. One computes

$$\begin{aligned} d(F_\theta) + [\theta, F_\theta] &= d(d\theta + \frac{1}{2}[\theta, \theta]) + [\theta, d\theta + \frac{1}{2}[\theta, \theta]] \\ &= \frac{1}{2}[d\theta, \theta] - \frac{1}{2}[\theta, d\theta] + [\theta, d\theta] + \frac{1}{2}[\theta, [\theta, \theta]] \\ &= [d\theta, \theta] + [\theta, d\theta] + \frac{1}{2}[\theta, [\theta, \theta]] = 0. \end{aligned}$$

□

Lemma 3.3.18. *If $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \mathfrak{X}(M)$ then*

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = &\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k) \\ &+ \sum_i (-1)^i \mathcal{L}_{X_i}(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \end{aligned}$$

where $\mathcal{L}_X$ denotes the Lie derivative.

Proof. The proof of this theorem is in the book "Topology and Geometry" by Glen E. Bredon [19]. □

Lemma 3.3.19. *If $X, Y \in \mathfrak{X}(M)$ so that X has a local flow φ_t then $[X, Y](p) = \frac{d}{dt}\Big|_{t=0} \varphi_t^*(Y)(p)$.*

Proof. The proof of this theorem is in the book "Introduction to smooth manifolds" by John M. Lee [18]. □

Lemma 3.3.20. *The curvature of a connection θ on a principal bundle satisfies the following properties:*

- *i F_θ is an equivariant and horizontal form.*
- *ii θ is flat if and only if $Ker(\theta)$ is a foliation.*

Proof. Let us start by proving the first statement. Every field in $\mathfrak{X}(P)$ can be expressed as a sum of vertical and horizontal fields. Then to prove that F is a horizontal form we will consider the following cases:

Case 1 X and Y are vertical fields.
There exits $v, w \in \mathfrak{g}$ so that $X = v^*$ and $Y = w^*$.

$$\begin{aligned} F(X,Y) &= \theta([X,Y]) + \mathcal{L}_X(\theta(Y)) - \mathcal{L}_Y(\theta(X)) + [\theta(X), \theta(Y)] \\ &= \theta([w^*, v^*]) + \mathcal{L}_{v^*}\theta(w^*) - \mathcal{L}_{w^*}\theta(v^*) + [\theta(v^*), \theta(w^*)] \end{aligned}$$

Since $\theta(w^*)$ and $\theta(v^*)$ are constant then $\mathcal{L}_{v^*}\theta(w^*) = \mathcal{L}_{w^*}\theta(v^*) = 0$.

$$F(X,Y) = \theta([v,w]^*) + [v,w] = [w,v] + [v,w] = 0.$$

Case 2 X is a vertical field and Y is a horizontal field. Then there exists $v \in \mathfrak{g}$ so that $X = v^*$.

$$F(v^*, Y) = \theta([Y, v^*]) + \mathcal{L}_{v^*}\theta(Y) - \mathcal{L}_Y\theta(v^*) + [\theta(v^*), \theta(Y)]$$

where $\mathcal{L}_{v^*}\theta(y) = \mathcal{L}_Y\theta(v^*) = [\theta(v^*), \theta(Y)] = 0$. Let φ_t be a flow of v^*, then for $p \in P$.

$$\begin{aligned} \theta([v^*, y])(p) &= \theta(p)\left(\frac{d}{dt}\bigg|_{t=0} (\varphi_t)^*(Y)(p)\right) = \theta(p)\left(\frac{d}{dt}\bigg|_{t=0} D(\varphi_t)Y(\varphi_t^{-1}(p))\right) \\ &= \theta(p)\left(\frac{d}{dt}\bigg|_{t=0} D\varphi_t Y(p\cdot exp(-tv))\right) = \theta(p)\frac{d}{dt}\bigg|_{t=0} D(R_{exp(tv)})(Y(p\cdot exp(-tv)) \\ &= \frac{d}{dt}\bigg|_{t=0} \theta(p)D(R_{exp(tv)})(Y(p\cdot exp(-tv))) = \frac{d}{dt}\bigg|_{t=0} Ad(exp(-tv))\theta(Y(p\cdot exp(-tv))) \end{aligned}$$

recall that if $Y \in Ker(\theta)$ then $\theta([v^*, Y])(p) = 0$.

Case 3 X and Y are horizontal field.

$$F(X,Y) = \theta[X,Y] + \mathcal{L}_X\theta(Y) - \mathcal{L}_Y\theta(X) + [\theta(X), \theta(Y)] = \theta([Y,X]).$$

Now, we will prove that F_θ is a equivariant form, i.e for $X, Y \in \mathfrak{X}(M)$ satisfies the formula $F_\theta(DR_gX, DR_gY) = Ad(g^{-1})F_\theta(X,Y)$.

If X and Y are horizontal,

$$F(DR_gX, DR_gY) = \theta([DR_gY, DR_gX]) = \theta(DR_g([Y,X])) = Ad(g^{-1})\theta([Y,X]) = Ad(g^{-1})F(X,Y).$$

In the case when X and Y are vertical fields then $F(DR_gX, DR_gY) = 0$ and $Ad(g^{-1})F(X,Y) = 0$.
For the second statement, let X and Y be horizontal vector fields. If $Ker(\theta)$ is a foliation then $[X,Y] \in Ker(\theta)$ hence that $F(X,Y) = 0$. Conversely, if θ is flat, then $F(X,Y) = 0$ then $\theta([X,Y]) = 0$ and $[X,Y] \in Ker(\theta)$.

□

Chapter 4

On Chern-Gauss-Bonnet theorem

4.1 Chern-Weil Homomorphism

The purpose of this section is to give a brief introduction to the Chern-Weil theory of characteristic classes, which was developed by Shiing-Shen Chern and André Weil in the first half of the 20th century. For each principal G-bundle over a manifold M we want to construct a homomorphism from the algebra of G- invariant polynomials on the Lie algebra to the cohomology of M.

Definition 4.1.1. *Let $\pi : P \to M$ be a principal G-bundle over M together with a connection θ. Then the curvature F_θ determines a homomorphism of algebras given by:*

$$\begin{array}{cccc} \Phi : & S(\mathfrak{g}^*) & \longrightarrow & \Omega(P) \\ & \alpha_1 \odot \cdots \odot \alpha_k & \mapsto & \phi(\alpha_1) \wedge \cdots \wedge \phi(\alpha_k) \end{array}$$

where $\phi(\alpha_i)(X,Y) = \alpha_i(F_\theta(X,Y))$ and $S(\mathfrak{g}^*)$ denotes the symmetric algebra of $\mathfrak{g}^*$.

Lemma 4.1.2. *Let $\pi : P \to M$ be a principal G-bundle and a representation of G $\rho : G \to GL(V)$. Then horizontal and equivariant forms of $\Omega(P,V)$ are in a one to one correspondence with forms of $\Omega(M, \rho(P))$, where $\rho(P)$ denotes the bundle $P \times_G V$.*

Proof. Consider the function:

$$\left\{ \begin{array}{c} \omega \in \Omega(P,V) \quad \text{such that} \\ \omega \quad \text{is horizontal} \\ \text{and equivariant} \end{array} \right\} \overset{\psi}{\to} \Omega(M, \rho(P)); \quad \theta \ \mapsto \ \psi(\theta)$$

defined by $\psi(\theta)(y)(X_1, ..., X_k) = [(p, \theta(p)(\tilde{X}_1, ..., \tilde{X}_k)]$ where $p \in \pi^{-1}(y)$ and $D\pi(\tilde{X}_i) = X_i$. The well definition of this function is given by the condition of θ is a horizontal and equivariant form. Note that the pullback of differential forms is the inverse function to ψ.

□

Lemma 4.1.3. *The set of horizontal and equivariant forms of P denoted by $\Omega^{bas}(P)$ is isomorphic to $\Omega(M)$.*

Proof. Consider the principal G-bundle $\pi : P \to M$ then there exists an algebra homomorphism $\pi^* : \Omega(M) \to \Omega(P)$. In fact, π^* is injective because if

$$\pi^*(\theta)(p)(X_1, ..., X_k) = 0 \text{ for every } p \in P \text{ and } X_1, ..., X_k \in T_pP,$$
$$\theta(\pi(p))(D\pi(X_1, ..., D\pi(X_k))) = 0$$

then $\theta(\pi(p))(Y_1, ..., Y_k) = 0$ every $\pi(p) \in M$ and $Y_1, ..., Y_k \in T_\pi \in T_{\pi(p)}M$.

The homomorphism π^* is surjective when we restrict the codomain to the set of horizontal and equivariant forms. Its surjectivity is guaranteed by Lemma 4.1.2 when we consider the trivial representation.

□

Definition 4.1.4. *Let V be a vector space, the symmetric algebra on V, denoted as $S(V)$ is defined by the quotient $S(V) = TV/I$, where TV is the free algebra on V and I is the ideal generated by the differences of products $v \otimes w - w \otimes v$.*

We will be interested in the case $V = \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of Lie group G. This symmetric algebra $S(\mathfrak{g}^*)$ has a action of the group G vía the adjoint representation. For a element $g \in G$ and $\alpha_1 \odot \cdots \odot \alpha_k \in S(\mathfrak{g}^*)$ the action is given by $g \cdot (\alpha_1 \odot \cdots \odot \alpha_k) = \alpha_1 \circ Ad(g^{-1}) \odot \cdots \odot \alpha_k \circ Ad(g^{-1})$. Therefore, it makes sense to define the G-invariant algebra,

$$S(\mathfrak{g}^*)^G = \{\alpha \in S(\mathfrak{g}^*) | g \cdot \alpha = \alpha \, \forall g \in G\}.$$

Theorem 4.1.5. *Let $\pi : P \to M$ be a principal G-bundle together with a connection θ. Then the homomorphism Φ defined above satisfies:*

1. $\Phi(S(\mathfrak{g}^*)^G) \subseteq \Omega(P)^G$.
2. *Each form of* $\Phi(S(\mathfrak{g})^*)$ *is horizontal.*
3. $d(\Phi(S(\mathfrak{g}^*))) = 0$.

Proof. For the first statement we will prove that $\Phi : S(\mathfrak{g}^*) \to \Omega(P)$ is equivariant. In particular, $\Phi(S(\mathfrak{g}^*)^G) \subseteq \Omega(P)^G$. To see this, one computes:

$$\begin{aligned}\Phi(g \cdot \alpha)(X, Y) &= (g \cdot \alpha)(F(X, Y)) = \alpha \circ Ad(g^{-1})F(X, Y) \\ &= \alpha(F(DR_gX, DR_gY)) = g \cdot (\Phi(\alpha))(X, Y).\end{aligned}$$

Let us now consider the second statement. It is enough to note that for $\alpha \in \mathfrak{g}^*$, $\Phi(\alpha)$ is horizontal. Then for any X vertical, $\Phi(\alpha)(X, Y) = \varphi(F(X, Y)) = \alpha(0) = 0$.

Let $\alpha \in \mathfrak{g}^*$ and X_1, X_2, X_3 horizontal vectors, then

$$\begin{aligned}&d\big(\Phi(\alpha)\big)(X_1, X_2, X_3) = \alpha\big(dF(X_1, X_2, X_3)\big) = -\frac{1}{2}\alpha\big([\theta, F](X_1, X_2, X_3)\big) \\ &= -\frac{1}{2}\alpha\big([\cancelto{0}{\theta(X_1)}, F(X_2, X_3)] - [\cancelto{0}{\theta(X_3)}, F(X_1, X_2)] + [\cancelto{0}{\theta(X_2)}, F(X_3, X_1)]\big) = 0.\end{aligned}$$

By induction, it follows $d\big(\Phi(\alpha_1) \wedge \cdots \wedge \Phi(\alpha_k)\big) = 0$.

□

In the following steps we will construct the Chern-Weil homomorphism:

- Given a connection θ on a principal G-bundle with curvature F_θ, there exists a algebras homomorphism $S(\mathfrak{g}^*) \xrightarrow{\Phi} \Omega(P)$.
- We proved that this homomorphism is equivariant and its image consists of basic forms.
- Use the isomorphism $\Omega(P)^{bas} \cong \Omega(M)$ to extender Φ:

$$\begin{tikzcd} S(\mathfrak{g}^*) \arrow[r] \arrow[rr, bend left, "\Phi"] & \Omega(P)^{bas} \arrow[r, "\cong"'] & \Omega(M). \end{tikzcd}$$

- By the last theorem, the image of Φ is a set of the closed forms in $\Omega(M)$ and therefore we can extend the function Φ to the De Rham cohomology of M via natural projection:

$$\begin{tikzcd} S(\mathfrak{g}^*)^G \arrow[r] \arrow[rrr, bend left, "\Phi"] & \Omega(P)^{bas} \arrow[r, "\cong"'] & \Omega(M) \arrow[r] & H^*(M). \end{tikzcd}$$

Now we will focus on proving that the Chern-Weil homomorphism Φ does not depend on the chosen curvature.

Definition 4.1.6. *A morphism between two principal G-bundles $\pi : P \to M$ and $\pi' : P' \to M'$ is given by a smooth function f and a equivariant function $\hat{f}$ such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, "\pi"] & P' \arrow[d, "\pi'"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

Proposition 4.1.7. *Let $\pi : P \to N$ and $\pi' : P' \to M$ be are G-principal bundles. If $f : N \to M$ and $\hat{f} : P \to P'$ define a morphism of principal G-bundles and θ is a connection on P' then $f^*(\theta)$ is a connection on P.*

Proof. $f^*(\theta)$ is a vertical form.
For $v \in \mathfrak{g}$:

$$Df(v^*(p)) = Df\left(\left.\frac{d}{dt}\right|_{t=0} p \cdot exp(tv)\right) = \left.\frac{d}{dt}\right|_{t=0} f\left(p \cdot exp(tv)\right) = \left.\frac{d}{dt}\right|_{t=0} f(p) \cdot exp(tv) = v^*(f(p))$$

Therefore:

$$f^*(\theta)(v^*(p)) = \theta(Df(v^*(p))) = \theta(v^*(f(p))) = v.$$

$f^*(\theta)$ is a equivariant.

$$\begin{aligned} f^*(\theta)(DR_g(p) &= \theta(Df \circ DR_g(p)) = \theta(D(f \circ R_g)(p)) = \theta(D(R_g \circ f)(p) \\ &= \theta(DR_g \circ Df(p)) = Ad(g^{-1})\theta(Df(p)) = Ad(g^{-1})f^*(\theta)(p). \end{aligned}$$

□

Lemma 4.1.8. *Let $\pi : P \to N$ and $\hat{\pi} : P' \to M$ be are G-principal bundles. If $f : N \to M$ and $\hat{f} : P \to P'$ define a morphism of principal G-bundles and θ is a connection on P'. Then $\Phi_{\hat{f}(\theta)} = \hat{f}^* \circ \Phi_\theta$, meaning that the following diagram commutes:*

$$\begin{tikzcd}
S(\mathfrak{g}^*) \arrow[r, "\Phi_\theta"] \arrow[dr, "\Phi_{\hat{f}^*(\theta)}"'] & \Omega(P') \arrow[d, "\hat{f}^*"] \\
 & \Omega(P)
\end{tikzcd}$$

Proof. For $\alpha \in \mathfrak{g}^*$, $\Phi_{\hat{f}^*(\theta)}(\alpha) = \alpha \circ F_{\hat{f}^*(\theta)}$ and $\hat{f}^*(\alpha \circ F_\theta) = \alpha \circ \hat{f}^*(F_\theta)$. In fact, $F_{\hat{f}^*(\theta)} = \hat{f}^*(F_\theta)$ because $\hat{f}^*(F_\theta) = \hat{f}^*\big(d\theta + \frac{[\theta,\theta]}{2}\big) = d(\hat{f}^*(\theta)) + \frac{[\hat{f}^*(\theta), \hat{f}^*(\theta)]}{2}$. □

Theorem 4.1.9. *If $\pi : P \to M$ is a principal G-bundle with two connections on P called θ_0 and θ_1. Then $\Phi_{\theta_0} = \Phi_{\theta_1}$.*

Proof. We consider the natural projection $\tau : M \times [0,1] \to M$ then $\tau^*(P)$ is a principal G-bundle over $M \times [0,1]$. One defines a connection on $\tau^*(P)$ given by $\theta(p,t) = (1-t)\theta_0 + t\theta_1$, then we can obtain the connections by the pullback of functions ι_0 and ι_1, i.e $\iota_0^*(\theta) = \theta_0$ and $\iota_1^*(\theta) = \theta_1$, where:

$$\begin{array}{rrcl}
\iota_0 & : M & \longrightarrow & M \times [0,1] \\
 & x & \mapsto & (x,0)
\end{array}
\qquad\qquad
\begin{array}{rrcl}
\iota_1 & : M & \longrightarrow & M \times [0,1] \\
 & x & \mapsto & (x,1).
\end{array}$$

Therefore

$$\Phi_{\theta_1} = \Phi_{\iota_1^*(\theta)} = \iota_1^* \circ \Phi_\theta = \iota_0^* \circ \Phi_\theta = \Phi_{\iota_0^*(\theta)} = \Phi_{\theta_0}.$$

□

In view of the above, we will omit the connection from the notation and simply write Φ_P.

Corollary 4.1.10. *If there are a morphism $f : N \to M$ between two principal bundles $\pi : P \to N$ and $\pi' : P' \to M$ then $f^* \circ \Phi_{P'}(\alpha) = \Phi_P(\alpha)$ for every $\alpha \in S(\mathfrak{g}^*)^G$, in other words the following diagram commutes:*

$$\begin{tikzcd}
S(\mathfrak{g}^*)^G \arrow[r, "\Phi_{P'}"] \arrow[dr, "\Phi_P"'] & \Omega(P') \arrow[d, "f^*"] \\
 & \Omega(P)
\end{tikzcd}$$

Definition 4.1.11. *Let V be a finite dimensional vector space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) then the space of polynomials on V is defined by $P(V)$=\{ f is a function from V to $\mathbb{K}$ /given a base $\{e_1, ..., e_n\}$ there exist a polynomial $p(t_1, ..., t_n)$ such that $f(t_1e_1 + \cdots + t_ne_n) = p(t_1, ..., t_n)$ \}.*

Definition 4.1.12. *Let V a vector space with an action of group G. A G-invariant polynomial is $f \in P(V)$ such that for every $g \in G$ and $v \in V$, we have $(g \cdot f)(v) = f(g^{-1}(v))$.*

Proposition 4.1.13. *If V is a vector space with an action of G. Then the symmetric algebra on the dual of V which is invariant under the action of group G, denoted as $S(V^*)^G$, corresponds to the G-invariant polynomials on V.*

Proof. Consider the function ξ.

$$\begin{array}{rccl} \xi : & S(V^*) & \longrightarrow & fun(V) \\ & \alpha_1 \odot \cdots \odot \alpha_k & \mapsto & \xi(\alpha_1 \odot \cdots \odot \alpha_k)(v) = \alpha_1(v) \cdots \alpha_k(v). \end{array}$$

Let $\{e_1, \ldots, e_n\}$ be a base to the vector space V, then

$$\xi(\alpha_1 \odot \cdots \odot \alpha_k)(v) = \xi(\alpha_1 \odot \cdots \odot \alpha_k)\left(\sum_i t_i e_i\right) = \sum_i t_i^k \alpha_1(e_i) \cdots \alpha_k(e_i)$$

where $\sum_i t_i^k \alpha_1(e_i) \cdots \alpha_k(e_i)$ is a polynomial, then $\xi(S(V^*)) \subseteq P(V)$.

Conversely, let $t_1^{p_1} \cdots t_n^{p_n}$ be a monomial, it defines the function $f(t_1e_1 + \cdots + t_ne_n) = t_1^{p_1} \cdots t_n^{p_n}$. The we define $\beta := \underbrace{\beta_1 \odot \cdots \odot \beta_1}_{p_1-copies} \odot \cdots \odot \underbrace{\beta_n \odot \cdots \odot \beta_n}_{p_n-copies}$ where $\{\beta_1, \ldots, \beta_n\}$ is the dual basis for the basis $\{e_1, \ldots, e_n\}$, then $\xi(\beta) = f$.

Therefore the image of ξ is $P(V)$.

Also, we will prove that ξ is a equivariant function.

$$\begin{aligned} \xi(g \cdot (\alpha_1 \odot \cdots \odot \alpha_k))(v) &= \xi(\alpha_1 \circ g^{-1} \odot \cdots \odot \alpha_k \circ g^{-1})(v) \\ &= \alpha_1 \circ g^{-1}(v) \cdots \alpha_k \circ g^{-1}(v) \\ &= \xi(\alpha_1 \odot \cdots \odot \alpha_k)(g^{-1}(v)) \\ &= g \cdot (\xi(\alpha_1 \odot \cdots \odot \alpha_k))(v). \end{aligned}$$

The injectivity of the function is easy to verify, then $S(V^*)^G \cong P(V)^G$. □

In our case $V = \mathfrak{g}$, we can identify $S(\mathfrak{g}^*)^G$ with $P(\mathfrak{g})^G$ then we can write the Chern-Weil homomorphism as $\Phi : P(\mathfrak{g})^G \to H^*(M)$.

Definition 4.1.14. *Let $\mathfrak{so}(2n)$ be the Lie algebra of skew symmetric matrices. The Pfaffian polynomial:*

$$\mathsf{Pf} : \mathfrak{so}(2n) \to \mathbb{R},$$

is defined by the formula:

$$\mathsf{Pf}(A) = \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} sg(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

The Pfaffian polinomial has the following properties:

For $B \in \mathsf{End}(\mathbb{R}^{2n})$ and $A \in \mathfrak{so}(2n)$,

(a)
$$\mathsf{Pf}(BAB^t) = \det(B)\mathsf{Pf}(A).$$

(b)
$$\mathsf{Pf}(A)^2 = \det(A).$$

In particular, if $B \in SO(n)$ then:

$$\mathsf{Pf}(BAB^{-1}) = \mathsf{Pf}(A).$$

The previous properties show that $\mathsf{Pf} \in P(\mathfrak{so}(2n))^{SO(2n)}$.

Definition 4.1.15. *The Euler class $e(P)$ is defined as the image of Pfaffian polynomial under the Chern-Weil homomorphism.*

Example 4.1.16. The Chern class
Let $\pi : P \to M$ be a principal bundle with structutal group $U(n)$.For every $A \in \mathfrak{u}(n)$ the characteristic polynomial $char(A) = det(tI - \frac{1}{2\pi i}A) = c_1t^0 + c_2t^1 + \cdots + c_1t^{n-1} + t^n$ is invariant under the action conjugation of group G, then the coefficients of polynomial also are invariant.
The polynomial $c_1t^0 + c_2t^1 + \cdots + c_1t^{n-1} + t^n$ is called the Chern polynomial and the i-th class of Chern c_i is the i-th coefficient of Chern polynomial.

Definition 4.1.17. *A polynomial $p(x_1, \dots, x_n)$ is called symmetric if for every $\sigma \in S_n$, $p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = p(x_1, \dots, x_n)$. The elementary symmetric polynomials, written $e_k(x_1, \dots, x_n)$ for $k = 0, 1, \dots, n$ are defined by:*

$$\begin{aligned} e_k(x_1, \dots, x_n) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} && \text{for} \quad k \ge 0 \\ e_k(x_1, \dots, x_n) &= 0 && \text{for} \quad k > n. \end{aligned}$$

Any symmetric polynomial can be expressed in terms of elementary symmetric polynomials. Also, $char(A) = (t - \lambda_1) \cdots (t - \lambda_n) = t^n + \sum_{i=1}^{n} (-1)^i e_i(\lambda_1, \dots, \lambda_n) t^{n-i}$.
Since that $char(A) = c_1t^0 + c_2t^1 + \cdots + t^n$ therefore $c_{i+1} = (-1)^{n-i} e_{n-i}(\lambda_1, \dots, \lambda_n)$

The Chern classes satisfy the following properties:

There exists an algebra isomorphism between the symmetric polynomials with complex coefficients and $S(\mathfrak{g}^*)^G$, for $G = U(n)$.

Theorem 4.1.18. *If $\pi : \tau \to \mathbb{C}P^1$ is a line principal $U(n)$-bundle then $\displaystyle\int_{\mathbb{C}P^1} c_1(\tau) = -1$.*

Proof. By the Chern-Weil homomorphism the first Chern class is given as $c_1(\tau) = \frac{i}{2\pi}[\Omega] \in H^2(M, \mathbb{C})$, where Ω is the curvature form. therefore $c_1(\tau) = \dfrac{-i}{2\pi} \dfrac{dw \wedge d\bar{w}}{(1 + w\bar{w})^2}$.
If $w = x + iy$ and $\bar{w} = x - iy$ then

$$\int_{\mathbb{C}P^1} \frac{dw \wedge d\bar{w}}{(1 + w\bar{w})^2} = \int_{\mathbb{C}^2} \frac{-2idx \wedge dy}{(1 + x^2 + y^2)^2}.$$

Using polar coordinates:

$$\int_{\mathbb{C}^2} \frac{-2idx \wedge dy}{(1+x^2+y^2)^2} = -2i \int_0^{2\pi} \int_0^{\infty} \frac{r}{(1+r^2)^2} drd\theta = -2\pi i.$$

Therefore

$$\int_{\mathbb{C}P^1} c_1(\tau) = \frac{-i}{2\pi} \int_{\mathbb{C}P^1} \frac{dw \wedge d\bar{w}}{(1+w\bar{w})^2} = \left(\frac{-i}{2\pi}\right)(-2\pi i) = -1.$$

□

4.2 Reduction of principal bundles

Definition 4.2.1. *Let $\pi : P \to M$ be a G-principal bundle and a subgroup H of G. A reduction of the structural group for P is a H-principal bundle $\tilde{\pi} : P' \to M$ together with a smooth function $\iota : P' \to P$ which is H-equivariant and the following diagram commutes:*

$$\begin{tikzcd}
P' \arrow[rr, "\iota"] \arrow[dr, "\tilde{\pi}"'] & & P \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

Lemma 4.2.2. *If an H-principal bundle $\tilde{\pi} : P' \to M$ together with a function $\iota : P' \to P$ is a reduction of group of the G-principal bundle $\pi : P \to M$. Then ι is a embedding.*

Proof. Let θ be a connection on P, $q \in P'$ and $p = \iota(q)$. We will show that $\mathsf{dim}(D\iota(q)) = \mathsf{dim}(P') = \mathsf{dim}(H) + \mathsf{dim}(M)$. Denote $a = \mathsf{dim}(H)$ and $b = \mathsf{dim}(M)$. Since ι is equivariant then for every $v \in \mathfrak{h}$, $D\iota(v^*) = v^*$because:

$$D\iota(q)(v^*(q)) = D\iota(q)\left(\frac{d}{dt}\bigg|_{t=0} q \cdot exp(tv)\right) = \frac{d}{dt}\bigg|_{t=0} \iota(q{\cdot}exp(tv)) = \frac{d}{dt}\bigg|_{t=0} \iota(q){\cdot}exp(tv) = v^*(\iota(q)).$$

If $v_1, ..., v_a$ is a base of $\mathfrak{h}$ then $D\iota(v_j\iota^*(q)) = v_j^*(p)$ and $v_1, ..., v_a$ are linearly independent.

Now, fix a base $\beta_1, ..., \beta_b$ of $T_{\tilde{\pi}(q)}M$ then there exists vectors $w_1, ..., w_b$ such that $D\tilde{\pi}(w_j) = \beta_j$. Consider the vectors $z_1, ..., z_b$ such that $z_j \in D\iota(q)(w_j)$ then $z_1, ..., z_b$ are linearly independent because $D\pi(z_j) = \beta_j$.

Therefore $\{z_1, ..., z_b, v_1, ..., v_a\}$ are linearly independent. □

Lemma 4.2.3. *Let $\tilde{\pi} : P' \to M$ be an H-principal bundle together with a function $\iota : P' \to P$ a reduction of group to a G-principal bundle $\pi : P \to M$. If θ is a connection on P then $\iota^*(\theta)$ is a connection on P'.*

Proof. Let $T = Ker(\theta)$, we will define a distribution $\tilde{T}$ on P. For a fixed point $p \in P$ there exists $g \in G$ such that $p = \iota(x) \cdot g^{-1}$.
Define

$$\tilde{T}_p = D(R_{g^{-1}}) \circ D\iota(T_x).$$

$\tilde{T}_p$ is well defined because if g' is other choice then $p \cdot g = \iota(x)$ and $p \cdot g' = \iota(xh)$.

$$\begin{aligned}\tilde{T}_p &= D(R_{(g')^{-1}}) \circ D\iota(T_{xh}) = D(R_{(g')^{-1}}) \circ D\iota \circ D(R_h)(T_x) \\ &= D(R_{(g')^{-1}}) \circ D(R_h) \circ D\iota(T_x) = D(R_{h(g')^{-1}}) \circ D\iota(T_x).\end{aligned}$$

Then it is enough show that $g^{-1} = h(g')^{-1}$, but remember that $pg = \iota(x)$ and $pg' = \iota(xh)$ then $pg'h^{-1} = \iota(x) = pg$.

It is easily verified that $\tilde{T}_p \oplus Ker(d\pi(p)) = T_pP$. Then we will show that $\iota^*(\theta) = \tilde{\theta}$.

For $x \in P'$ we consider the cases:

- If v is horizontal, $\tilde{\theta}(x)(v) = 0$ and $\iota^*(\theta)(v) = \theta(D\iota(v)) = 0$ because $D\iota(v) \in \tilde{T}_{\iota(x)} = Ker(\theta)$.
- If $v = w^*$ for $w \in \mathfrak{h}$, $\tilde{\theta}(x)(w^*) = w$ and $\iota^*(\theta)(x)(w^*) = \theta(D\iota(w^*)) = \theta(w^*) = w$.

□

Theorem 4.2.4. *If an H-principal bundle $\tilde{\pi} : P' \to M$ together with a function $\iota : P' \to P$ is a reduction of group to a G-principal bundle $\pi : P \to M$ then the following diagram commute:*

$$\begin{tikzcd} S(\mathfrak{g}^*)^G \arrow[r, "\Phi_P"] \arrow[d, "\Phi_P"'] & H^*(M) \\ S(\mathfrak{h}^*)^H \arrow[ur, "\Phi_{P'}"'] & \end{tikzcd}$$

Proof. Let θ be a connection on P' then there exist a unique connection $\tilde{\theta}$ on P such that $\iota^*(\theta) = \tilde{\theta}$ and $\iota^*(F_\theta) = F_{\tilde{\theta}}$.

We will check the following diagram commute:

$$\begin{tikzcd} S(\mathfrak{g}^*) \arrow[r, "\Phi_\theta"] \arrow[d, "\iota^*"'] & \Omega(P) \arrow[d, "\iota^*"] \\ S(\mathfrak{h}^*) \arrow[r, "\Phi_{\tilde{\theta}}"'] & \Omega(P') \end{tikzcd}$$

For $\alpha \in \mathfrak{g}^*$:

$$\Phi_{\tilde{\theta}} \circ \iota^*(\alpha) = \iota^*(\alpha) \circ F_{\tilde{\theta}} = \iota^*(\alpha) \circ \iota^*(F_\theta) = \iota^*(\alpha \circ F_\theta) = \iota^* \circ \Phi_\theta(\alpha).$$

Now since $\iota^*(S(\mathfrak{g}^*)^G) \subset S(\mathfrak{h}^*)^H$ and $\iota : P' \to P$ is a embedding we obtain the result.

□

4.3 The product bundle

Suppose that $\pi_1 : P_1 \to M$ and $\pi_2 : P_2 \to M$ are principal bundles with structural group G_1 and G_2 respectively then it makes sense to consider the product of bundles:

$$P_1 \times_M P_2 = \{(a,b) \in P \times_M \tilde{P} | \pi_1(a) = \pi_2(b)\}$$

In other words the following diagram commutes:

$$\begin{tikzcd}
P_1 \times_M P_2 \arrow[r] \arrow[d] & P_2 \arrow[d, "\pi_2"] \\
P_1 \arrow[r, "\pi_1"'] & M
\end{tikzcd}$$

The manifold $P_1 \times P_2$ has the free action of $G_1 \times G_2$ given by $(a,b) \cdot (g_1, g_2) = (ag_1, bg_2)$. Therefore the submersion:

$$\begin{array}{rccc} \pi : & P_1 \times_M P_2 & \longrightarrow & M \\ & (a,b) & \mapsto & \pi_1(a) = \pi_2(b) \end{array}$$

define a principal bundle over M with structural group $G_1 \times G_2$.

Lemma 4.3.1. *Let $\pi : P_1 \times_M P_2 \to M$ be a product bundle with structural group $G_1 \times G_2$, so that provides of the principal bundles P_1, P_2 over M.If θ_1 and θ_2 are connections of P_1 and P_2 respectively, then $\theta = \pi_1^*(\theta_1) + \pi_2^*(\theta_2)$ is a connection on $P_1 \times_M P_2$.*

$$\begin{tikzcd}
P_1 \times_M P_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P_2 \arrow[d] \\
P_1 \arrow[r] & M
\end{tikzcd}$$

Proof. First we prove that θ is an equivariant form. Since $\theta = \pi_1^*(\theta_1) + \pi_2^*(\theta_2)$ it is enough to prove that $\pi_1^*(\theta_1)$ is equivariant.
Proving that $\pi_1^*(\theta_1)$ is equivariant with respect to G_2 is equivalent to proving that the following diagram commutes:

$$\begin{tikzcd}
T_{(a,b)}(P_1 \times_M P_2) \arrow[r, "\pi_1^*(\theta_1)"] \arrow[d, "DR_g"'] & \mathfrak{g}_1 \oplus \mathfrak{g}_2 \arrow[d, "Ad(g^{-1})"] \\
T_{(a,bg)}(P_1 \times_M P_2) \arrow[r, "\pi_1^*(\theta_1)"'] & \mathfrak{g}_1 \oplus \mathfrak{g}_2
\end{tikzcd}$$

Note that $T_{(a,b)}(P_1 \times_M P_2) = \{(v,w) \in T_aP_1 \oplus T_bP_2 | D\pi(v) = D\pi(w)\}$. Then

$$\begin{array}{rccc} D\pi_1 : & T_{(a,b)}(P_1 \times_M P_2) & \longrightarrow & T_aP_1 \\ & (v,w) & \mapsto & v \end{array} \qquad \begin{array}{rccc} D\pi_2 : & T_{(a,b)}(P_1 \times_M P_2) & \longrightarrow & T_bP_2 \\ & (v,w) & \mapsto & w. \end{array}$$

We compute:

$$Ad(g^{-1}) \circ \pi_1^*(\theta_1)(v,w) = Ad(g^{-1})(\theta_1(v))$$

And

$$\pi_1^*(\theta_1)(DR_g)(v,w) = \pi_1^*(\theta_1)(v, DR_g w) = \theta_1(v) = Ad(g^{-1})(\theta_1(v))$$

which implies that $\pi_1^*(\theta_1)$ is equivariant with respect to G_2.

Now we will show that $\pi_1^*(\theta_1)$ is equivariant with respect to G_1.

$$Ad(g^{-1})(\pi_1^*(\theta_1))(v,w) = Ad(g^{-1})(\pi_1^*(\theta_1(v))$$

Also

$$\pi_1^*(\theta_1)(DR_g)(v,w) = \pi_1^*(\theta_1)(DR_g v, w) = \theta_1(DR_g v) = Ad(g^{-1})(\theta_1(v)).$$

In the other hand, we will show that for $v \in \mathfrak{g}_1 \oplus \mathfrak{g}_2$ obtains $\theta(v^*) = v$.

One can compute:

$$\pi_1^*(\theta_1)(v^*) = \theta_1(D\pi_1(v^*)) = \theta_1(v^*) = v \qquad \pi_2^*(\theta_2)(v^*) = \theta_2(D\pi_2(v^*)) = 0.$$

This calculation implies that if $v \in \mathfrak{g}_1$ then $\pi_1^*(\theta_1)(v^*) = v$ and $\pi_2^*(\theta_2)(v^*) = 0$. ☐

Proposition 4.3.2. *If P_1 and P_2 are principal bundles over M then:*

$$\begin{tikzcd}
S((\mathfrak{g}_1 \oplus \mathfrak{g}_2)^*)^{G_1 \times G_2} \arrow[r, "\Phi_{P_1 \times_M P_2}"] & H(M) \\
S(\mathfrak{g}_1^*)^{G_1} \otimes S(\mathfrak{g}_2^*)^{G_2} \arrow[u, "\cong_\varphi"'] \arrow[r, "\Phi_{P_1} \otimes \Phi_{P_2}"'] & H(M) \otimes H(M) \arrow[u, "f"]
\end{tikzcd}$$

where the function f is defined by $f(\omega_1, \omega_2) = \omega_1 \wedge \omega_2$

Proof. Since the morphisms are algebra morphisms it is enough prove $\Phi_{P_1 \times_M P_2}(\varphi)(\alpha) = f \circ (\Phi_{P_1} \otimes \Phi_{P_2})(\alpha)$ for $\alpha = 1 \otimes \beta \in S(\mathfrak{g}_1^*) \otimes S(\mathfrak{g}_2^*)$ and $\beta \in \mathfrak{g}_2^*$.

Consider $\theta = \pi_1^*(\theta_1) + \pi_2^*(\theta_2)$ a connection on $P_1 \times_M P_2$ and compute:

$$\Phi_{P_1 \times_M P_2}(\varphi)(\alpha) = \Phi_{P_1 \times_M P_2}(\beta) = \beta \circ F_\theta = \beta \circ \pi_1^*(F_{\theta_1}) + \beta \circ \pi_2^*(F_{\theta_2}) = 0 + \beta \circ \pi_2^*(F_{\theta_2})) = \beta(F_{\theta_2})$$

And

$$f \circ (\Phi_{P_1} \otimes \Phi_{P_2})(1 \otimes \beta) = f(1 \otimes \Phi_{P_2})(\beta) = \Phi_{P_2}(\beta) = \beta(F_{\theta_2}).$$

☐

Theorem 4.3.3. *Let $\pi_1 : E \to M$ and $\pi_2 : F \to M$ be principal bundles with structural group $U(n)$. Then $c(E \oplus F) = c(E)c(F)$.*

Proof. Consider the Chern class to calculate $c_i(E \oplus F)$ then by Proposition 4.3.2:

$$\begin{tikzcd}
S((\mathfrak{g}l(k+r)^*)^{GL(k+r)} \arrow[r, "\Phi_{Fr(E \oplus F)}"] & H(M) \\
S(\mathfrak{g}l(k)^*)^{GL(k)} \otimes S(\mathfrak{g}l(r)^*)^{GL(r)} \arrow[u] \arrow[r, "\Phi_{Fr(E)} \otimes \Phi_{Fr(F)}"'] & H(M) \otimes H(M) \arrow[u, "f"]
\end{tikzcd}$$

Then

$$c_k(E \oplus F) = f \circ (\Phi_{Fr(E)} \otimes \Phi_{Fr(F)})(\sum_{i+j=k} y_i x_j) = f(\sum_{i+j=k} c_i(E) \otimes c_j(F)) = \sum_{i+j=k} c_i(E)c_j(F).$$

We can conclude that if $c(E) = 1 + c_1(E) + c_2(E) + \cdots$ then $c(E \oplus F) = c(E)c(F)$. □

4.4 Chern-Gauss-Bonnet theorem

The Gauss-Bonnet theorem provides a formula for the Euler characteristic of a closed oriented surface Σ:

$$\chi(\Sigma) = \frac{1}{2\pi} \int_\sigma K dA.$$

Here K denotes the Gaussian curvature of Σ associated to a riemannian metric and dA is the volume form determined by the metric and the orientation. This formula is remarkable because, while the left hand side evidently depends only on the topology of Σ, the right hand side is a priori a geometric quantity. The extension of this formula to higher dimensions had to wait almost 200 years until the language of differential geometry was developed and Chern [5] proved his generalised version of the Gauss-Bonnet theorem.

Theorem (Chern-Gauss-Bonnet)**.** Let (M, g) be a closed oriented Riemannian manifold of dimension $d = 2n$ and let Ω be the curvature form of Levi Civita Connection, i.e Ω is an $\mathfrak{so}(2n)$ valued 2 form that comes from a connection in $Fr^{SO}(TM)$. Then:

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M \mathsf{Pf}(\Omega).$$

Remember that to a oriented compact manifold M the Euler characteristic is defined by the formula:

$$\chi(M) = \sum_i (-1)^i \mathsf{dim}(H^i(M)).$$

Definition 4.4.1. *The set of differential forms with compact support is defined as* $\Omega_c(M) = \{\omega \in \Omega(M) / \quad supp(\omega) \quad \text{is compact}\}$.

Definition 4.4.2. *Let* $\pi : E \to M$ *a vector bundle. The algebra of differential forms with compact vertical support is* $\Omega_{cv}(E) = \{\omega \in \Omega(E)/\omega\big|_{E_p} \text{has compact support, for each} p \in M\}$. *then* $H_{cv}(E) := H^*(\Omega_{cv}(E))$.

Lemma 4.4.3. *Let* $\pi : E \to M$ *be an oriented vector bundle of rank* k. *Then there exits a function* $\pi_* : \Omega^l_{cv}(E) \to \Omega^{l-k}(M)$ *defined in the following way:*

We fix a metric g *on* E *which induces a isomorphism.*

$$T_e E \cong E_{\pi(e)} \oplus T_{\pi(e)} M$$

We define

$$\pi_*(\omega)(p)(X_1, \ldots, X_{l-k}) = \int_{E_p} i_{\tilde{X}_{l-k}} \cdots i_{\tilde{X}_1}(\omega)$$

where $X_1, \ldots, X_{l-k} \in T_pM$ *and* $\tilde{X}_i \in \Gamma(E_p, TE)$ *so that* $\tilde{X}_i(e) = X_i$.

Theorem 4.4.4 (Thom isomorphism)**.** *Let* $\pi : E \to M$ *be a oriented vector bundle of rank* k *over a compact manifold* M. *Then the homomorphism* $\pi_* : H^l_{cv}(E) \to H^{l-k}(M)$ *is a isomorphism.*

Proof. The proof of this theorem is in the book "Differential Forms in Algebraic Topology" by Bott, R.; Tu, L [2]. □

Definition 4.4.5. *Let* $\pi : E \to M$ *be a oriented vector bundle of rank* k. *The Thom class is cohomology class* $\Psi(E) := (\pi_*)^{-1}(1) \in H^k_{cv}(E)$. *This class is characterized by the property:*

$$\int_{E_p} \iota_p^* \Psi(E) = 1.$$

where ι_p *is the inclusion in the fiber.*

Definition 4.4.6. *Let* $\pi : E \to M$ *be a vector bundle over a compact manifold* M *with. We define the topological Euler class as* $\tilde{e}(E) = s^*(\Psi(E))$, *where* s *is zero section.*

The next goal is prove that definition of topological Euler class agrees with the previous definition of the Euler class except for a factor.

Theorem 4.4.7 (Poincaré duality)**.** *Let* M *be a oriented closed manifold. The bilinear form:*

$$\begin{array}{rrcl} \int : & H^*(M) \otimes H^*(M) & \to & \mathbb{R} \\ & ([\alpha] \otimes [\beta]) & \mapsto & \int_M \alpha \wedge \beta \end{array}$$

is nondegenerate i.e the induced function $\psi : H^i(M) \to (H^{n-i}(M))^\vee$ *is a isomorphism. Here* $(H^{n-i}(M))^\vee$ *denote the dual of* $(H^{n-i}(M))$.

Proof. The proof of this theorem can be found in [2]. □

Definition 4.4.8. *If* $S \subseteq M$ *is a oriented closed submanifold of dimension* k. *Then we can define the functional:*

$$\begin{array}{rrcl} f : & H^k(M) & \to & \mathbb{R} \\ & [\alpha] & \mapsto & \int_S \alpha. \end{array}$$

Then $f \in (H^k(M))^\vee$ *and since* $(H^k(M))^\vee \cong H^{m-k}(M)$, *where* m *is dimension of* M. *The dual Poincaré of* S *is cohomology class* $[f] \in H^{m-k}(M)$.

Therefore the Poincaré dual of $S \subseteq M$ *is cohomology class* $\beta \in H^{m-k}(M)$ *so that*

$$\int_S \alpha = \int_M \alpha \wedge \beta.$$

Lemma 4.4.9. *Let $\pi : E \to M$ be an oriented real vector bundle over a compact manifold M. If $s \in \Gamma(E)$ such that $im(s) \pitchfork im(s_0)$, where s_0 is the zero section. Then the Poincaré dual of the submanifold $Z_s = \{p \in M | s(p) = 0\} \cong im(s) \pitchfork im(s_0)$ is the euler class $\tilde{e}(E)$.*

Proof. Let $\eta = s_0^*(\Psi) = \tilde{e}(E)$ and $\alpha \in H(M)$ then

$$\int_{Z_s} \alpha = \int_{Z_s} \pi^*(\alpha) = \int_E \pi^*(\alpha) \wedge \eta_{Z_s}$$

where η_{Z_s} is the Poincaré dual of E. Since $im(s) \pitchfork im(s_0)$ then $\eta_{Z_s} = \eta_{im(s)} \wedge \eta_{im(s_0)} = \eta_{im(s_0)} \wedge \eta_{im(s_0)}$.

Furthermore $\pi^*(s_0^*(\Psi)) = \Psi + d\xi$ then

$$\begin{aligned}\int_{Z_s} \alpha &= \int_E \pi^*(\alpha) \wedge \eta_{Z_s} \\ &= \int_E \pi^*(\alpha) \wedge \pi^* s_0^*(\Psi) \wedge \Psi + \int_E \pi^*(\alpha) \wedge d\xi\Psi \\ &= \int_M \pi_*(\pi^*(\alpha \wedge s_0^*(\Psi)) \wedge \Psi) = \int_M \alpha \wedge \eta \wedge \pi_*(\Psi) = \int_M \alpha \wedge \eta.\end{aligned}$$

☐

Theorem 4.4.10 (Classification theorem of vector bundles)**.** *If $\pi : E \to M$ is a complex vector bundle of rank 1 over a compact manifold M, then there exists an integer n and function $f : M \to \mathbb{C}P^n$ such that $E \cong f^*(\tau)$, where τ represents the tautological line bundle.*

Proof. Let $U_1, ..., U_n$ be the cover of local trivialization and sections $\alpha_i \in \Gamma(E, U_i)$ such that $\alpha_i(p) \neq 0$ for every $p \in U_i$.
If $\{\rho_i\}_{i \in I}$ is a partition of unity subordinate to the open cover then for each $1 \leq i \leq n$ there is a section $\beta_i \in \Gamma(E)$ defined by

$$\beta_i(p) = \begin{cases} 0 & \text{si } p \notin U_i \\ \rho_i(p)\alpha_i(p) & \text{si } p \in U_i \end{cases} \tag{4.1}$$

therefore the following function is surjective

$$\begin{array}{rccl} \lambda : & M \times \mathbb{C}^n & \to & E \\ & (p, x_1, ..., x_n) & \mapsto & x_1\beta_1(p) + \cdots + x_n\beta_n(p). \end{array}$$

Define the function $f : M \to \mathbb{C}P^{n-1}$ by $f(p) = E_p$ then $f^*(\tau) \cong E$ because the inclusion $\iota : E \to M \times \mathbb{C}^n$ induce a function that preserve the fibers. ☐

Remark 4.4.11. A complex vector bundle of rank 1 can be thought as an oriented real vector bundle of rank 2. Every fiber has structure of complex of dimension 1, if we chosen a complex basis $\{a\}$, then we define the orientation to the underlying real vector space of dimension 2 choosing the basis $\{a, ia\}$. Remember that $GL(n, \mathbb{C})$ is a connected group, then every complex basis can be continuously connected by a path on $GL(n, \mathbb{C})$, therefore this is well defined.

Proposition 4.4.12. *Let S be oriented submanifold of a oriented compact manifold M. If the dimension of S is zero then*

$$\int_{\mathbb{C}P^1} \eta_S = \sum_{i=1}^{n} sign(p_i)$$

where η_S is the Poincaré dual of S and $S = \{p_1, \ldots, p_n\}$.

Theorem 4.4.13. *Let $\pi : E \to M$ be an oriented bundle of rank k over a oriented compact manifold of dimension k. If s is a section of E with a finite number of zeros. Then the Euler class of E is Poincaré dual to the zeros of s.*

Proof. The prove of this theorem is in the book "Differential forms in algebraic topology" by Bott and Tu [2]. ☐

Lemma 4.4.14. *Let $\pi : \tau \to \mathbb{C}P^1$ be the complex tautological line bundle, then $\tilde{e}(\tau) = c_1(\tau)$.*

Proof. Consider $\pi : \tau^* \to \mathbb{C}P^1$ the complex dual tautological bundle. Every linear function $\rho : \mathbb{C}^2 \to \mathbb{C}$ defines a section $\tilde{\rho} \in \Gamma(\tau^*)$ of the form $\tilde{\rho}[z_0, z_1] = \rho\big|_{[z_0,z_1]}$.The linear function $p : \mathbb{C}^2 \to \mathbb{C}$, $p(x_0, x_1) = x_0$ induces a section $\bar{p} \in \Gamma(\tau^*)$, where the zeros of $\tilde{p}$ is the set $\{[0, 1]\}$. Therefore by previous theorem 4.4.13

$$\int_{\mathbb{C}P^1} \tilde{e}(\tau^*) = 1.$$

Remember that by the Poincaré duality there exists the isomorphism $H^2(\tau) \to \mathbb{R}$, $\eta \mapsto \int_{\mathbb{C}P^1} \eta$. Then to prove $\tilde{e}(\tau) = c_1(\tau)$ is enough prove that

$$\int_{\mathbb{C}P^1} \tilde{e}(\tau) = \int_{\mathbb{C}P^1} c_1(\tau).$$

We have seen that $\int_{\mathbb{C}P^1} c_1(\tau) = -1$ then $c_1(\tau) = -\tilde{e}(\tau^*) = \tilde{e}(\tau)$. ☐

Proposition 4.4.15. *Let $\pi : E \to M$ be a complex vector bundle of rank 1 over a compact manifold M, then $c_1(E) = \tilde{e}(E)$.*

Proof. By the classification theorem of vector bundles 4.4.10 there is a function $f : M \to \mathbb{C}P^n$ such that $E \cong f^*(\tau)$. we have seen that $\tilde{e}(\tau) = c_1(\tau)$, then by the naturality of classes

$$\tilde{e}(E) = \tilde{e}(f^*(\tau)) = f^*(\tilde{e}(\tau)) = f^*(c_1(\tau)) = c_1(f^*(\tau)) = c_1(E).$$

☐

Theorem 4.4.16 (Leray-Hirsch theorem)**.** *Let $\pi : P \to M$ be a fiber bundle so that is locally trivial. If there exists a finite set of elements $\beta_i \in H^*(P)$, for $1 \leq i \leq n$, such that for each point $p \in M$ the restriction of the each β_i form a basis to cohomology of the fiber. Then $H^*(P)$ is a free modulo over $H^*(M)$ with generators $\beta_1, \ldots, \beta_n$.*

Proof. The prove of this theorem is in the book of Algebraic Topology, Hatcher [17]. ☐

Theorem 4.4.17 (Real Splitting principal). *For any oriented real vector bundle E over M there exits a manifold X and smooth proper map $f : X \to M$ such that*

1. $f^* : H^*(M) \to H^*(X)$ *is injective.*
2. *If* $\mathsf{dim}E = 2n$ *then* $f^*(E) = L_1 \oplus \cdots \oplus L_n$ *and if* $\mathsf{dim}(E) = 2n + 1$ *then* $f^*(E) = L_1 \oplus \cdots \oplus L_n \oplus \varepsilon^1$*; Where* $L_1, \ldots, L_n$ *are vector bundles of rank* 2 *and* ε^1 *is the trivial line bundle.*

Proof. The prove of this theorem is in the book "From calculus to cohomology: De Rham cohomology and characteristic classes" by Madsen and Tornehave [20]. ☐

Definition 4.4.18. *Let $S \subseteq M$ be a submanifold of M , the normal bundle of S is the vector bundle $N(S) = \iota^*(TM)/TS$ such that $\iota : S \to M$ is the immersion map.*

Let $S \subseteq M$ be a submanifold closed and oriented of M such that $\iota : S \to M$ is a immersion, then there exists a open U of M and a diffeomorphism $\varphi : N(S) \to U$ such that U contain to S and the following diagram commutes:

$$\begin{tikzcd}
N(S) \arrow[rr, "\varphi"] & & U \\
& S. \arrow[ul, "\iota_0"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

where ι_0 denotes the zero section.
The open U is called a tubular neighborhood of S.

Lemma 4.4.19. *Let M be a oriented compact manifold. If $\{\alpha_i\}$ is a basis of the algebra $H^*(M)$ and $\{\beta_j\}$ is the dual basis respect to Poincaré duality $\left(i.e \int_M \alpha_i \wedge \beta_j = \delta_{ij}\right)$. Then the Poincaré dual of the diagonal of M is given by*

$$\eta_\triangle = \sum_i (-1)^{|\alpha_i|} \pi_1^* \alpha_i \wedge \pi_2^* \beta_i.$$

Proof. The prove of this lemma is in the book "Differential forms in algebraic topology" by Boot and Tu [2]. ☐

Lemma 4.4.20. *If M is a oriented compact manifold then*

$$\int_{M \times M} \eta_\triangle \times \eta_\triangle = \chi(M).$$

Proof.

$$\begin{aligned}\int_{\triangle\times\triangle}\eta_\triangle\times\eta_\triangle=\int_\triangle\eta_\triangle&=\sum_i{}_\triangle(-1)^{|\alpha_i}\pi_1^*(\alpha_i)\wedge\pi_2^*(\beta_i)\\&=\sum_i(-1)^{|\alpha_i}\int_M\iota^*\pi_1^*(\alpha_i)\wedge\iota^*\pi_2^*(\beta_i)\\&=\sum_i\int_M\alpha_i\wedge\beta_i\\&=\sum_l(-1)^{|\alpha_l|}=\sum_q\mathsf{dim}(H^q(M))=\chi(M).\end{aligned}$$

□

Theorem 4.4.21. *Is M is a oriented compact manifold then*

$$\int_M\tilde{e}(TM)=\chi(M).$$

Proof. Consider a tubular neighborhood of $\triangle$ then the following short exact sequence splits:

$$0\to T_q(\triangle)\to T_pM\oplus T_pM\to T_pM\to 0.$$

By the previous sequence $N(S)\cong TM$ then

$$\int_M\tilde{e}(TM)=\int_M\tilde{e}(N(\triangle))=\int_M\iota^*(\Psi(N(\triangle)))=\int_M\iota^*(\eta_\triangle)=\int_\triangle\iota^*(\eta_\triangle)=\int_{M\times M}\eta_\triangle\times\eta_\triangle=\chi(M).$$

□

Lemma 4.4.22. *For the tautological line bundle, $\tau_1:E(\tau_1)\to\mathbb{C}P^1$, $e(\tau_1)=(-2\pi)^1\tilde{e}(\tau_1)$.*

Proof. The proof of this lemma we will be divided in several steps:

Step 1 $\displaystyle\int_{\mathbb{C}P^1}e(\tau_1)=2\pi.$

Consider the principal bundle $\pi:Fr^U(\tau_1)\to\mathbb{C}P^1$ and note that $Fr^U(\tau_1)\cong St_1(\mathbb{C}^2)$. The Stiefel bundle has a natural connection θ given by $\theta(A)(v)=A^*v$, i.e. θ is a differential form with values in $\mathfrak{u}(1)\cong i\mathbb{R}$, the form can be extended to $\mathbb{C}^2$ by $\theta=\bar{z}_1dz_1+\bar{z}_2dz_2$, then its curvature $F=d\theta+\cancelto{0}{\frac{1}{2}[\theta,\theta]}=d\theta$.

We want calculate $\displaystyle\int_{\mathbb{C}P^1}e(Fr^U(\tau_1))=\int_{\mathbb{CP}^1}\alpha$ where $F=\pi^*(\alpha)$.
Even though $\pi:St_1(\mathbb{C}^2)\to\mathbb{C}P^1$ does not admit sections because it is a non trivial bundle, when the base is restricted to $\mathbb{C}$ it does admit sections.

$$\begin{tikzcd} & St_1(\mathbb{C}^2) \arrow[d, "\pi"] \\ C \arrow[ur, bend left, "\sigma"] \arrow[r, "\iota"'] & B \end{tikzcd}$$

Where the section σ is given by $\varphi(w) = \left(\frac{1}{\sqrt{1+w\bar{w}}}, \frac{w}{\sqrt{1+w\bar{w}}}\right)$.
Therefore if we denote $z_1 = \frac{1}{1+w\bar{w}}$ and $z_2 = \frac{w}{1+w\bar{w}}$ then

$$\begin{aligned}\int_{\mathbb{CP}^1} \alpha = \int_{\mathbb{C}^2} \sigma^*(F) = \int_{\mathbb{C}^2} \sigma^*(d\theta) = \int_{\mathbb{C}^2} d(\varphi^*(\bar{z}_1 dz_1 + \bar{z}_2 dz_2)) \\ = \int_{\mathbb{C}^2} d(\sigma^*(\bar{z}_1) d(\sigma^* z_1)) + d(\sigma^*(\bar{z}_2)\sigma^*(dz_2)) \\ = \int_{\mathbb{C}^2} d(\sigma^*(\bar{z}_2) \wedge \sigma^*(dz_2)) = \int_{\mathbb{C}^2} d\left(\frac{\bar{w}}{\sqrt{1+w\bar{w}}} \wedge d\left(\frac{w}{\sqrt{1+w\bar{w}}}\right)\right) \\ = \int_{\mathbb{C}^2} \frac{d\bar{w} \wedge dw}{(1+w\bar{w})^2} = 2\pi i.\end{aligned}$$

Now we will see $St_1(\mathbb{C}^2) \to \mathbb{C}P^1$ as a $SO(2)$-principal bundle. There is exists an isomorphism of Lie groups:

$$\begin{array}{ccc} SO(2) & \longrightarrow & \mathbb{S}^1 \\ \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} & \mapsto & cos\theta + isin\theta, \end{array}$$

on the Lie algebra level we have the isomorphism:

$$\begin{array}{ccc} \mathfrak{so}(2) & \longrightarrow & i\mathbb{R} \\ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} & \mapsto & ia, \end{array}$$

Since that $d\theta = \frac{1}{(1+w\bar{w})^2} dw \wedge d\bar{w} = \frac{2i}{(1+x^2+y^2)^2} dx \wedge dy$ is a $i\mathbb{R}$-valued 2-form then

$$d\theta = \begin{bmatrix} 0 & \frac{2}{(1+x^2+y^2)^2} dx \wedge dy \\ \frac{-2}{(1+x^2+y^2)^2} dx \wedge dy & 0 \end{bmatrix},$$

it is seen as a $\mathfrak{so}(2n)$-valued 2-form for this case $\mathsf{Pf}(d\theta) = \frac{2}{(1+x^2+y^2)^2} dx \wedge dy$.
Therefore

$$\int_{\mathbb{C}P^1} e(\tau_1) = \int_{\mathbb{R}^2} \frac{2}{(1+x^2+y^2)^2} dx \wedge dy = 2\pi.$$

Step 2 $T\mathbb{C}P^n \oplus \mathbb{C} \cong \underbrace{\tau_1^* \oplus \cdots \oplus \tau_1^*}_{n+1-copies}$, here τ_1^* denote the dual of tautological bundle.

Consider ω^n the orthogonal complement of the line tautological bundle, then $\tau_1 \oplus \omega^n$ is the

trivial complex bundle over $\mathbb{C}P^n$ of rank $n+1$.
Note that $Hom_{\mathbb{C}}(\tau_1, \omega^n)$ we can identified with $T\mathbb{C}P^n$ and $\mathbb{C} \cong Hom_{\mathbb{C}}(\tau_1, \tau_1)$ then

$$\begin{aligned}
T\mathbb{C}P^n &\cong Hom(\tau_1, \omega^n) \\
T\mathbb{C}P^n \oplus \mathbb{C} &\cong Hom(\tau_1, \omega^n) \oplus \mathbb{C} \cong Hom(\tau_1, \omega^n) \oplus Hom(\tau_1, \tau_1) \\
T\mathbb{C}P^n \oplus \mathbb{C} &\cong Hom(\tau_1, \omega^n \oplus \tau_1) \cong Hom(\tau_1, \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n+1-copies}) \\
T\mathbb{C}P^n \oplus \mathbb{C} &\cong Hom(\tau_1, \mathbb{C}) \oplus \cdots \oplus Hom(\tau_1, \mathbb{C}) \cong \underbrace{\tau_1^* \oplus \cdots \oplus \tau_1^*}_{n+1-copies}
\end{aligned}$$

Step 3 $\tilde{e}(T\mathbb{C}P^1) = -2\tilde{e}(\tau_1)$.

Using properties of Chern classes:
$c(T\mathbb{C}P^1 \oplus \mathbb{C}) = c(T\mathbb{C}P^1)$ then $c_1(T\mathbb{C}P^1) = \tilde{e}(T\mathbb{C}P^1)$ and $c(\tau_1^* \oplus \tau_1^*) = (1 - c_1(\tau_1))^2$ then $c_1(\tau_1^* \oplus \tau_1^*) = -2c_1(\tau_1) = -2\tilde{e}(\tau_1)$.

Step 4 Remember that by Poincaré duality $H^2(\tau_1) \to \mathbb{R}$, $\eta \mapsto \int_{\mathbb{C}P^1} \eta$ is an isomorphism then to prove that $(-2\pi)^1\tilde{e}(\tau_1) = e(\tau_1)$ it is enough to prove that:

$$\int_{\mathbb{C}P^1} \tilde{e}(\tau_1) = \left(\frac{-1}{2\pi}\right)^1 \int_{\mathbb{C}P^1} e(\tau_1).$$

We have seen $\int_{\mathbb{C}P^1} \tilde{e}(T\mathbb{C}P^1) = \chi(\mathbb{C}P^1) = \chi(\mathbb{S}^2) = 2$, by step 3 $\int_{\mathbb{C}P^1} \tilde{e}(T\mathbb{C}P^1) = -2\int_{\mathbb{C}P^1} \tilde{e}(\tau_1)$ then $\int_{\mathbb{C}P^1}(-2\pi)\tilde{e}(\tau_1) = 2\pi$.

□

Lemma 4.4.23. *The following properties hold:*

1. $e(\tau_n) = (-2\pi)\tilde{e}(\tau_n)$, *where* $\pi : \tau_n \to \mathbb{C}P^n$ *is the tautological bundle over* $\mathbb{C}P^n$.
2. e *and* $\tilde{e}$ *are exponential classes, i.e* $e(E \oplus F) = e(E) \wedge e(F)$ *and* $\tilde{e}(E \oplus F) = \tilde{e}(E) \wedge \tilde{e}(F)$.
3. $e(L) = (-2\pi)\tilde{e}(L)$ *for a complex line bundle* $\pi : L \to M$.

Proof. 1. Consider the inclusion map $\iota : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n$, then $\iota^* : H^2(\mathbb{C}P^n, \mathbb{R}) \to H^2(\mathbb{C}P^1, \mathbb{R})$ is an isomorphism and there is the following commutative diagram:

$$\begin{tikzcd}
\tau_1 \cong \iota^*(\tau_1) \arrow[r, "\sim"] \arrow[d] & \tau_n \arrow[d] \\
\mathbb{C}P^1 \arrow[r, "\iota"] & \mathbb{C}P^n
\end{tikzcd}$$

By the naturality of the classes and the previous lemma

$$\iota^*(e(\tau_n)) = e(\iota^*(\tau_n)) = e(\tau_1) = (-2\pi)\tilde{e}(\tau_1) = (-2\pi)\tilde{e}(\iota^*(\tau_n)) = \iota^*((-2\pi)\tilde{e}(\tau_n)).$$

Hence that $e(\tau_n) = (-2\pi)\tilde{e}(\tau_n)$.

2. Let E and F be a oriented vector bundles over M with Thom class Ψ_1 and Ψ_2 respectively, if we consider the projections $\pi_1 : E \oplus F \to E$ and $\pi_2 : E \oplus F \to F$ then $\Psi(E \oplus F) = \pi_1^*(\Psi_1) \wedge \pi_2^*(\Psi_2)$ because

$$\int_{(E\oplus F)_p} \pi_1^*(\Psi_1) \wedge \pi_2^*(\Psi_2) = \int_{E_p \oplus F_p} \pi_1^*(\Psi_1) \wedge \pi_2^*(\Psi_2) = \int_{E_p} \Psi_1 \int_{F_p} \Psi_2 = 1$$

Hence

$$\tilde{e}(E\oplus F) = \iota^*(\Psi(E\oplus F)) = \iota^*(\pi_1^*(\Psi_1)\wedge\pi_2^*(\Psi_2)) = \iota^*(\pi_1^*(\Psi_1))\wedge\iota^*(\pi_2^*(\Psi_2)) = \tilde{e}(E)\wedge\tilde{e}(F).$$

On the other hand, the following diagram is a group reduction :

$$\begin{tikzcd} Fr(E) \times_M Fr(F) \arrow[r, "\iota"] \arrow[d, "\pi"'] & Fr(E \oplus F) \arrow[d, "\pi"] \\ M \arrow[r] & M \end{tikzcd}$$

given by the inclusion:

$$\begin{array}{rcl} \iota \ : Fr(E) \times_M Fr(F) & \longrightarrow & Fr(E \times F) \\ (\alpha, \beta) & \mapsto & \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \end{array}$$

and since if A and B are skew-symmetric matrices, $\mathsf{Pf}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \mathsf{Pf}(A)\mathsf{Pf}(B)$ then $e(E \oplus F) = e(E) \cdot e(F)$.

3. Suppose that $\pi : L \to M$ is a complex vector bundle of rank 1, using the classification theorem of vector bundles there exits a big number n and a function $f : X \to \mathbb{C}P^n$ such that $E \cong f^*(\tau_n)$.
Then by naturality of classes:

$$e(L) = e(f^*(\tau_n)) = f^*(e(\tau_n))$$

$$(-2\pi)\tilde{e}(L) = (-2\pi)^n\tilde{e}(f^*(\tau_n)) = f^*((-2\pi)\tilde{e}(\tau_n))$$

since $e(\tau_n) = (-2\pi)\tilde{e}(\tau_n)$ by (1), then $e(L) = (-2\pi)\tilde{e}(L)$.

□

Theorem 4.4.24. *Let M be a closed oriented Riemannian manifold of dimension $d = 2n$, then $e(TM) = k\tilde{e}(TM)$ for a constant k.*

Proof. Note that the oriented real vector bundles of rank 2 are identified with complex bundles of rank 1, then using the splitting principle for the tangent vector bundle $\pi : TM \to M$, there is exists $f : X \to M$ such that holds the properties (1) and (2), calculated

$$f^*(e(E)) = e(f^*(E)) = e(L_1 \oplus \cdots \oplus L_n) = \prod_{i=1}^{n} e(L_i)$$

$$f^*((-2\pi)^n \tilde{e}(E)) = (-2\pi)^n \tilde{e}(f^*(E)) = (-2\pi)^n \tilde{e}(L_1 \oplus \cdots \oplus L_n) = \prod_{i=1}^{n} (-2\pi)\tilde{e}(L_i).$$

Since $(-2\pi)\tilde{e}(L_i) = e(L_i)$ and the induced cohomology homomorphism f^* is injective, $e(E) = (-2\pi)^n \tilde{e}(E)$.

□

Theorem 4.4.25 (Chern-Gauss-Bonnet). *Let (M, g) be a closed oriented Riemannian manifold of dimension $d = 2n$ and let Ω be the curvature form of Levi Civita Connection, i.e Ω is an $\mathfrak{so}(2n)$ valued 2 form that comes from a connection in $Fr^{SO}(TM)$. Then:*

$$\chi(M) = \left(\frac{-1}{2\pi}\right)^n \int_M e(TM).$$

Proof. We have seen that $\chi(M) = \int_M \tilde{e}(TM)$ and by the previous theorem $\tilde{e}(TM) = \left(\frac{-1}{2\pi}\right)^n e(TM)$, the we can conclude:

$$\chi(M) = \int_M \tilde{e}(TM) = \left(\frac{-1}{2\pi}\right)^n \int_M e(TM).$$

□

Chapter 5

Chern conjecture for affine manifolds

5.1 Affine manifolds

Definition 5.1.1. *A diffeomorphism φ between open subsets of $\mathbb{R}^m$ is affine if it has the form:*

$$\varphi(x) = Ax + b,$$

where $A \in GL(m, \mathbb{R})$ and $b \in \mathbb{R}^m$.

Definition 5.1.2. *An affine structure on a manifold is an atlas such that all transition functions are affine and it is maximal with this property. An affine manifold is a manifold together with an affine structure.*

It is possible to characterise affine structures in a more intrinsic manner that can be expressed without reference to an atlas, as the following lemma shows:

Lemma 5.1.3. *Let M be a manifold. There is a natural bijective correspondence between affine structures on M and flat torsion free connections on TM.*

Proof. Let $(U_\alpha, \varphi_\alpha)$ be an affine structure on M. There is a unique connection ∇ on TM whose Christoffel symbols vanish in affine coordinates. Conversely, given a flat torsion free connection ∇ on TM, the set of coordinates for which the Christoffel symbols of ∇ vanish gives an affine structure on M. $\square$

Example 5.1.4. The torus $\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m$ has a natural affine structure for which the projection map $\pi : \mathbb{R}^m \to \mathbb{T}^m$ is an affine local difeomorphism.

Example 5.1.5 (Hopf manifolds)**.** Let us fix a real number $\lambda > 1$ and consider the action of the group $\mathbb{Z}$ on $\mathbb{R}^m - \{0\}$ given by:

$$n \star x := \lambda^n x.$$

Since the action is free and proper, the quotient is a smooth manifold called the Hopf manifold Hopf^m_λ. Since the group $\mathbb{Z}$ acts by affine transformations the quotient space is an affine manifold. Topologically, these manifolds are the union of two circles for $m = 1$ and diffeomorphic to $S^{m-1} \times S^1$ for $m > 1$.

Definition 5.1.6. *An affine structure on a Lie group G is called left invariant if for all $g \in G$:*

$$(L_g)^*(\nabla) = \nabla,$$

where L_g denotes the diffeomorphism given by left multiplication by g.

Definition 5.1.7. *Let V be a real vector space. A bilinear map:*

$$\beta : V \otimes V \to V;\ v \otimes w \mapsto v \cdot w$$

is called left symmetric if:

$$v \cdot (w \cdot z) - (v \cdot w) \cdot z = w \cdot (v \cdot z) - (w \cdot v) \cdot z.$$

Definition 5.1.8. *An affine structure on a finite dimensional real Lie algebra $\mathfrak{g}$ is a left symmetric bilinear form on $\mathfrak{g}$ such that for all $v, w \in \mathfrak{g}$:*

$$[v, w] = v \cdot w - w \cdot v.$$

Lemma 5.1.9. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a natural bijective correspondence between left invariant affine structures on G and affine structures on $\mathfrak{g}$.*

Proof. For any $v \in \mathfrak{g} = T_eG$ denote by $\hat{v}$ the corresponding left invariant vector field. Given a left invariant affine structure on G we define a bilinear form on $\mathfrak{g}$ by:

$$v \cdot w := \nabla_{\hat{v}}\hat{w}(e).$$

Since ∇ is torsion free we know that:

$$v \cdot w - w \cdot v = \nabla_{\hat{v}}\hat{w}(e) - \nabla_{\hat{w}}\hat{v}(e) = [v, w].$$

Using the fact that ∇ is flat and left invariant, we compute:

$$\begin{aligned} v \cdot (w \cdot z) - (v \cdot w) \cdot z - w \cdot (v \cdot z) + (w \cdot v) \cdot z &= \nabla_{\hat{v}}(\nabla_{\hat{w}}(\hat{z})) \\ &\quad -\nabla_{\nabla_{\hat{v}}(\hat{w})}(\hat{z}) - \nabla_{\hat{w}}(\nabla_{\hat{v}}(\hat{z})) + \nabla_{\nabla_{\hat{w}}(\hat{v})}(\hat{z}) \\ &= \nabla_{[\hat{v},\hat{w}]}(\hat{z}) - \nabla_{[\hat{v},\hat{w}]}(\hat{z}) \\ &= 0. \end{aligned}$$

So we conclude that the bilinear form is left symmetric. Conversely, given an affine structure on $\mathfrak{g}$ there is a unique left invariant connection ∇ such that:

$$\nabla_{\hat{v}}\hat{w}(e) = v \cdot w.$$

The computations above show that this connection is flat and torsion free. □

Example 5.1.10. The Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ admits a natural affine structure given by:

$$A \cdot B := AB.$$

We conclude that the Lie group $GL(n, \mathbb{R})$ admits a left invariant affine structure.

The question of which Lie groups admit left invariant affine structures is an interesting problem. In [12], Milnor asked whether solvable Lie algebras admit affine structures. In [3], Burde showed that the answer to this question is negative.

5.2 Complete manifolds and the developing map

In this section we introduce the developing map of a simply connected affine manifold and use it to characterise complete affine manifolds.

Theorem 5.2.1. *Let M be an affine manifold of dimension m and G be the group $\mathsf{Aff}(\mathbb{R}^m)$ seen as a discrete group. There is a natural principal G-bundle $\pi : \tau(M) \to M$ such that sections of π are in natural bijective correspondence with affine immersions from M to $\mathbb{R}^m$.*

Proof. For each $p \in M$ we define:

$$C_p = \{\varphi : U \to V \subseteq \mathbb{R}^m : \varphi \text{ is an affine diffeomorphism and } p \in U\}.$$

There is an equivalence relation $\sim$ on C_p given by declaring that $\varphi \sim \varphi'$ if an only if there exists an open subset $W \subseteq U \cap U'$ such that $p \in W$ and $\varphi|_W = \varphi'|_W$. Let us denote by L_p the set of equivalence classes of elements in C_p and set:

$$\tau(M) := \coprod_p L_p.$$

There is a natural map:

$$\pi : \tau(M) \to M.$$

The Lie group G acts on each set L_p by composition:

$$g * \varphi := g \circ \varphi,$$

and this action is free and transitive. Moreover, an affine chart $\varphi : U \to V \subseteq \mathbb{R}^m$ induces a natural identification:

$$\hat{\varphi} : \mathsf{Aff}(\mathbb{R}^m) \times U \to \pi^{-1}(U);\ (g, p) \mapsto [g \circ \varphi]_p,$$

where $[g \circ \varphi]_p$ denotes the class of $g \circ \varphi$ in L_p. There is a unique topology on $\tau(M)$ such that $\hat{\varphi}$ is a homeomorphism for all affine diffeomorphisms φ. The map $\pi : \tau(M) \to M$ is a principal G bundle with respect to this topology. Let σ be a section of π. Then we define $\tilde{\sigma} : M \to \mathbb{R}^m$ by:

$$\tilde{\sigma}(p) := \sigma(p)(p).$$

By construction, the map $\tilde{\sigma}$ is an affine immersion. Conversely, an affine immersion $f : M \to \mathbb{R}^m$ defines a section σ_f given by:

$$\sigma_f(p) := [f]_p.$$

□

Corollary 5.2.2. *Let M be a simply connected affine manifold. Any affine chart $\varphi : U \to V \subseteq \mathbb{R}^m$ extends uniquely to an affine immersion from M to $\mathbb{R}^m$.*

Proof. Since M is simply connected, the covering space $\pi : \tau(M) \to M$ is trivial and therefore any local section extends uniquely to a global one. □

Definition 5.2.3. *A developing map for an affine simply connected manifold is an affine immersion into* $\mathbb{R}^m$*.*

Corollary 5.2.4. *If* M *is an affine connected manifold with finite fundamental group then* M *is not compact.*

Proof. If M is compact then so is its universal cover $\tilde{M}$ which is simply connected and therefore admits an immersion to $\mathbb{R}^m$. This is impossible. □

Definition 5.2.5. *An affine manifold* M *with affine connection* ∇ *is called complete if all geodesics can be extended to arbitrary time.*

Lemma 5.2.6. *Let* M *and* N *be connected affine manifolds of the same dimension and* $f, g : M \to N$ *affine immersions. If* f *and* g *coincide on a nonempty open subset of* M *then they are equal.*

Proof. First we observe that the lemma is true if M and N are open subsets of $\mathbb{R}^m$. For the general case, let $X \subset M$ be the subset of M that consists of points $p \in M$ such that f and g coincide in an open neighborhood of p. Clearly, X is open. It suffices to show that X is closed. Fix $p \in X$ and $q \in M$ arbitrary. Since M is connected, there is a path $\gamma : I \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Set

$$Y := \gamma^{-1}(M - X).$$

We would like to show that $Y = \emptyset$. Suppose that Y is nonempty and set $y := \inf(Y)$ and $z = \gamma(y)$. Choose affine coordinates $\varphi : U \to V$ around z and $\psi : W \to Z$ around $f(z)$ such that $f(U) \subseteq W$ and $g(U) \subseteq W$. Since p is in X there exists $l < y$ such that $\gamma(l) \in U \cap X$. Then $f|_U$ and $g|_U$ coincide in an open neighborhood. Since both U and W are isomorphic to open subsets of $\mathbb{R}^m$ we conclude that $f|_U = g|_U$. This is a contradiction. □

Lemma 5.2.7. *Let* M *be a simply connected complete affine manifold of dimension* m*. Then any affine coordinate* $\varphi : U \to V \subseteq \mathbb{R}^m$ *can be extended uniquely to a diffeomorphism from* M *to* $\mathbb{R}^m$*.*

Proof. Fix a point $p \in M$ and affine coordinates $\varphi : U \to V$. By Corollary 5.2.2, φ can be extended uniquely to an affine immersion $\tilde{\varphi} : M \to \mathbb{R}^m$. We will prove that $\tilde{\varphi}$ is a diffeomorphism. Since M is complete, the exponential map is defined on the whole tangent space T_pM:

$$\mathsf{Exp} : T_pM \to M.$$

On the other hand, the derivative of φ^{-1} at $x = \varphi(p)$ is an isomorphism from $\mathbb{R}^m$ to T_pM. We claim that the map:

$$\psi(y) := \mathsf{Exp} \circ D(\varphi^{-1})(y - x)$$

is the inverse of $\tilde{\varphi}$. By Lemma 5.2.7, it suffices to show that the functions are inverses to each other in a small open neighborhood. For $q \in U$ we compute:

$$\psi(\tilde{\varphi}(q)) = \mathsf{Exp} \circ D(\varphi^{-1})(\varphi(q) - x) = q.$$

Conversely, for $y \in V$ we have:

$$\tilde{\varphi} \circ \psi(y) = \varphi \circ \mathsf{Exp} \circ D(\varphi^{-1})(y - x) = \mathsf{Exp}(y - x) = y.$$

□

Proposition 5.2.8. *Let M be an affine manifold. The following statements are equivalent:*

1. *M is complete.*
2. *There is an affine diffeomorphism $M \cong \mathbb{R}^m/\Gamma$, where Γ is a discrete subgroup of $\mathsf{Aff}(\mathbb{R}^m)$ and the projection $\pi : \mathbb{R}^m \to M$ is the universal cover of M.*

Proof. Let us assume that M is complete. Then $\tilde{M}$ is also complete and simply connected. By Proposition 5.2.8 there is an affine difeomorphism $\varphi : \tilde{M} \to \mathbb{R}^m$. Let us fix a point $p \in M$. Then there is an action of $\pi_1(M, p)$ on $\tilde{M}$ and therefeore on $\mathbb{R}^m$. Since this action is by affine transformations, it defines a homomorphism $\rho : \pi_1(M, p) \to \mathsf{Aff}(\mathbb{R}^m)$. If we set $\Gamma := \mathsf{Im}(\rho)$ we get by construction that $M \cong \mathbb{R}^m/\Gamma$. Conversely, any manifold of the form $\mathbb{R}^m/\Gamma$ is complete.

□

Chapter 6

Milnor's inequalities and the conjecture in dimension d=2

Throughout this section we will set $G = GL^{+}(2, \mathbb{R})$, $S = SO(2) = U(1)$ and Σ will be a closed oriented surface of positive genus g. Oriented rank two vector bundles over Σ are classified by their Euler class. Milnor considered the question of which of those bundles admit a flat connection. The answer is given by his famous inequalities, which are the following.

Definition 6.0.1. *Let $\pi : E \to \Sigma$ be a rank two oriented vector bundle. The degree of E is the number:*

$$D(E) = \int_{\Sigma} e(E).$$

Here $e(E)$ denotes the Euler class of E.

Theorem. A rank two oriented vector bundle $\pi : E \to \Sigma$ admits a flat connection if and only if:

$$|D(E)| < g.$$

Milnor's inequalities give in particular a positive answer to Chern's conjecture in dimension $d = 2$, a result that had been previously obtained by Benzecri [1].

The classification of rank two oriented bundles is of course the same as the classification of $GL^{+}(2, \mathbb{R})$ principal bundles.

Lemma 6.0.2. *The inclusion $\iota : S \hookrightarrow G$ is a homotopy equivalence and therefore the classifying spaces BS and BG are homotopy equivalent.*

Proof. By the spectral theorem every nonsingular matrix γ can be written uniquely in the form $\gamma = OS$, where O is orthogonal matrix and S is symmetric positive definite matrix. Then the following function define a retraction:

$$\begin{aligned} \theta : G &\longrightarrow S \\ \gamma &\longmapsto O. \end{aligned}$$

$\square$

Lemma 6.0.3. *Let* $\mathsf{Bun}(\Sigma)$ *be the set of isomorphism classes of rank two oriented vector bundles over* Σ*. The degree:*

$$D : \mathsf{Bun}(\Sigma) \to \mathbb{Z};\ E \mapsto D(E)$$

is bijection.

Proof. The set $\mathsf{Bun}(\Sigma)$ is in bijective correspondence with the set of isomorphism classes of principal G bundles over Σ. These are classified by the set $[\Sigma, BG]$ of homotopy classes of maps to BG. Moreover:

$$[\Sigma, BG] \cong [\Sigma, BS] \cong [\Sigma, \mathbb{CP}^\infty] = [\Sigma, K(\mathbb{Z}, 2)] \cong H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}.$$

By definition of the Euler class, this bijection is given by the degree. □

Remark 6.0.4. Fix a point $p \in \Sigma$ and a homomorphism $\rho : \pi_1(\Sigma, p) \to G$. The group $\pi = \pi_1(\Sigma, p)$ acts on the right on the universal cover $\tilde{\Sigma}$ and on the left on $\mathbb{R}^2$ via the representation ρ. Let $\tilde{\Sigma} \times_\rho V$ be the quotient of $\tilde{\Sigma} \times V$ via the equivalence relation:

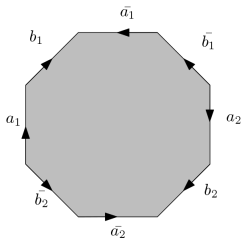
$$(mg, v) \sim (m, \rho(g)v).$$

Then $\tilde{\Sigma} \times_\rho V$ has a natural structure of a flat oriented vector bundle over Σ. We will denote this vector bundle E^ρ and call it the bundle associated to ρ.

Lemma 6.0.5. *A vector bundle over* Σ *admits a flat connection if an only if it is isomorphic to one of the form* E^ρ*.*

Proof. By the previous exercise, E^ρ admits a flat connection. Assume that E admits a flat connection ∇ and fix a point $p \in \Sigma$ and an isomorphism $E_p \cong \mathbb{R}^2$. The holonomy of ∇ gives a representation $\rho : \pi_1(\Sigma, p) \to G$ and an isomorphism $E \cong E^\rho$. □

An oriented closed surface Σ of genus g admits a CW structure with one zero-dimensional cell, $2g$ one-dimensional cells and one two-dimensional cell. For example, the surface of genus 2 can be obtained by the following identification:



In general, this decomposition gives the following presentation of the fundamental group of Σ:

$$\pi_1(\Sigma) = \left\langle a_1, \ldots, a_g, b_1, \ldots, b_g \quad | \quad \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle,$$

here $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. Thus, a representation ρ of $\pi_1(\Sigma)$ on G is the same as a choice of matrices $A_1, \dots, A_g, B_1, \dots B_g \in G$ such that

$$\prod_{i=1}^{g} [A_i, B_i] = \mathrm{id}.$$

We have seen that such a representation gives rise to a vector bundle E^ρ and the natural question arises of computing the degree of E^ρ in terms of the matrices A_i, B_i.

Let $\tilde{G}$ be the universal cover of G and consider the short exact sequence

$$0 \to \pi_1(G) \xrightarrow{\iota} \tilde{G} \xrightarrow{p} G \to 0.$$

Consider also the map $\phi : \mathbb{R} \to S$ given by:

$$\alpha \quad \longmapsto \quad \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

The map ϕ is the universal covering map of S and therefore the retraction map $\theta : G \to S$ can be lifts naturally to a map:

$$\hat{\theta} : \tilde{G} \to \mathbb{R}.$$

Thus we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \pi_1(G) \arrow[r] \arrow[d, "\iota"] & \mathbb{Z} \arrow[d, "2\pi"] \\
 & \tilde{G} \arrow[r, "\tilde{\theta}"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "\phi"] \\
\pi_1(\Sigma) \arrow[r, "\rho"] & G \arrow[r, "\theta"] & S
\end{tikzcd}$$

Lemma 6.0.6. *Given a homomorphism $\rho : \pi_1(\Sigma) \to G$ determined by matrices $A_1, \dots, A_g, B_1, \dots, B_g$ we define the number $\delta(\rho)$ by:*

$$\delta(\rho) := \frac{1}{2\pi}\tilde{\theta}(\prod_{i=1}^{g}[\alpha_i, \beta_i]) \in \mathbb{Z}.$$

Here α_i, β_i are elements in $\tilde{G}$ such that $p(\alpha_i) = A_i$ and $p(\beta_i) = B_i$. This number depends only on ρ.

Proof. Let us first proof that for any choice of the α_i and β_i the number $\delta(\rho)$ is an integer. By construction, $\prod_{i=1}^{g}[\alpha_i, \beta_i] \in \mathsf{ker}(p)$. This implies that $\hat{\theta}(\prod_{i=1}^{g}[\alpha_i, \beta_i])$ is a multiple of 2π and the result follows. Let us now show that the number is well defined. Suppose that α_i' and β_i' is a different choice. Then, for each i we know that:

$$\alpha_i' = \alpha_i x_i \;\; \beta_i' = \beta_i y_i,$$

where the x_i and the y_i are in the kernel of p which is contained in the center of $\tilde{G}$. This implies that:

$$[\alpha_i, \beta_i] = [\alpha_i', \beta_i'],$$

and the result follows. □

Lemma 6.0.7. *Given a representation $\rho : \pi_1(\Sigma) \to G$ we have:*

$$\delta(\rho) = D(E^{\rho}).$$

Proof. This will be added later. □

Remark 6.0.8. We can show that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G,$$

then

$$\theta(A) = \frac{1}{\sqrt{x^2 + y^2}} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

Here $x = a + d$ and $y = b - c$. Moreover, show that:

1. If $A \in G$ has positive trace, then $\theta(A)$ has positive trace.
2. If A and B are symmetric and positive definite matrices, then AB has positive trace.
3. If $A, B \in G$ are symmetric and positive definite matrices, then $\theta(AB)$ has positive trace.
4. $\theta(A^{-1}) = \theta(A)^{-1}$.

Lemma 6.0.9. *The function $\tilde{\theta} : \tilde{G} \to \mathbb{R}$ satisfies the property:*

$$|\tilde{\theta}(\alpha\beta) - \tilde{\theta}(\alpha) - \tilde{\theta}(\beta)| < \pi/2.$$

Proof. Given $A, B \in G$, then there exists symmetric positive definite matrices R, T such that:

$$AB = \theta(A)R\theta(B)T.$$

If we set $X = \theta(B)^{-1}R\theta(B)$ then:

$$AB = \theta(A)\theta(B)XT.$$

This implies that:

$$\theta(AB) = \theta(A)\theta(B)\theta(XT).$$

Therefore:

$$\theta(B)^{-1}\theta(A)^{-1}\theta(AB) = \theta(XT).$$

Since X, T are positive definite, the previous Exercise 6.0.8 implies that $\theta(B)^{-1}\theta(A)^{-1}\theta(AB)$ has positive trace. Fix $\alpha, \beta \in \tilde{G}$ such that $p(\alpha) = A$ and $p(\beta) = B$ and set $\Delta = \tilde{\theta}(\alpha\beta) - \tilde{\theta}(\alpha) - \tilde{\theta}(\beta)$. Then:

$$\phi(\Delta) = \begin{bmatrix} \cos\Delta & \sin\Delta \\ -\sin\Delta & \cos\Delta \end{bmatrix} = \theta(B)^{-1}\theta(A)^{-1}\theta(AB)$$

has positive trace. So $cos\Delta > 0$. Since Δ is a continuous function of α and β which vanishes when $\alpha = \beta = 1$ we conclude that $|\Delta| < \frac{\pi}{2}$.

□

Lemma 6.0.10. *Given a representation $\rho : \pi_1(\Sigma) \to G$ we have:*

$$|\delta(\rho)| < g.$$

Proof. Applying Lemma 6.0.9 $4g-1$ times one obtains:

$$|\tilde{\theta}([\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]) - \tilde{\theta}(\alpha_1) - \tilde{\theta}(\beta_1) - \tilde{\theta}(\alpha_1^{-1}) - \tilde{\theta}(\beta_1^{-1}) \dots - \tilde{\theta}(\alpha_g) - \tilde{\theta}(\beta_g) - \tilde{\theta}(\alpha_g^{-1}) - \tilde{\theta}(\beta_g^{-1})|$$

$$< (4g-1)\frac{\pi}{2}$$

By Exercise 6.0.8 we know that

$$\tilde{\theta}(\alpha) + \tilde{\theta}(\alpha^{-1}) = 0,$$

so that the inequality above becomes:

$$|\tilde{\theta}([\alpha_1, \beta_1] \dots [\alpha_g, \beta_g])| < (4g-1)\frac{\pi}{2}.$$

Dividing by 2π one obtains:

$$|\delta(\rho)| < g - \frac{1}{4} < g.$$

□

The lemma gives one direction of Milnor's inequalities. We will now show that the converse is also true.

Remark 6.0.11. Let G be a connected Lie group with universal cover $\tilde{G}$. If $g \in \tilde{G}$ is such that $p(g) \in Z(G)$ then $g \in Z(\tilde{G})$.

Let us set $A_0 := \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $\alpha_0 \in p^{-1}(A_0)$ so that $\tilde{\theta}(\alpha_0) = 0$. We will denote by K and $\tilde{K}$ the conjugacy classes of A_0 and α_0 respectively.

Since $\mathbb{R}$ is simply connected, the group homomorphism $\phi : \mathbb{R} \to G$ lifts to a homomorphism $\mathbb{R} \to \tilde{G}$ and therefore $\mathbb{R}$ acts on $\tilde{G}$ and also on conjugacy classes of $\tilde{G}$.

Lemma 6.0.12. *Any element of $\pi\tilde{K}$ can be written as a product of two elements in $\tilde{K}$.*

Proof. Consider the matrices:

$$A_1 = \begin{pmatrix} -5/2 & 9/2 \\ -3 & 5 \end{pmatrix}$$

and

$$A_2 = A_0A_1 = \begin{pmatrix} -5 & 9 \\ -3/2 & 5/2 \end{pmatrix}.$$

Since the trace and determinant characterise the sets K and πK, then $A_0, A_1 \in K$ and $A_2 \in \pi K$. Let $\alpha_0, \alpha_1 \in \tilde{K}$ be such that $p(\alpha_i) = A_i$ and $\tilde{\theta}(\alpha_i) = 0$. Then $\alpha_0\alpha_1 \in n\pi\tilde{K}$ for some odd integer n. Let us show that $n = \pm 1$. Given $\alpha \in \tilde{K}$ we know that $p(\alpha)$ has positive trace and therefore $\cos(\tilde{\theta}(\alpha)) > 0$. Therefore, if $\beta \in n\pi\tilde{K}$ with $|n| > 1$ then:

$$|\tilde{\theta}(\beta)| > \frac{5\pi}{2}.$$

Applying Lemma 6.0.9 we obtain:

$$|\tilde{\theta}(\alpha_0\alpha_1)| < |\tilde{\theta}(\alpha_0)| + |\tilde{\theta}(\alpha_1)| + \frac{\pi}{2} < \frac{3}{2}\pi$$

We conclude that $n = \pm 1$.

- If $n = 1$ we set:
$$\gamma(\alpha_0\alpha_1)\gamma^{-1} = (\gamma\alpha_0\gamma^{-1})(\gamma\alpha_1\gamma^{-1}) \in \tilde{K}\tilde{K}.$$
- If $n = -1$ then $\gamma(\alpha_0\alpha_1)\gamma^{-1} \in -\pi\tilde{K}$ and $\gamma(\alpha_0\alpha_1)^{-1}\gamma^{-1} \in \pi\tilde{K}$. We then write:
$$\gamma(\alpha_0\alpha_1)^{-1}\gamma^{-1} = (\gamma\alpha_1^{-1}\gamma^{-1})(\gamma\alpha_0^{-1}\gamma^{-1}) \in \tilde{K}\tilde{K}.$$

☐

Lemma 6.0.13. *For every $\alpha \in \pi\tilde{K}$ there exists $\beta_1\beta_2 \in \tilde{G}$ so that $\alpha = \beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$.*

Proof. By the previous lemma there exists $\beta_1, \beta_3 \in \tilde{K}$ so that $\alpha = \beta_1\beta_3$. Since $\alpha_1^{-1} \in \tilde{K}$ there exists $\beta_2 \in \tilde{G}$ so that $\beta_2\alpha_1^{-1}\beta_2^{-1} = \beta_3$.
Then

$$\alpha = \beta_1\beta_3 = \beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$$

☐

Lemma 6.0.14. *For every $n \geq 1$ there exists $\gamma_1, \ldots, \gamma_{n+1} \in \tilde{K}$ so that $\gamma_1 \ldots \gamma_{n+1} = n\pi\alpha_0$.*

Proof. We will argue by induction. The case $n = 1$ is Lemma 7.1.4. Now assume that the lemma holds for n so that we can write:

$$\gamma_1 \ldots \gamma_{n+1} = n\pi\alpha_0.$$

Then:

$$(n+1)\pi\alpha_0 = \pi(n\pi)\alpha_0 = \pi(\gamma_1 \ldots \gamma_{n+1}) = (\pi\gamma_1) \ldots \gamma_{n+1} = \gamma\gamma'\gamma_2 \ldots \gamma_{n+1}.$$

Here $\gamma, \gamma' \in \tilde{K}$ and we have applied the Lemma product to $\pi\gamma_1$.

☐

Theorem 6.0.15. *A rank two oriented vector bundle $\pi : E \to \Sigma$ admits a flat connection if and only if:*

$$|D(E)| < g.$$

Proof. One direction is guaranteed by Lemma 6.0.10. We want to prove that a bundle E with $|D(E)| < g$ admits a flat connection. It suffices to show that there is some representation $\rho : \pi_1(\Sigma) \to G$ such that $\delta(\rho) = D(E)$. Since the existence of a flat connection does not depend on the orientation we may assume that $D(E) > 0$. Since some of the matrices in the representation can be set to id it suffices to show the case where $D(E) = g - 1$ and $g > 1$. By Lemma 6.0.14 there exist $\gamma_1, ..., \gamma_{g-1} \in \tilde{K}$ so that:

$$\gamma_1 \dots \gamma_{g-1} = (g-2)\pi\alpha_0.$$

If we set $\gamma_g = \alpha_0^{-1}$ then:

$$\gamma_1 \dots \gamma_{g-1}\gamma_g = (g-2)\pi.$$

Now by Lemma 6.0.13, for every $1 \leqslant i \leqslant g$ there exists $\alpha_i, \beta_i \in \tilde{G}$ so that

$$[\alpha_i, \beta_i] = \pi\gamma_i.$$

Then

$$\prod_{i=1}^{g} [\alpha_i, \beta_i] = 2\pi(g-1).$$

This implies that by setting $A_i = p(\alpha_i)$ and $B_i = p(\beta_i)$ one obtains a representation ρ such that:

$$\delta(\rho) = \frac{1}{2\pi}\prod_{i=1}^{g} [\alpha_i, \beta_i] = g - 1.$$

□

Corollary 6.0.16. *The torus is the only closed oriented surface that admits an affine structure. In particular, Chern's conjecture holds in dimension $d = 2$.*

Proof. Since S^2 is simply connected, a flat connection on TM would give a trivialization of TM, which does not exist because the Euler characteristic of the sphere is 2. Let us now assume that $g > 1$. Then $D(TM) = \chi(\Sigma) = 2 - 2g$. Then:

$$|D(TM)| = 2|(1-g)| = 2g - 2 \geq g.$$

□

Chapter 7

The counterexamples of Smillie

Chern's conjecture asks about the Euler characteristic of closed affine manifolds. For a while, it was not known whether the condition that the connection is torsion free was essential.

Definition 7.0.1. *A manifold M is said to be flat if its tangent bundle admits a flat connection.*

A strong version of Chern's conjecture asking whether the Euler characteristic of a closed flat manifold vanishes was an open problem for a while. This question was answered by Smillie [15], who proved the following:

Theorem (Smillie)**.** For each $n > 1$ there are closed flat manifolds M^{2n} which have non-zero Euler characteristic.

This construction is based in the ideas of almost trivial vector bundles and almost parallelizable manifolds.

7.1 Almost trivial vector bundles

Definition 7.1.1. *We will say that a vector bundle $\pi : E \to M$ is almost trivial if*

$$E \oplus \mathbb{R} \simeq \mathbb{R}^m.$$

We will say that M is almost parallelizable if TM is almost trivial.

Lemma 7.1.2. *An oriented vector bundle $\pi : E \to M$ is almost parallelizable if and only if there exists a function $f : M \to S^m$ such that:*

$$f^*(TS^m) \simeq E.$$

Proof. Since TS^m is almost parallelizable, so is $f^*(TS^m)$. On the other hand, suppose that E is almost parallelizable. Once we fix an isomorphism

$$\varphi : E \oplus \mathbb{R} \simeq \mathbb{R}^{m+1},$$

there is a unique smooth function $f : M \to S^m$ such that:

- $f(p) \in \varphi(p)(E_p)^{\perp}$.
- If $\{e_1, \dots, e_m\}$ is an oriented basis for E_p then

  $$\{f(p), \varphi(p)(e_1), \dots, \varphi(p)(e_m)\}$$

  is an oriented basis for $\mathbb{R}^{m+1}$.

By construction φ restricts to an isomorphism from E to $f^*(TS^m)$. □

Remark 7.1.3. There exists an embedding:

$$\iota : S^m \times S^n \to \mathbb{R}^{m+n+1}$$

with trivial normal bundle. Conclude that $S^m \times S^n$ is almost parallelizable.

Lemma 7.1.4. *Let $\pi_1 : E \to M$ and $\pi_2 : E' \to N$ be almost trivial vector bundles. Then the vector bundle ${p_1}^*E \oplus {p_2}^*E'$ over $M \times N$ is almost trivial.*

Proof. By Lemma 7.1.2 there exist functions $f : M \to S^m$ and $g : N \to S^n$ such that:

$$E \simeq f^*(TS^m); \text{ and } E' \simeq g^*(TS^n).$$

By remark 7.1.3 there exists a function $h : S^m \times S^n \to S^{m+n}$ such that:

$$h^*(TS^{m+n}) = T(S^m \times S^n).$$

Then:

$$(h \circ (f \times g))^*(TS^{m+n}) = (f \times g)^*(h^*(TS^{m+n})) = (f \times g)^*(T(S^m \times S^n)) = {p_1}^*E \oplus {p_2}^*E.$$

By Lemma 7.1.2 we conclude that ${p_1}^*E \oplus {p_2}^*E$ is almost trivial. □

Definition 7.1.5. *Let M, N be closed oriented manifolds of dimension m and $f : M \to N$ be a smooth function. The degree of f is the number:*

$$\mathsf{deg}(f) := \int_M f^*(\omega)$$

where ω is any m-form on N such that:

$$\int_N \omega = 1.$$

Remark 7.1.6. The number $\mathsf{deg}(f)$ defined above is an integer that does not depend on the choice of ω. Show that if f and g are smoothly homotopic then they have the same degree.

It turns out that the degree is a complete invariant of maps from a closed oriented m-manifold to S^m. This is a theorem of Heinz Hopf whose proof can be found in [2].

Theorem 7.1.7 (Hopf degree Theorem)**.** *Let M be a closed, connected, oriented m-dimensional manifold and $f, g : M \to S^m$ be smooth maps. Then f and g are smoothly homotopic if and only if they have the same degree.*

Lemma 7.1.8. *Let M be a closed oriented manifold of dimension m and E, E' oriented almost trivial vector bundles of rank m. Then E and E' are isomorphic if and only if they have the same Euler class.*

Proof. Clearly, if the bundles are isomorphic they have the same Euler class. Let us prove the converse. By Lemma 7.1.2 we know that there are functions $f, g : M \to S^m$ such that:

$$f^*(TS^m) \simeq E; \text{ and } g^*(TS^m) \simeq E'.$$

Since E and E' have the same Euler class we conclude that f and g have the same degree. The Hopf degree theorem implies that f and g are smoothly homotopic and therefore E is isomorphic to E'. □

7.2 Almost parallelizable manifolds

Lemma 7.2.1. *The connected sum of almost parallelizable manifolds is almost parallelizable.*

Proof. Let $f : M \to S^m$ and $g : N \to S^m$ be smooth maps such that:

$$f^*(TS^m) \simeq TM; \text{ and } g^*(TS^m) \simeq TN.$$

Fix $p \in M$ and $q \in N$ and coordinates $\varphi : U \to \mathbb{R}^m$ and $\phi : W \to \mathbb{R}^m$ around p and q respectively. Set $X = \varphi^{-1}(B(0,1)) \subseteq U$ and $Y = \phi^{-1}(B(0,1)) \subseteq W$. Since $\pi_{m-1}(S^m) = 0$ we may assume that:

$$(f \circ \varphi^{-1})|_{S^{m-1}} = (g \circ \phi^{-1})|_{S^{m-1}}.$$

Then there is a well defined map

$$h : M\#N \to S^m$$

such that $h|_{M-X} = f$ and $h|_{N-Y} = g$. We conclude that

$$h^*(TS^m) \simeq T(M\#N)$$

and therefore $M\#N$ is almost parallelizable. □

Theorem 7.2.2 (Smillie)**.** *Let Σ_g be the closed oriented surface of genus g and $P = S^1 \times S^3$. Then:*

$$M^4 = (\Sigma_3 \times \Sigma_3)\#\underbrace{P\#\cdots\#P}_{6 \text{ times}}$$

and

$$M^6 = ((\Sigma_3 \times \Sigma_3)\#\underbrace{P\#\cdots\#P}_{9 \text{ times}}) \times \Sigma_3$$

are closed flat manifolds with non-zero Euler characteristic. By taking products of M^4 and M^6 one obtains closed flat manifolds with nonzero Euler characteristic in all even dimensions greater than $d = 2$.

Proof. The Euler characteristic of a connected sum of closed even dimensional manifolds satisfies:

$$\chi(M\#N) = \chi(M) + \chi(N) - 2.$$

Using this formula we compute:

$$\chi(M^4) = \chi(\Sigma_3 \times \Sigma_3) + \chi(\underbrace{P\# \cdots \# P}_{\text{6 times}}) - 2 = 16 - 10 - 2 = 4.$$

and

$$\chi(M^6) = (\chi(\Sigma_3 \times \Sigma_3) + \chi(\underbrace{P\# \cdots \# P}_{\text{9 times}}) - 2) \times \chi(\Sigma_3) = (16 - 16 - 2) \times (-4) = 8.$$

It remains to prove that M^4 and M^6 are flat manifolds. Lemmas 7.2.1 and 7.1.4 imply that M^4 and M^6 are almost trivial. Let $h : \Sigma_3 \to S^2$ be a degree one map and set $E = h^*(TS^2)$. Then E is almost parallelizable and:

$$\int_{\Sigma_3} e(E) = \int_{S^2} e(TS^2) = 2.$$

Lemma 7.1.4 implies that $\pi^*E \oplus \pi^*E$ is an almost trivial bundle over $\Sigma_3 \times \Sigma_3$. Let

$$f : M^4 \to \Sigma_3 \times \Sigma_3$$

be the map that sends

$$\underbrace{P\# \cdots \# P}_{\text{6 times}}$$

to a point. Then f has degree 1 and therefore:

$$\int_{M^4} e(f^*(E \oplus E)) = \int_{M^4} f^*(e(\pi(E \otimes E)) = \int_{\Sigma_3 \times \Sigma_3} \pi^*(e(E)) \wedge \pi^*(e(E)) = \Big(\int_{\Sigma_3} e(E)\Big)^2 = 4.$$

We conclude that:

$$e(TM^4) = e(\pi^*E \oplus \pi^*(E))$$

and therefore, by Lemma 7.1.8:

$$TM^4 \simeq {\pi_1}^*E \oplus {\pi_2}^*(E).$$

Milnor's inequality 6.0.15 implies that E admits a flat connection and therefore, so does TM^4. Similarly, Let

$$g : \Sigma_3 \times \Sigma_3 \# \underbrace{P\# \cdots \# P}_{\text{9 times}} \to \Sigma_3 \times \Sigma_3$$

be the map that sends

$$\underbrace{P\# \cdots \# P}_{\text{9 times}}$$

to a point. Then g has degree one and therefore the map:

$$z := g \times \mathsf{id} : M^6 \to \Sigma_3 \times \Sigma_3 \times \Sigma_3$$

also has degree one. This implies that:

$$\int_{M^6} e(z^*(\pi^*E \oplus \pi^*E \oplus \pi^*E)) = \Big(\int_{\Sigma_3} e(E)\Big)^3 = 8.$$

We conclude that:

$$e(TM^6) = e(\pi^*E \oplus \pi^*E \oplus \pi^*E)$$

and therefore:

$$TM^6 \simeq \pi^*E \oplus \pi^*E \oplus \pi^*E.$$

Theorem 6.0.15 implies that E admits a flat connection and therefore, so does TM^6.

$\square$

Chapter 8

Complete manifolds, after Kostant and Sullivan

The conjecture of Chern has been proved by Konstant and Sullivan in [9] for complete affine manifolds. The idea of their proof is that even though the connection on M may not admit a compatible metric, the Chern-Weil theory of characteristic classes can still be used to prove that the Euler class vanishes. They use the geometric characterization of complete affine manifolds given Proposition 5.2.8.

In this section we present the proof of Konstant and Sullivan with greater detail.

8.1 Previous lemmas

Definition 8.1.1. *We will say that a subgroup G of $Gl(m, \mathbb{R})$ is 1-spectral if for every $A \in G$,*

$$\mathsf{Det}(A - \mathsf{id}) = 0.$$

Definition 8.1.2. *We will say that a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(m, \mathbb{R})$ is singular if for all $v \in \mathfrak{g}$,*

$$\mathsf{Det}(\mathsf{v}) = 0.$$

The key observation of Sullivan and Kostant is that if the frame bundle of TM admits a reduction to a connected compact and 1-spectral subgroup of $GL^+(m, \mathbb{R})$ then the Euler class of TM vanishes.

Lemma 8.1.3. *If G is a 1-spectral Lie subgroup of $GL(m, \mathbb{R})$ then its Lie algebra $\mathfrak{g}$ is singular.*

Proof. It is enough to prove that there is an open subset $W \subseteq \mathfrak{g}$ such that $0 \in W$ and all matrices of W are singular. Choose W such that the exponential map $\mathsf{Exp} : W \to U \subseteq G$ is a diffeomorphism with inverse $\mathsf{Log} : U \to W$. For $v \in W$, $A \in U$ these functions are given explicitly by:

$$\mathsf{Exp}(v) = \sum_{k \geqslant 0} \frac{x^k}{k!},$$

$$\mathsf{Log}(A) = \sum_{k \geqslant 1} \frac{(-1)^{k+1}(A - \mathsf{id})^k}{k}.$$

Any $v \in W$ is of the form $v = \mathsf{Log}(A)$ with $\mathsf{Det}(A - \mathsf{id}) = 0$. Thus there exists a vector $x \in \mathbb{R}^m$ such that $Ax = x$. Then:

$$v(x) := \sum_{k \geqslant 1} \frac{(-1)^{k+1}(A - \mathsf{id})^k}{k}(x) = \sum_{k \geqslant 1} \frac{(-1)^{k+1}(A - \mathsf{id})^k(x)}{k} = 0.$$

We conclude that v is singular. ☐

Lemma 8.1.4. *Let M be a manifold of dimension m and P a $GL^+(m, \mathbb{R})$ principal bundle over M that admits a reduction to a subgroup G which is compact, connected and 1-spectral. Then the Euler class of P vanishes.*

Proof. The Chern-Weil homomorphism

$$\mu_P : H(BGL^+(m, \mathbb{R})) \to H(M)$$

factors through the restriction map:

$$\iota^* : H(BGL^+(m, \mathbb{R})) \to H(BG).$$

Moreover, there is a commutative diagram:

$$\begin{tikzcd}
H(BGL^+(m, \mathbb{R})) \arrow[r, "\iota^*"] \arrow[d, "\simeq"] & H(BG) \arrow[d, "\simeq"] \\
S(\mathfrak{gl}(m, \mathbb{R})^*)^{GL^+(m,\mathbb{R})} \arrow[r, "\rho"] & S(\mathfrak{g}^*)^G
\end{tikzcd}$$

where the vertical arrows are isomorphisms and ρ is the restriction of polynomials. Since the Euler class corresponds to the Pfaffian polynomial, it suffices to prove that $\rho(\mathsf{Pf}) = 0$. For this we observe that since $\mathfrak{g}$ is singular:

$$[\rho(\mathsf{Pf})(v)]^2 = \mathsf{Det}(v)^2 = 0.$$

We conclude that $\rho(\mathsf{Pf}) = 0$ and therefore the Euler class of P vanishes. ☐

Lemma 8.1.5. *Let M be a manifold of dimension m and P is a $GL^+(m, \mathbb{R})$ principal bundle over M that admits a reduction to a subgroup G which is closed, connected and 1-spectral. Then the Euler class of P vanishes.*

Proof. Since G is a closed subgroup of the general lineal group, it is a Lie subgroup. Let $K \subseteq G$ be a maximal compact subgroup. Since G/K is contractible, the natural map $\pi : BK \to BG$ is a homotopy equivalence. The Chern-Weill homomorphism

$$\mu_P : H(BGL^+(m, \mathbb{R})) \to H(M)$$

factors through the restriction map:

$$\iota^* : H(BGL^+(m, \mathbb{R})) \to H(BG)$$

so it suffices to show that ι^* sends the Euler class to zero. There is also a commutative diagram:

$$\begin{tikzcd}
H(BGL^+(m,\mathbb{R})) \arrow[r, "\pi^*\circ\iota^*"] \arrow[d, "\simeq"] & H(BK) \arrow[d, "\simeq"] \\
S(\mathfrak{gl}(m,\mathbb{R})^*)^{GL^+(m,\mathbb{R})} \arrow[r, "\rho"] & S(\mathfrak{K}^*)^K
\end{tikzcd}$$

By the Lemma 8.1.4 we know that $\rho(\mathsf{Pf}) = 0$ and therefore $\pi^* \circ \iota^*$ sends the Euler class to zero. Since π^* is an isomorphism, we conclude that ι^* sends the Euler class to zero.

$\square$

Lemma 8.1.6. *Let M be a manifold of dimension m and P be a $GL^+(m)$ principal bundle such that admits a reduction to a subgroup $G \subseteq GL^+(m,\mathbb{R})$ which is 1-spectral and closed. If G has finite number of connected components then the Euler class of P is zero.*

Proof. As in the previous lemmas, it suffices to show that the restriction map:

$$\iota^* : H(BGL^+(m,\mathbb{R})) \to H(BG)$$

sends the Euler class to zero. Let G' be the connected component of the identity in G. Then the natural map:

$$\pi : BG' \to BG$$

is a finite covering and therefore it induces an injective map in cohomology. Therefore it suffices to show that the map:

$$\pi^* \circ \iota^* : H(BGL^+(m,\mathbb{R})) \to H(BG')$$

sends the Euler class to zero. This is guaranteed by Lemma 8.1.5.

$\square$

In view of the lemma above, we are left with the problem of showing that the frame bundle of a closed complete affine manifold admits a reduction to a closed 1-spectral group with finitely many connected components.

Lemma 8.1.7. *Let G be a 1 spectral subgroup of $GL(m,\mathbb{R})$ and $\bar{G}$ its Zariski closure. Then:*

1. *$\bar{G}$ is a subgroup of $GL(m,\mathbb{R})$.*
2. *$\bar{G}$ is a closed Lie group in $GL(m,\mathbb{R})$.*
3. *$\bar{G}$ is 1-spectral.*
4. *$\bar{G}$ has finitely many connected components.*

Proof. Since the multiplication and inverse functions are algebraic operations, they are continuous in the Zariski topology. Fix $x \in G$ and consider the homeomorphism

$$\begin{array}{rccc} L_x : GL(m,\mathbb{R}) & \longrightarrow & GL(m,\mathbb{R}) \\ y & \longmapsto & xy \end{array}.$$

Then

$$x\bar{G} = \overline{xG} \subseteq \bar{G}.$$

Fix now $y \in \bar{G}$ and consider the hoeomorphism R_y given by right multiplication by y. Then:

$$\bar{G}y = \overline{Gy} \subseteq \bar{G}.$$

We conclude that $\bar{G}$ is closed with respect to the product. The map $x \mapsto x^{-1}$ is a homeomorphism of $GL(m, \mathbb{R})$ and therefore:

$$\bar{G}^{-1} = \overline{G^{-1}} = \overline{G}.$$

We conclude that G is a group. The second statement is true because $\bar{G}$ is closed in the Zariski topology and in therefore also in the smooth topology. The third statement holds because the determinant is a continuous function. The last statement is true because any real algebraic set has finitely many connected components.

□

8.2 Proof of Chern conjecture for complete manifolds

Theorem 8.2.1 (Kostant-Sullivan [9]). *If M is a closed affine complete manifold then* $\chi(M) = 0$.

Proof. In view of Lemma 8.1.6 it suffices to show that the frame bundle of TM admits a reduction to a closed 1-spectral subgroup of $GL^+(m, \mathbb{R})$ which has finitely many connected components. By Proposition 5.2.8 we know that M is the quotient $\mathbb{R}^m/\Gamma$ where $\Gamma \subset \mathsf{Aff}(\mathbb{R}^m)$ is isomorphic to the fundamental group of M. Consider the natural homomorphism

$$\lambda : \mathsf{Aff}(\mathbb{R}^m) \to GL(m, \mathbb{R});\ Ax + b \mapsto A$$

and let G be the group $\lambda(\Gamma)$. We claim that G is 1-spectral. Take $g = Ax + b \in \mathsf{Aff}(\mathbb{R}^m)$. If A is not 1-spectral then the equation $Ax+b = x$ has a solution, which is imposible because Γ acts freely on $\mathbb{R}^m$. Lemma 8.1.7 guarantees that $\bar{G}$ is a closed 1-spectral group with finitely many connected components. Therefore, it suffices to prove that the frame bundle of TM admits a reduction to the group G and therefore to $\bar{G}$. Consider the projection

$$\pi : \mathbb{R}^m \to M$$

and for each $p \in M$, the following subset of the frame bundle of T_pM:

$$S_p := \{\phi : \mathbb{R}^m \to T_pM : \phi \text{ is a linear isomorphism of the form } \phi = D\pi(x) \text{ for } x \in \pi^{-1}(p)\}$$

The group G acts on the right by composition and this action is free and transitive. If we set:

$$S := \coprod_p S_p \subset \mathsf{Fr}(TM)$$

we obtain a reduction of the structure group of the frame bundle of TM to G. □

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