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Calabi-Yau property for graded skew PBW extensions

Propiedad Calabi-Yau para extensiones PBW torcidas graduadas

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Abstract. Graded skew PBW extensions were defined by the first author as a generalization of graded iterated Ore extensions [36]. The purpose of this paper is to study the Artin-Schelter regularity and the (skew) Calabi-Yau condition for this kind of extensions. We prove that every graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra is also an Artin-Schelter regular algebra and, as a consequence, every graded quasi-commutative skew PBW extension of a connected skew Calabi-Yau algebra is skew Calabi-Yau. Finally, we prove that graded skew PBW extensions of a finitely presented connected Auslander-regular algebra are skew Calabi-Yau.

Key words and phrases. Graded skew PBW extensions, AS-regular algebras, skew Calabi-Yau algebras.

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Resumen. Las extensiones PBW torcidas graduadas fueron definidas por el primer autor como una generalización de las extensiones de Ore iteradas graduadas [36]. El propósito de este artículo es estudiar las condiciones Artin-Schelter regular y Calabi-Yau (torcida) para esta clase de extensiones. Demostramos que cada extensión PBW torcida cuasi-conmutativa graduada de un álgebra Artin-Schelter regular también es Artin-Schelter regular, y, como consecuencia, que cada extensión PBW torcida cuasi-conmutativa graduada de un álgebra conexa Calabi-Yau torcida es Calabi-Yau torcida. Finalmente, mostramos que las extensiones PBW torcidas graduadas de álgebras Auslander-regular finitamente presentadas y conexas son Calabi-Yau torcidas.

Palabras y frases clave. Extensiones PBW torcidas graduadas, álgebras AS-regular, álgebras Calabi-Yau torcidas.

1. Introduction

In the schematic approach to non-commutative algebraic geometry, some important classes of non-commutative algebras like Koszul algebras, Artin–Schelter regular algebras, Calabi-Yau algebras (see for example [3, 4, 5, 6, 10, 14, 34]) arise, and related with them, the skew PBW extensions (see [18, 20]). Koszul algebras were introduced by Priddy in [22]; regular algebras were defined by Artin and Schelter in [3] and they are now known in the literature as Artin-Schelter regular algebras (we denote these algebras in short as AS-regular algebras); Calabi-Yau algebras were defined by Ginzburg in [10], and as a generalization of them, were defined the skew (also named twisted) Calabi-Yau algebras. Skew PBW extensions were introduced in [8]. The first author in [36] defined the graded skew PBW extensions and showed that if R is a finite presented Koszul algebra, then every graded skew PBW extension of R is Koszul. The class of graded skew PBW extensions is more general that the class of graded iterated Ore extensions, for example, the homogenized enveloping algebra $\mathcal{A}(\mathcal{G})$ and the diffusion algebra are graded skew PBW extensions but these are not iterated Ore extensions. In [37] were illustrated through examples, known in the literature, some relationships between Koszul algebras, AS-regular algebras, Calabi-Yau algebras and the skew PBW extensions. Formally, there are some relationships between the above algebras, for example:

(1) Connected graded Calabi-Yau algebras are AS-regular (see [12]).

(2) For connected algebras, skew Calabi-Yau algebras and AS-regular algebras match ([33], Lemma 1.2).

(3) Semi-commutative skew PBW extensions of a field are Koszul algebras ([39], Corollary 3.13).

(4) Graded skew PBW extensions of finitely presented Koszul algebras are Koszul ([36], Theorem 5.5).

We will prove that graded skew PBW extensions of a finitely presented connected Auslander-regular algebra are skew Calabi-Yau. Reyes, Rogalski and J. Zhang in [33] proved that for connected algebras, skew Calabi-Yau property is equivalent to AS-regular property, therefore, graded quasi-commutative skew PBW extensions of connected skew Calabi-Yau algebras are skew Calabi-Yau. We will also show that graded skew PBW extensions of Auslander-regular algebras are skew Calabi-Yau. Moreover, it is clear that Calabi-Yau algebras are skew Calabi-Yau, but we will exhibit examples of graded skew PBW extensions which are skew Calabi-Yau, but not Calabi-Yau (one of them is the Jordan Plane).

2. Graded skew PBW extensions

In this section we present some definitions, properties and examples related with skew PBW extensions. For more details and to check other recent properties related to skew PBW extensions, see [1, 2, 9, 7, 17, 18, 19, 23, 24, 26, 25, 27, 37, 16], [28, 29, 30, 31, 36, 38], [39, 40], and [32].

Definition 2.1. Let R and A be rings. We say that A is a *skew PBW extension of* R if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist finitely many elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the set of standard monomials

$$\mathrm{Mon}(A) := \{x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2}$$

Under these conditions we will write $A = \sigma(R)\langle x_1, \dots, x_n\rangle$.

Proposition 2.2 ([8], Proposition 3)**.** *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exists an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \qquad r \in R. \tag{3}$$

The notation $\sigma(R)\langle x_1, \dots, x_n\rangle$ and the name of skew PBW extensions are due to Proposition 2.2.

Definition 2.3. Let A be a skew PBW extension of R, $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$, where σ_i and δ_i $(1 \leq i \leq n)$ are as in Proposition 2.2.

(a) A is called *pre-commutative* if the conditions (iv) in Definition 2.1 are replaced by:

For any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R x_1 + \cdots + R x_n. \tag{4}$$

(b) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r = c_{i,r} x_i; \tag{5}$$

(iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j. \tag{6}$$

(c) A is called *bijective* if σ_i is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(d) If $\sigma_i = \mathrm{id}_R$ for every $\sigma_i \in \Sigma$, we say that A is a skew PBW extension of *derivation type*.

(e) If $\delta_i = 0$ for every $\delta_i \in \Delta$, we say that A is a skew PBW extension of *endomorphism type*.

(f) Any element r of R such that $\sigma_i(r) = r$ and $\delta_i(r) = 0$ for all $1 \leq i \leq n$, will be called a *constant*. A is called *constant* if every element of R is constant.

(g) A is called *semi-commutative* if A is quasi-commutative and constant.

Examples of the above classes of skew PBW extensions can be found in [39].

For the rest of the paper, we fix a field $\mathbb{K}$, all algebras are $\mathbb{K}$-algebras and the dimension of a $\mathbb{K}$-vector space is denoted by $\dim_{\mathbb{K}}$. Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension and let $x^\alpha \in \mathrm{Mon}(A)$ with $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then we establish the notation $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Proposition 2.4 ([36], Proposition 2.7)**.** *Let $R = \bigoplus_{m \geq 0} R_m$ be a $\mathbb{N}$-graded algebra and let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a bijective skew PBW extension of R satisfying the following two conditions:*

(i) *σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation for all $1 \leq i \leq n$, where σ_i and δ_i are as in Proposition 2.2.*

(ii) *$x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (2) and $c_{i,j} \in R_0$.*

For $p \geq 0$, let A_p the $\mathbb{K}$-space generated by the set

$$\Big\{ r_t x^\alpha \mid t + |\alpha| = p, \ r_t \in R_t \text{ and } x^\alpha \in \mathrm{Mon}(A) \Big\}.$$

Then A is a $\mathbb{N}$-graded algebra with graduation

$$A = \bigoplus_{p \geq 0} A_p. \tag{7}$$

Definition 2.5. Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a $\mathbb{N}$-graded algebra $R = \bigoplus_{m\geq 0} R_m$. We say that A is a *graded skew PBW extension* if A satisfies the conditions (i) and (ii) in Proposition 2.4.

Proposition 2.6. *Quasi-commutative skew PBW extensions with the trivial graduation of R are graded skew PBW extensions. If we assume that R has a different graduation from the trivial graduation, then A is a graded skew PBW extension if and only if σ_i is graded and $c_{i,j} \in R_0$, for $1 \leq i, j \leq n$.*

Proof. Let $R = R_0$ and $r \in R = R_0$. From (5) we have that $x_i r = c_{i,r} x_i = \sigma_i(r)x_i$. So, $\sigma_i(r) = c_{i,r} \in R = R_0$ and $\delta_i = 0$, for $1 \leq i \leq n$. Therefore σ_i is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded σ_i-derivation for all $1 \leq i \leq n$. On the other hand, from (6) we have that $x_j x_i - c_{i,j} x_i x_j = 0 \in R_2 + R_1 x_1 + \cdots + R_1 x_n$ and $c_{i,j} \in R = R_0$. If R has a nontrivial graduation, then we get the result from relations (5), (6) and Definition 2.5. ☑

An algebra is called *Noetherian* if it is right and left Noetherian. It is known that a graded algebra A is right (left) Noetherian if and only if it is *graded right* (left) *Noetherian*, which means that every graded right (left) ideal is finitely generated ([15], Proposition 1.4).

Proposition 2.7. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a graded skew PBW extension.*

(i) *If R is a graded (left) Noetherian algebra, then every graded skew PBW extension A of R is graded (left) Noetherian.*

(ii) *If A is quasi-commutative, then A is isomorphic to a graded iterated Ore extension of endomorphism type $R[z_1; \theta_1]\cdots[z_n; \theta_n]$, where θ_i is bijective, for each i; $\theta_1 = \sigma_1$;*

$$\theta_j : R[z_1; \theta_1]\cdots[z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1]\cdots[z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j} z_i$ ($c_{i,j} \in R_0$ as in (2)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$.

Proof. (i) Since A is bijective, then by [19] Corollary 2.4, we have that A is a (left) Noetherian algebra. As A is graded then A is graded left Noetherian.

(ii) By [19], Theorem 2.3 and its proof, we have that A is isomorphic to an iterated Ore extension of endomorphism type $R[z_1; \theta_1]\cdots[z_n; \theta_n]$, where θ_i is bijective; $\theta_1 = \sigma_1$;

$$\theta_j : R[z_1; \theta_1]\cdots[z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1]\cdots[z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j} z_i$ ($c_{i,j} \in R$ as in (2)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Since A is graded then σ_i is graded and $c_{i,j} \in R_0$.

Moreover, since $\theta_i(r) = \sigma_i(r)$, then θ_i is a graded automorphism for each i. Note that z_i has graded 1 in A, for all i. Thus, $A \cong R[z_1; \theta_1] \cdots [z_n; \theta_n]$ is a graded iterated Ore extension.

☑

Example 2.8. Examples of graded skew PBW extensions can be found in [36]; we present next some examples of graded quasi-commutative skew PBW extensions.

(1) The Sklyanin algebra is the algebra $S = \mathbb{K}\langle x, y, z\rangle/\langle ayx+bxy+cz^2, axz+bzx+cy^2, azy+byz+cx^2\rangle$, where $a, b, c \in \mathbb{K}$. If $c \neq 0$ then S is not a skew PBW extension. If $c = 0$ and $a, b \neq 0$ then in S: $yx = -\frac{b}{a}xy$; $zx = -\frac{a}{b}xz$ and $zy = -\frac{b}{a}yz$, therefore $S \cong \sigma(\mathbb{K})\langle x, y, z\rangle$ is a skew PBW extension of $\mathbb{K}$, and we call this algebra a *particular Sklyanin algebra*. The particular Sklyanin algebra is a graded quasi-commutative skew PBW extension of $\mathbb{K}$.

(2) For a fixed $q \in \mathbb{K} \setminus \{0\}$, the *algebra of linear partial q-dilation operators* with polynomial coefficients is $\mathbb{K}[t_1, \dots, t_n][H_1^{(q)}, \dots, H_m^{(q)}]$, $n \geq m$, subject to the relations: $t_jt_i = t_it_j$, $1 \leq i < j \leq n$; $H_i^{(q)}t_i = qt_iH_i^{(q)}$, $1 \leq i \leq m$; $H_j^{(q)}t_i = t_iH_j^{(q)}$, $i \neq j$; $H_j^{(q)}H_i^{(q)} = H_i^{(q)}H_j^{(q)}$, $1 \leq i < j \leq m$. This algebra is a graded quasi-commutative skew PBW extension of $\mathbb{K}[t_1, \dots, t_n]$, where $\mathbb{K}[t_1, \dots, t_n]$ is endowed with the usual graduation.

(3) The *multiplicative analogue of the Weyl algebra* $\mathcal{O}_n(\lambda_{ji})$ is the algebra generated by $x_1, \dots, x_n$ subject to the relations: $x_jx_i = \lambda_{ji}x_ix_j$, $1 \leq i < j \leq n$, $\lambda_{ji} \in \mathbb{K} \setminus \{0\}$. Thus $\mathcal{O}_n(\lambda_{ji}) \cong \sigma(\mathbb{K}[x_1])\langle x_2, \dots, x_n\rangle$.

(4) Let $n \geq 1$ and $\mathbf{q}$ be a matrix $(q_{ij})_{n\times n}$ with entries in a field $\mathbb{K}$, where $q_{ii} = 1$ y $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. Then *multi-parameter quantum affine n-space* $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ is defined to be the $\mathbb{K}-$algebra generated by $x_1, \dots, x_n$ with the relations $x_jx_i = q_{ij}x_ix_j$, for all $1 \leq i, j \leq n$.

***Remark* 2.9.** The algebra of shift operators is defined by $S_h := \mathbb{K}[t][x_h; \sigma_h]$, where $\sigma_h(p(t)) := p(t-h)$, with $h \in \mathbb{K}$ and $p(t) \in \mathbb{K}[t]$. Notice that $x_ht = (t-h)x_h$ and $x_hp(t) = p(t-h)x_h$. Thus, $S_h \cong \sigma(\mathbb{K}[t])\langle x_h\rangle$ is a quasi-commutative skew PBW extension of $R := \mathbb{K}[t]$. S_h is a graded quasi-commutative skew PBW extension if $\mathbb{K}[t]$ is endowed with trivial graduation. But if $h \neq 0$ and $\mathbb{K}[t]$ is endowed with the usual graduation, i.e. $R_0 = \mathbb{K}$, R_1 is the subspace generated by t, R_2 is the subspace generated by t^2, etc., then S_h is not a graded skew PBW extension, since $\sigma_h(t) = t - h \notin R_1$, i.e, σ_h is not graded.

Proposition 2.10. *Let B be a connected $\mathbb{N}$-graded algebra. B is finitely generated as algebra if and only if $B = \mathbb{K}\langle x_1, \dots, x_m\rangle/I$, where I is a proper homogeneous two-sided ideal of $\mathbb{K}\langle x_1, \dots, x_m\rangle$. Moreover, for every $n \in \mathbb{N}$, $dim_{\mathbb{K}}B_n < \infty$, i.e., B is locally finite.*

Let B be a finitely graded algebra; it is said that B is *finitely presented* if the two-sided ideal I of relations in Proposition 2.10 is finitely generated.

***Remark* 2.11.** Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a graded skew PBW extension, we recall next some properties of A derived from the definition (see [36], Remark 2.10):

(i) A is a $\mathbb{N}$-graded algebra and $A_0 = R_0$.

(ii) R is connected if and only if A is connected.

(iii) If R is finitely generated then A is finitely generated.

(iv) For (i), (ii) and (iii) above we have that if R is a finitely graded algebra (see [34]), then A is a finitely graded algebra.

(v) If R is locally finite, A as algebra is locally finite.

(vi) If A is quasi-commutative and R is concentrated in degree 0, then $A_0 = R$.

(vii) If R is finitely presented then A is finitely presented. Indeed: by Proposition 2.10, $R = \mathbb{K}\langle t_1, \dots, t_m\rangle/I$ where

$$I = \langle r_1, \dots, r_s\rangle \tag{8}$$

is a two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m\rangle$ generated by a finite set $r_1, \dots, r_s$ of homogeneous polynomials in $\mathbb{K}\langle t_1, \dots, t_m\rangle$. Then

$$A = \mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle/J,$$

where

$$J = \langle r_1, \dots, r_s,\ f_{hk},\ g_{ji} \mid 1 \leq i, j, h \leq n,\ 1 \leq k \leq m\rangle \tag{9}$$

is the two-sided ideal of $\mathbb{K}\langle t_1, \dots, t_m, x_1, \dots, x_n\rangle$ generated by a finite set of homogeneous elements $r_1, \dots, r_s$, f_{hk}, g_{ji} where $r_1, \dots, r_s$ are as in (8);

$$f_{hk} := x_h t_k - \sigma_h(t_k)x_h - \delta_h(t_k) \tag{10}$$

with σ_h and δ_h as in Proposition 2.2;

$$g_{ji} := x_j x_i - c_{i,j} x_i x_j - (r_{0_{j,i}} + r_{1_{j,i}} x_1 + \cdots + r_{n_{j,i}} x_n) \tag{11}$$

as in (2) of Definition 2.1.

(viii) The class of graded iterated Ore extensions is strictly contained in the class of graded skew PBW extensions (see [36], Remark 2.11).

3. Artin-Schelter regular algebras

Let B be a ring and let M be a B-module. We let $\mathrm{pd}_B(M)$ denote the projective dimension of M and $\mathrm{injdim}_B(M)$ the injective dimension of M. Let $\mathrm{lgld}(B)$ ($\mathrm{rgld}(B)$) be the left (right) global dimension of B. Not always the left and right global dimensions of B are equal. However, if the ring B is Noetherian the equality holds. If B is Noetherian and if $\mathrm{injdim}({}_BB) < \infty$ and $\mathrm{injdim}(B_B) < \infty$, then $\mathrm{injdim}({}_BB) = \mathrm{injdim}(B_B)$. We say that B has *finite global dimension* (resp. finite injective dimension) if the left and right global dimensions of B are finite and equal (resp. the modules ${}_BB$ and B_B have finite injective dimensions which are equal). In such case we denote these numbers by $\mathrm{gld}(B)$ (resp. $\mathrm{injdim}(B)$).

Let M be a B-module. The *grade number* of M is $j_B(M) := \min\{p \mid \mathrm{Ext}^p_B(M,B) \neq 0\}$ or ∞ if no such p exists. Notice that $j_B(0) = +\infty$. When B is Noetherian, $j_B(M) \leq \mathrm{pd}_B(M)$, and if furthermore $\mathrm{injdim}(B) = q < \infty$, we have $j_B(M) \leq q$ for all non-zero finitely generated B-module M (see [15]). A graded ring B has finite graded injective dimension q if ${}_BB$ and B_B are both of injective dimension q in the category of graded B-modules. We then write $\mathrm{grinjdim}(B) = q$. If M and N are graded B-modules, we use $\underline{\mathrm{Hom}}^d_B(M,N)$ to denote the set of all B-module homomorphisms $h : M \to N$ such that $h(M_i) \subseteq N_{i+d}$. We set $\underline{\mathrm{Hom}}_B(M,N) = \bigoplus_{d\in\mathbb{Z}} \underline{\mathrm{Hom}}^d_B(M,N)$, and we denote the corresponding derived functors by $\underline{\mathrm{Ext}}^i_B(M,N)$. Given any graded B-module M, for the graded case, the *grade number* (j-number) of M is $j_B(M) = \min\{p \mid \underline{\mathrm{Ext}}^p_B(M,B) \neq 0\}$ or ∞ if no such p exists. In particular, if $M = 0$, then $j_B(M) = 0$. For finitely graded algebras, we have two additional remarks: Let B be a finitely graded algebra and let M, N be $\mathbb{Z}$-graded B-modules. Then there is a natural inclusion $\underline{\mathrm{Hom}}_B(M,N) \hookrightarrow \mathrm{Hom}_B(M,N)$. If M is a B-module finitely generated, then $\underline{\mathrm{Hom}}_B(M,N) \cong \mathrm{Hom}_B(M,N)$ and $\underline{\mathrm{Ext}}^i_B(M,N) \cong \mathrm{Ext}^i_B(M,N)$.

Definition 3.1 ([15], Definition 2.1)**.** Let B be a Noetherian ring.

(i) A B-module M satisfies the *Auslander-condition*, if $\forall p \geq 0$, $j_B(N) \geq p$ for all B-submodules N of $\mathrm{Ext}^p_B(M,B)$.

(ii) The ring B is said to be *Auslander-Gorenstein* of dimension q, if $\mathrm{injdim}(B) = q < \infty$, and every left or right finitely generated B-module satisfies the Auslander-condition.

(iii) The ring B is said to be *Auslander-regular* of dimension q if $\mathrm{gld}(B) = q < \infty$ and every left or right finitely generated B-module satisfies the Auslander-condition.

(iv) Let B be an algebra. If $\mathrm{GKdim}(B) = j_B(M)+\mathrm{GKdim}(M)$ for every non-zero Noetherian B-module M, then B is called *Cohen-Macaulay.*

For the case of graded modules, in Definition 3.1, one can define the notion of a *graded-Auslander-Gorenstein* ring, or *graded-Auslander-regular* ring. The Noetherian graded ring B is Auslander-Gorenstein (resp. regular) if and only if B is graded-Auslander-Gorenstein (resp. regular) (see [15], Theorem 3.1). Let B be a graded Noetherian ring and let $\mathrm{grgld}(B)$ and $\mathrm{grinjdim}(B)$ be a graded global dimension and graded injective dimension of B respectively. It is known that $\mathrm{gld}(B)$ (resp. $\mathrm{injdim}(B)$) is finite if and only if $\mathrm{grgld}(B)$ (resp. $\mathrm{grinjdim}(B)$) is finite. Furthermore we have bounds: $\mathrm{grinjdim}(B) \leq \mathrm{injdim}(B) \leq \mathrm{grinjdim}(B) + 1$ (see [15], p. 281). Let B be a positively graded Noetherian ring. Then $\mathrm{injdim}(B) < \infty$ if and only if $\mathrm{grinjdim}(B) < \infty$, in which case these two numbers are equal (see [15], Lemma 3.3). When B is graded, one can define a *graded Cohen-Macaulay* property by taking $M \neq 0$ as a graded finitely generated B-module.

Definition 3.2. Let $B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$ be a finitely presented graded algebra over a field $\mathbb{K}$. The algebra B will be called *AS-regular* if it has the following properties:

(i) B has finite global dimension d.

(ii) B has finite Gelfand-Kirillov dimension (GK-dimension).

(iii) B is *Gorenstein*, meaning that $\mathrm{Ext}^i_B(\mathbb{K}, B) = 0$ if $i \neq d$, and $\mathrm{Ext}^d_B(\mathbb{K}, B) \cong \mathbb{K}(l)$, for some integer l.

***Remark* 3.3.** Let B be an algebra.

(i) If B is a graded right Noetherian algebra and B_0 is finite dimensional, then B is locally finite ([35], p. 1).

(ii) ([35], Theorem 2.4) Every connected graded right Noetherian algebra with finite global dimension has finite GK-dimension.

Proposition 3.4 ([19], Theorem 4.2)**.** *Let A be a bijective skew PBW extension of a ring R. Then* $\mathrm{lgld}(A) \leq \mathrm{lgld}(R) + n$*, if* $\mathrm{lgld}(R) < \infty$*. If A is quasi-commutative, then* $\mathrm{lgld}(A) = \mathrm{lgld}(R) + n$*.*

Proposition 3.5. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a graded skew PBW extension of a connected algebra R.*

(i) *If R is Noetherian with graded finite global dimension, then A has graded finite global dimension and finite GK-dimension.*

(ii) *If R is graded Auslander-Gorenstein and graded Cohen-Macaulay then A is graded Cohen-Macaulay.*

(iii) *If R is finitely presented connected Auslander-regular, then A is AS-regular.*

Proof. (i) Since A is a bijective skew PBW extension, then by Proposition 2.7-(i) we have that A is a Noetherian algebra. By Proposition 3.4 we have that $\operatorname{lgld}(A) < \infty$. Now, by Remark 2.11-(ii) we have that A is a connected algebra. So, by Remark 3.3-(ii) we have that GK-dimension of A is finite.

(ii) Since A is bijective, R is a $\mathbb{N}$-graded algebra, connected and each σ_i is graded, i.e., $\sigma_i(R_m) \subseteq R_m$ for each $m \geq 0$ and $1 \leq i \leq n$, then by [20], Theorem 3.9, we have that A is Cohen-Macaulay.

(iii) Since A is bijective, then by [20], Theorem 2.9, we have that A is Auslander-regular. Now, since R is Noetherian, then by Proposition 2.7-(i) we have that A is Noetherian. Therefore, from [15] Theorem 3.1, we have that A is graded Auslander-regular. From part (i) above, we have that A has graded finite global dimension and finite GK-dimension. Then by [15], Theorem 6.3, we have that A is AS-regular.

☑

Theorem 3.6. *Let R be an AS-regular algebra and let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ be a graded quasi-commutative skew PBW extension. Then A is AS-regular.*

Proof. Let $R = \mathbb{K} \bigoplus R_1 \bigoplus R_2 \bigoplus \cdots$ be an AS-regular algebra and let A be a graded skew PBW extension of A. Since R is finitely presented, then by Proposition 2.10, we know that $R = \mathbb{K}\langle t_1, \ldots, t_m\rangle/I$, where I is a proper two-sided ideal of $\mathbb{K}\langle t_1, \ldots, t_m\rangle$ generated by finite homogeneous polynomials $r_1, \ldots, r_s$ in $\mathbb{K}\langle t_1, \ldots, t_m\rangle$ (it is assumed that t_j has grade 1, $1 \leq j \leq m$). Then $A = \mathbb{K}\langle t_1, \ldots, t_m, x_1, \ldots, x_n\rangle/J$, where J is a two-sided ideal of $\mathbb{K}\langle t_1, \ldots, t_m, x_1, \ldots, x_n\rangle$, generated by a finite set of homogeneous polynomials $r_1, \ldots, r_s$, f_{hk} and g_{ji}, where the polynomials f_{hk} are as in (10) and g_{ji} are as in (11). Now, by Remark 2.11-(ii), we have that A is connected. So, by Proposition 2.4 and Remark 2.11-(vii), we know that $A = \mathbb{K} \bigoplus A_1 \bigoplus A_2 \bigoplus \cdots$ is a finitely presented graded algebra.

(i) Since R has finite global dimension, say e, then by Proposition 3.4 we know that $\operatorname{lgld}(A) = e + n = d$, i.e., A has finite global dimension.

(ii) Let V be a subspace of R generated by $\{t_1, \ldots, t_m\}$. Note that V is a finite dimensional generating subspace of R. As σ_i is graded for all i, then $\sigma_n(V) \subseteq V$. Now, as A is bijective and R has finite GK-dimension then by [23], Theorem 14, we have that $\operatorname{GKdim}(A) = \operatorname{GKdim}(R) + n$, i.e., A has finite GK-dimension.

(iii) From Proposition 2.7-(ii) and his proof, we know that A is isomorphic to a graded iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where θ_i is bijective, for each i, $\theta_1 = \sigma_1$, $\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to$

$R[z_1;\theta_1]\cdots[z_{j-1};\theta_{j-1}]$ is such that $\theta_j(z_i)=c_{i,j}z_i$, $c_{i,j}\in\mathbb{K}\setminus\{0\}$ and $\theta_i(r)=\sigma_i(r)$, for $r\in R$. Then $z_1r=\theta_1(r)z_1=\sigma_1(r)z_1\in Rz_1$ and $rz_1=z_1\theta_1^{-1}(r)=z_1\sigma_1^{-1}(r)\in z_1R$. Hence $z_1\in A_1$ is a nonzero normal element of $A^{(1)}:=R[z_1;\theta_1]$ and $A^{(1)}/\langle z_1\rangle=R$. From the Rees lemma (see [15], Proposition 3.4-(b)) we have that

$$\operatorname{Ext}^j_{A^{(1)}}(\mathbb{K},A^{(1)})\cong\operatorname{Ext}^{j-1}_{A^{(1)}/\langle z_1\rangle}(\mathbb{K},A^{(1)}/\langle z_1\rangle)=\operatorname{Ext}^{j-1}_R(\mathbb{K},R).$$

By Proposition 3.4, we have that $d_1:=\operatorname{gld}(A^{(1)})=e+1$. Since R is Gorenstein then $\operatorname{Ext}^i_R(\mathbb{K},R)=0$, if $i\neq e$, and $\operatorname{Ext}^e_R(\mathbb{K},R)\cong\mathbb{K}$, i.e., $\operatorname{Ext}^{i+1}_{A^{(1)}}(\mathbb{K},A^{(1)})=0$ if $i+1\neq e+1=d_1$ and $\operatorname{Ext}^{d_1}_{A^{(1)}}(\mathbb{K},A^{(1)})\cong\mathbb{K}$. Then $A^{(1)}=R[z_1;\theta_1]$ is Gorenstein. Now, $z_2\in A_1$ is a nonzero normal element of $A^{(2)}:=A^{(1)}R[z_2;\theta_2]=R[z_1;\theta_1][z_2;\theta_2]$ and $A^{(2)}/\langle z_2\rangle=A^{(1)}$. Thus, with the above procedure we have that $R[z_1;\theta_1][z_2;\theta_2]$ is Gorenstein. Now, $z_n\in A_1$ is a nonzero normal element of $A^{(n)}:=A^{(n-1)}R[z_n;\theta_n]=R[z_1;\theta_1]\cdots[z_{n-1};\theta_{n-1}][z_n;\theta_n]=A$, $A/\langle z_n\rangle=A^{(n-1)}$, $\operatorname{lgld}(A^{(n-1)})=e+n-1:=d_{n-1}$ and $\operatorname{lgld}(A^{(n)})=\operatorname{lgld}(A^{(n)})=e+n:=d_n=d$. Assuming that $A^{(n-1)}$ is Gorenstein, we have that $\operatorname{Ext}^{i-1}_{A^{(n-1)}}(\mathbb{K},A^{(n-1)})=0$ if $i-1\neq e+n-1$ and $\operatorname{Ext}^{e+n-1}_{A^{(n-1)}}(\mathbb{K},A^{(n-1)})\cong\mathbb{K}$. From the Rees lemma, we have that $\operatorname{Ext}^i_{A^{(n)}}(\mathbb{K},A^{(n)})=\operatorname{Ext}^i_A(\mathbb{K},A)=0$ if $i\neq e+n=d$ and $\operatorname{Ext}^d_A(\mathbb{K},A)\cong\mathbb{K}$. Thus $A^{(n)}:=R[z_1;\theta_1]\cdots[z_n;\theta_n]\cong A$ is Gorenstein.

Therefore, A is an AS-regular algebra. ☑

4. Calabi-Yau algebras

The *enveloping algebra* of an algebra B is the tensor product $B^e=B\otimes B^{op}$, where B^{op} is the opposite algebra of B. Bimodules over B are essentially the same as modules over the enveloping algebra of B, so in particular, B and M can be considered as B^e-modules. Suppose that M and N are both B^e-modules. Then there are two B^e-module structures on $M\otimes N$, one is called the outer structure defined by $(a\otimes b)\cdot(m\otimes n)=^{out} am\otimes nb$, and the other is called the inner structure defined by $(a\otimes b)\cdot(m\otimes n)=^{int} ma\otimes bn$, for any $a,b\in B$, $m\in M$, $n\in N$. Since B^e is identified with $B\otimes B$ as a $\mathbb{K}$-module (${}_{\mathbb{K}}B^e={}_{\mathbb{K}}(B\otimes B^{op})={}_{\mathbb{K}}(B\otimes B)$), $B\otimes B$ endowed with the outer structure is nothing but the left regular B^e-module B^e. ${}_{B^e}(B\otimes B)=^{out}{}_{B^e}B^e$: In ${}_{B^e}(B\otimes B)$, $(a\otimes b)\cdot(x\otimes y)=a\cdot(x\otimes y)\cdot b=^{out} ax\otimes yb$, whereas that in ${}_{B^e}B^e$ $(a\otimes b)\cdot(x\otimes y)=ax\otimes b\circ y=ax\otimes yb$. $B\otimes B$ endowed with the inner structure is nothing but the right regular B^e-module B^e. ${}_{B^e}(B\otimes B)=^{int}B^e_{B^e}$: In ${}_{B^e}(B\otimes B)$, $(a\otimes b)\cdot(x\otimes y)=a\cdot(x\otimes y)\cdot b=^{int} xa\otimes by$, whereas that in $B^e_{B^e}$, $(x\otimes y)\cdot(a\otimes b)=xa\otimes y\circ b=xa\otimes by$. Hence, we often say B^e has the outer (left) and inner (right) B^e-module structures.

An algebra B is said to be *homologically smooth*, if as a B^e-module, B has a projective resolution that has finite length and is such that each term in the projective resolution is finitely generated. The length of this resolution is known as the *Hochschild dimension* of B. In the next definition, the outer structure on B^e is used when computing the homology $\operatorname{Ext}^*_{B^e}(B, B^e)$. Thus, $\operatorname{Ext}^*_{B^e}(B, B^e)$ admits a B^e-module structure induced by the inner one on B^e.

Let M be a B-bimodule, ν, $\mu : B \to B$ be two automorphisms, the skew B-bimodule ${}^{\nu}M^{\mu}$ is equal to M as a vector $\mathbb{K}$-space with $a \cdot m \cdot b := \nu(a) \cdot m \cdot \mu(b)$. Thus, M is a left B^e-module with product given by

$$(a \otimes b) \cdot m = a \cdot m \cdot b = \nu(a) \cdot m \cdot \mu(b).$$

In particular, for B and B^e we have the structure of left B^e-modules given by

$$(a \otimes b) \cdot x = \nu(a)x\mu(b),$$

$$(a \otimes b) \cdot (x \otimes y) = a \cdot (x \otimes y) \cdot b = \nu(a) \cdot (x \otimes y) \cdot \mu(b) = \nu(a)x \otimes y\mu(b).$$

Definition 4.1. A graded algebra B is called *skew Calabi-Yau* of dimension d if

(i) B is homologically smooth.

(ii) There exists an algebra automorphism ν of B such that

$$\operatorname{Ext}^i_{B^e}(B, B^e) \cong \begin{cases} 0, & i \neq d; \\ B^{\nu}(l), & i = d. \end{cases}$$

as B^e -modules, for some integer l. If ν is the identity, then B is said to be *Calabi-Yau*.

Ungraded Calabi-Yau algebras are defined similarly but without degree shift. The automorphism ν is called the *Nakayama* automorphism of B, and is unique up to inner automorphisms of B. Note that a skew Calabi-Yau algebra is Calabi-Yau if and only if its Nakayama automorphism is inner.

Proposition 4.2 ([33], Lemma 1.2)**.** *Let B be a connected graded algebra. Then B is skew Calabi-Yau if and only if it is AS-regular.*

A graded algebra is quadratic if $B = T(V)/\langle R\rangle$ where V is a finite dimensional $\mathbb{K}$-vector space, concentrated in degree l; $T(V)$ is the tensor algebra on V, with the induced grading, and $\langle R\rangle$ is the ideal generated by a subspace $R \subseteq V \otimes V$. The *dual* of such a quadratic algebra is $B^! := T(V^*)/\langle R^{\perp}\rangle$, where

$$R^{\perp} = \{\lambda \in V^* \otimes V^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

We identify $(V \otimes V)^*$ with $V^* \otimes V^*$ by defining $(\alpha \otimes \beta)(u \otimes v) := \alpha(u)\beta(v)$ for $\alpha, \beta \in V^*$ and $u, v \in V$.

Let $B = \mathbb{K} \bigoplus B_1 \bigoplus B_2 \bigoplus \cdots$ be a locally finite graded algebra and $E(B) = \bigoplus_{s,p} E^{s,p}(B) = \bigoplus_{s,p} \mathrm{Ext}_B^{s,p}(\mathbb{K},\mathbb{K})$ the associated bigraded Yoneda algebra, where s is the cohomology degree and $-p$ is the internal degree inherited from the grading on A. Let $E^s(B) = \bigoplus_p E^{s,p}(B)$. B is called *Koszul* if the following equivalent conditions hold:

(i) $\mathrm{Ext}_B^{s,p}(\mathbb{K},\mathbb{K}) = 0$ for $s \neq p$;

(ii) B is one-generated and the algebra $\mathrm{Ext}_B^*(\mathbb{K},\mathbb{K})$ is generated by $\mathrm{Ext}_B^1(\mathbb{K},\mathbb{K})$, i.e., $E(B)$ is generated in the first cohomological degree;

(iii) The module $\mathbb{K}$ admits a *linear free resolution*, i.e., a resolution by free B-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to \mathbb{K} \to 0$$

such that P_i is generated in degree i.

(iv) $\mathrm{Ext}^*(\mathbb{K},\mathbb{K}) \cong B^!$ as graded algebras.

Theorem 4.3 ([36], Theorem 5.5)**.** *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a graded skew PBW extension of a finitely presented Koszul algebra R, then A is Koszul.*

Proposition 4.4. *Let R be a Koszul AS-regular algebra of global dimension d with Nakayama automorphism σ.*

(i) ([13], Theorem 3.3) *The skew polynomial algebra $B = R[x;\sigma]$ is a Calabi-Yau algebra of dimension $d+1$.*

(ii) ([41], Remark 3.13) *There exists a unique skew polynomial extension B such that B is Calabi-Yau.*

(iii) ([41], Theorem 3.16) *If ν is a graded algebra automorphism of R, then $B = R[x;\nu]$ is Calabi-Yau if and only if $\sigma = \nu$.*

Note that the Calabi-Yau property is not preserved by skew PBW extensions. The Jordan plane $A = \mathbb{K}\langle x, y\rangle / \langle yx - xy - x^2\rangle \cong \sigma(\mathbb{K}[x])\langle y\rangle = \mathbb{K}[x][y;\sigma,\delta]$, where $\sigma(x) = x$ and $\delta(x) = x^2$, is a graded skew PBW extension of a Calabi-Yau algebra $\mathbb{K}[x]$, but A is not Calabi-Yau. Indeed: the Nakayama automorphism ν of the Jordan plane is given by $\nu(x) = x$ and $\nu(y) = 2x + y$ (see for example [21], Page 16) and this is not inner.

The Calabi-Yau and skew Calabi-Yau properties for graded skew PBW extensions will be next proved using the cited results presented in the literature and our previous results.

Theorem 4.5. *Let $A = \sigma(R)\langle x_1, \cdots, x_n\rangle$ be a graded skew PBW extension of an algebra R.*

(i) *If A is quasi-commutative and R is a connected skew Calabi-Yau algebra of global dimension d, then A is skew Calabi-Yau of global dimension $d+n$. Moreover, if R is Koszul and θ_i is the Nakayama automorphism of $R[x_1;\theta_1]\cdots[x_{i-1};\theta_{i-1}]$ for $1 \leq i \leq n$, then A is Calabi-Yau of dimension $d+n$ (θ_i as in Proposition 2.7-(ii), $x_0 = 1$).*

(ii) *If R is finitely presented, connected and Auslander-regular, then A is skew Calabi-Yau.*

Proof. (i) Since R is connected and skew Calabi-Yau, then by Proposition 4.2 we know that R is AS-regular. From Theorem 3.6 we have that A is AS-regular and, in particular, connected. Thus, using again Proposition 4.2, we have that A is a skew Calabi-Yau algebra. By the proof of Theorem 3.6 we have that the global dimension of A is $d+n$.

For the second part, we know that graded Ore extensions of Koszul algebras are Koszul algebras and, as a particular case of Theorem 3.6, we have that a graded Ore extension of an AS-regular algebra is an AS-regular algebra. Now, by Proposition 2.7-(ii) we have that A is isomorphic to a graded iterated Ore extension $R[x_1;\theta_1]\cdots[x_n;\theta_n]$. It is known that if A is a Calabi-Yau algebra of dimension d, then the global dimension of A is d (see for example [6], Remark 2.8). Then, using Proposition 4.4-(i) and applying induction on n we obtain that A is a Calabi-Yau algebra of dimension $d+n$.

(ii) From Proposition 3.5-(iii) we have that A is AS-regular. Since R is connected, then by Remark 2.11-(ii) we have that A is connected. Then from Proposition 4.2 we get that A is skew Calabi-Yau.

☑

Using the previous results we have the following examples of skew Calabi-Yau algebras and AS-regular algebras. For some of these algebras other authors had already studied the skew Calabi-Yau property and the Artin-Schelter regularity, but using other techniques. The novelty here consists in interpreting these algebras as skew PBW extensions and applying some its algebraic properties studied before in [19] and [36].

Example 4.6. From Theorem 3.6 we obtain that the algebra of linear partial q-dilation operators $\mathbb{K}[t_1,\dots,t_n][H_1^{(q)},\dots,H_m^{(q)}]$ (Example 2.8-2), multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$ (Example 2.8-3) and multi-parameter quantum affine n-space $\mathcal{O}_{\mathbf{q}}(\mathbb{K}^n)$ (Example 2.8-4), are AS-regular algebras. By Theorem 4.5-(i), we have that the above examples are also skew Calabi-Yau algebras.

Example 4.7. The following examples are graded skew PBW extensions of the classical polynomial ring R with coefficients in a field $\mathbb{K}$, which are not quasi-commutative and where R has the usual graduation (see [36], Example 2.9). In [8, 19] and [38] we can find further details of these algebras. By Theorem 4.5-(ii), these extensions are skew Calabi-Yau algebras, since R is a connected Auslander-regular algebra.

(1) The Jordan plane. $A = \mathbb{K}\langle x, y\rangle/\langle yx - xy - x^2\rangle \cong \sigma(\mathbb{K}[x])\langle y\rangle$.

(2) The homogenized enveloping algebra. $\mathcal{A}(\mathcal{G}) \cong \sigma(\mathbb{K}[z])\langle x_1, \dots, x_n\rangle$.

(3) The diffusion algebra 2. $A \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle D_1, \dots, D_n\rangle$.

(4) The algebra $U \cong \sigma(\mathbb{K}[x_1, \dots, x_n])\langle y_1, \dots, y_n; z_1, \dots, z_n\rangle$.

(5) Manin algebra. $\mathcal{O}(M_q(2)) \cong \sigma(\mathbb{K}[u])\langle x, y, v\rangle$.

(6) Algebra of quantum matrices. $\mathcal{O}_q(M_n(\mathbb{K})) \cong \sigma(\mathbb{K}[x_{im}, x_{jk}])\langle x_{ik}, x_{jm}\rangle$, for $1 \leq i < j, k < m \leq n$.

(7) Quadratic algebras. A quadratic algebra in 3 variables is a algebra generated by x, y, z subject to the relations

$$\begin{aligned} yx &= xy + a_1xz + a_2y^2 + a_3yz + \xi_1z^2, \\ zx &= xz + \xi_2y^2 + a_5yz + a_6z^2, \\ zy &= yz + a_4z^2. \end{aligned}$$

If $a_1 = a_4 = 0$ then the quadratic algebra is a graded skew PBW extension of $R = \mathbb{K}[y, z]$, and if $a_5 = a_3 = 0$ then quadratic algebras are graded skew PBW extensions of $R = \mathbb{K}[x, z]$.

Note that the above algebras are also AS-regular algebras (see proof of Theorem 4.5-(ii)).

It is possible that for algebras (1) - (7) above, the AS-regular and the skew Calabi-Yau properties may have not been yet studied.

***Remark* 4.8.** Every skew Calabi-Yau algebra may be extended to a Calabi-Yau algebra, i.e., if B is a skew Calabi-Yau algebra then $B[z, \sigma]$ is Calabi-Yau, where σ is the Nakayama automorphism (see [11], Theorem 1.1 and Remark 5.1).

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