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A convergent iterative method for a logistic chemotactic system

Un método iterativo convergente para un sistema logístico quimiotáctico

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Abstract. In this paper we study a nonlinear system of differential equations arising in chemotaxis. The system consists of a PDE that describes the evolution of a population and another which models the concentration of a chemical substance. In particular, we prove the existence and uniqueness of nonnegative solutions via an iterative method. First, we generate a Cauchy sequence of approximate solutions from a linear modification of the original system. Next, some uniform bounds on the solutions are used to find a subsequence that converges weakly to the solution of the original system.

Key words and phrases. reaction-diffusion equations, weak solution, convergence.

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Resumen. En este artículo estudiamos un sistema no lineal de ecuaciones diferenciales que aparecen en quimiotaxis. El sistema consiste de una EDP que describe la evolución de una población y otra que modela la concentración de una sustancia química. En particular, probamos la existencia y unicidad de soluciones no negativas vía un método iterativo. Primero generamos una sucesión de Cauchy de soluciones aproximadas a partir de una modificación lineal del sistema original. Luego, algunas cotas uniformes de las soluciones son usadas para encontrar una subsucesión débilmente convergente a la solución del sistema original.

Palabras y frases clave. ecuaciónes de reacción-difusión, solución débil, convergencia.

1. Introduction

Chemotaxis systems have received considerable attention because they describe several biological phenomena such as leukocyte movement, self-organization during embryonic development, wound healing and cancer growth [8, 9]. These are phenomena where a population of cells moves towards a chemical signal emitted by a substance, or another population, called chemoattractant. Various forms of the system and boundary condition have been studied (cf. [5, 3, 6, 12]).

Of special interest is the following Chemotaxis system:

$$\partial_t c - D_c \triangle c = \frac{s\rho}{\beta + \rho} - \gamma c, \qquad \text{in } \Omega \times (0,T), \tag{1}$$

$$\partial_t \rho - D_\rho \triangle \rho + \alpha \nabla \cdot (\rho \nabla c) = r\rho(\rho_\infty - \rho), \qquad \text{in } \Omega \times (0,T), \tag{2}$$

$$\frac{\partial c}{\partial \eta} = 0, \qquad \frac{\partial \rho}{\partial \eta} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{3}$$

$$c(x,0) = c_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad \text{on } \Omega. \tag{4}$$

where $\Omega \subset \mathbb{R}^N, (N = 1,2,3)$ is a bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial\eta$ denotes the derivative with respect to the outer normal of $\partial\Omega$ and $T > 0$ is a fixed time.

The above problem arises from the study of pattern formation on animal coats, where pigment cells both respond to and produce their own chemoattractant [11, 10, 7]. In the biological interpretation $\rho = \rho(\mathbf{x}, t)$ and $c = c(\mathbf{x}, t)$ represent the pigment cell density and the chemoattractant concentration respectively at position $\mathbf{x}$ and time t. The constants D_ρ and D_c are the cells and chemoattractant diffusion coefficient respectively, and α is the chemotaxis coefficient. It is assumed that cell population grows logistically where $r\rho_\infty$ is the linear mitotic growth rate with r and ρ_∞ both nonnegative constants. The chemoattractant production by the cells is given by a simple Michaelis-Menten kinetics and its consumption is linear. The constants s, β and γ are nonnegative.

Concerning to the well-posedness of the system (1)-(4) many advances have been done in the recent years [13, 2] and [1]. Specially, in [1] is proven the existence and uniqueness of classical solution for all positive values of α, ρ_∞ and r. The proof uses semigroup techniques, parabolic Schauder estimates and contraction arguments.

The aim of this paper is to get the local-in-time existence and uniqueness of a weak solution to (1)-(4) in one, two and three dimensions with proper assumptions on the initial data. Before stating our main results, we give the definition of a weak solution.

Definition 1.1. A weak solution of (1) - (4) is a pair (c, ρ) of functions satisfying the following conditions, $c(\mathbf{x}, t) \geq 0$ and $\rho(\mathbf{x}, t) \geq 0$, for a.e $(\mathbf{x}, t) \in$

$\Omega \times (0,T)$,

$$c, \rho \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$$
$$\partial_t c, \partial_t \rho \in L^2(0,T; L^2(\Omega))$$

and for all $\phi \in H^1(\Omega)$,

$$\int_\Omega \partial_t c\ \phi\ dx + \int_\Omega D_c \nabla c \nabla \phi\ dx + \int_\Omega \gamma c\ \phi\ dx = \int_\Omega \Big(\frac{s\rho}{\beta+\rho}\Big)\phi\ dx, \tag{5}$$
$$\int_\Omega \partial_t \rho\ \phi\ dx + \int_\Omega D_\rho \nabla \rho \nabla \phi\ dx - \int_\Omega \alpha(\rho\nabla c)\nabla \phi\ dx = \int_\Omega r\rho(\rho_\infty - \rho)\phi\ dx, \tag{6}$$

a.e. in $[0,T]$.

The main result is the following existence and uniqueness theorem for weak solutions.

Theorem 1.2. *If $c_0, \rho_0 \in H^3(\Omega)$ with $0 \le c_0$ and $0 \le \rho_0 \le \rho_\infty$ in Ω, then there exists $T > 0$ such that the system (1) - (4) has a unique weak solution in the sense of Definition 1.1. Furthermore, c and ρ belong to the space $L^2(0,T; H^4(\Omega)) \cap L^\infty(0,T; H^3(\Omega))$.*

Our proof is based on generate a convergent sequence of approximate solutions of the nonlinear system (1)-(4). To this aim, we perform a successive substitution strategy, such that the nonlinear system (1)-(4) is replaced by a sequence of linear partial differential equations.

We start taking as initial value of the iteration the weak solutions $c^0, \rho^0 \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ of the homogeneous system

$$\begin{cases} \partial_t c^0 - D_c \triangle c^0 + \gamma c^0 = 0, & \text{in } \Omega \times (0,T), \\ \dfrac{\partial c^0}{\partial \eta} = 0, \text{ on } \partial\Omega \times (0,T), & c^0(x,0) = c_0(x), \text{ for } x \in \Omega. \end{cases} \tag{7}$$

$$\begin{cases} \partial_t \rho^0 - D_\rho \triangle \rho^0 + \alpha \nabla \cdot (\rho^0 \nabla c^0) = 0 & \text{in } \Omega \times (0,T) \\ \dfrac{\partial \rho^0}{\partial \eta} = 0 \text{ on } \partial\Omega \times (0,T), & \rho^0(x,0) = \rho_0(x) \text{ for } x \in \Omega. \end{cases} \tag{8}$$

In addition, for $k \in \mathbb{N}_0$ let $c^{k+1}, \rho^{k+1} \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ be the weak solutions to the nonhomogeneous system

$$\begin{cases} \partial_t c^{k+1} - D_c \triangle c^{k+1} + \gamma c^{k+1} = \dfrac{s\rho^k}{\beta + \rho^k}, & \text{in } \Omega \times (0,T), \\ \dfrac{\partial c^{k+1}}{\partial \eta} = 0 \text{ on } \partial\Omega \times (0,T), & c^{k+1}(x,0) = c_0(x) \text{ for } x \in \Omega, \end{cases} \tag{9}$$

$$\begin{cases} \partial_t \rho^{k+1} \; - D_\rho \triangle \rho^{k+1} + \alpha \nabla \cdot (\rho^{k+1} \nabla c^{k+1}) = r\rho^k(\rho_\infty - \rho^k), \;\; \text{in } \Omega \times (0,T), \\ \dfrac{\partial \rho^{k+1}}{\partial \eta} = 0, \; \text{on } \partial\Omega \times (0,T), \qquad \rho^{k+1}(x,0) = \rho_0(x) \text{ for } x \in \Omega. \end{cases} \tag{10}$$

To prove theorem 1.2 we first prove existence and uniqueness of weak solutions to the homogeneous problems (7) and (8) by applying the standard theory for linear PDE. These solutions c^0 and ρ^0 are sufficient regular, that the standard theory for linear PDE guarantee the existence and uniqueness of the successive iterates (c^k, ρ^k) $k = 1, 2, \ldots$. Next, we show that the generated solutions sequence is a bounded Cauchy sequence, and its limit is the solution of (1)-(4).

2. Detail of Proof

Lemma 2.1. (Properties of iterative Sequence). *Under the assumptions of theorem 1.2, there exists $T > 0$ such that:*

(i) There exists a unique weak solution to the system (7)-(8) and (9)-(10) with conditions (3) and (4) and for every $k \in \mathbb{N}_0$ it holds that

$$c^k, \rho^k \in L^2(0,T; H^4(\Omega)) \cap L^\infty(0,T; H^3(\Omega)), \tag{11}$$

$$\partial_t c^k, \partial_t \rho^k \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)). \tag{12}$$

For adequate constants $\mathcal{C}(\Omega, T)$ the following estimates are satisfied

$$\begin{aligned} &\underset{t\in[0,T]}{\text{ess sup}} \Big[\|\partial_t c^k\|_{H^1(\Omega)} + \|c^k\|_{H^3(\Omega)}\Big] + \|c^k\|_{L^2(0,T;H^4(\Omega))} + \|\partial_t c^k\|_{L^2(0,T;H^2(\Omega))} \\ &\qquad \le \mathcal{C}(\Omega,T)\Big[\|c_0\|_{H^3\Omega)} + \|f\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t f\|_{L^2(0,T;L^2(\Omega))}\Big], \end{aligned} \tag{13}$$

$$\begin{aligned} &\underset{t\in[0,T]}{\text{ess sup}} \Big[\|\partial_t \rho^k\|_{H^1(\Omega)} + \|\rho^k\|_{H^3(\Omega)}\Big] + \|\rho^k\|_{L^2(0,T;H^4(\Omega))} + \|\partial_t \rho^k\|_{L^2(0,T;H^2(\Omega))} \\ &\qquad \le \mathcal{C}(\Omega,T)\Big[\|\rho_0\|_{H^3(\Omega)} + \|g\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t g\|_{L^2(0,T;L^2(\Omega))}\Big]. \end{aligned} \tag{14}$$

(ii) The functions ρ^k, c^k satisfy for all $k \in \mathbb{N}_0$, the following inequalities

$$0 \le c^k(\mathbf{x}, t), \qquad 0 \le \rho^k(\mathbf{x}, t) \le \rho_\infty \;\; \textit{for a.e } \mathbf{x} \in \Omega, t \in (0,T) \tag{15}$$

Proof. The proof is by induction on k.

Verification for k=0: We prove, that the lemma holds for the system (7)-(8). If we write $c^0(\mathbf{x}, t) = u(\mathbf{x}, t)e^{-\gamma t}$, then

$$(\partial_t u - \gamma u)e^{-\gamma t} = D_c e^{-\gamma t} \triangle u - \gamma u e^{-\gamma t}$$

which simplifies to

$$\partial_t u = D_c \triangle u.$$

Hence, $c^0(\mathbf{x},t)$ equals some solution $u(\mathbf{x},t)$ of the diffusion solution, multiplied by an exponentially decay term. Since $c_0 \in H^3(\Omega)$ and the compatibility conditions are fulfilled trivially, the regularity theory of linear parabolic equations [4] implies that

$$c^0 \in L^2(0,T;H^4(\Omega)) \cap L^\infty(0,T;H^3(\Omega)) \tag{16}$$

$$c^0_t \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \tag{17}$$

and

$$\begin{aligned} \operatorname*{ess\,sup}_{t\in[0,T]} \Big[\|c^0_t\|_{H^1(\Omega)} + \|c^0\|_{H^3(\Omega)}\Big] + \|c^0\|_{L^2(0,T;H^4(\Omega))} \\ + \|c^0_t\|_{L^2(0,T;H^2(\Omega))} \le \mathcal{C}(\Omega,T)\|c_0\|_{H^3(\Omega)}. \end{aligned} \tag{18}$$

That $c^0(\mathbf{x},t) \geq 0$ a.e in $\Omega \times (0,T)$ follows from the maximum principle for the diffusion equation.

To prove that ρ^0 satisfies the lemma, we start writing the equation (8) as follows

$$\frac{\partial \rho^0}{\partial t} - D_\rho \triangle \rho^0 + \alpha \nabla c^0 \cdot \nabla \rho^0 + \alpha \triangle c^0 \ \rho^0 = 0. \tag{19}$$

Since c^0 is known, equation (19) is linear. To show existence and uniqueness of $\rho^0 \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ it is sufficient to see that

(a) The coefficients $\alpha\nabla c^0$ and $\alpha \triangle c^0$ belong to $L^\infty(\Omega_T)$.

and

(b) There exist some $\mu > 0$ and $\kappa \geq 0$ such that for all $0 \leq t \leq T$

$$\mu\|\rho^0\|^2_{H^1(\Omega)} \le B[\rho^0,\rho^0;t] + \kappa\|\rho^0\|^2_{L^2(\Omega)}. \tag{20}$$

Where $B[\rho,v,t]$ denotes the bilinear form

$$B[\rho,v;t] := \int_\Omega \Big(D_\rho \nabla\rho^0 \ \nabla v + \alpha\nabla c^0 \cdot \nabla\rho^0 \ v + \alpha \ \triangle c^0 \ \rho^0 \ v\Big) d\mathbf{x}. \tag{21}$$

for $\rho, v \in H^1(\Omega)$, a.e. $0 \leq t \leq T$.

Item (a) follows from the fact that $c^0 \in L^2(0,T;H^4(\Omega)) \cap L^\infty(0,T;H^3(\Omega))$ and the Sobolev embedding of $H^2(\Omega)$ in $C(\bar{\Omega})$ for Ω open subset of $\mathbb{R}^N$, $N = 1,2,3$.

In order to prove (b), first observe that by the uniformly elliptic property, there exists a constant $\theta > 0$ such that

$$\int_\Omega D_\rho \nabla\rho^0 \nabla\rho^0 \ge \theta\|\nabla\rho^0\|^2_{L^2(\Omega)}.$$

Furthermore, for all $\varepsilon > 0$,

$$\int_\Omega (\nabla c^0 \cdot \nabla \rho^0)\rho^0 \; d\mathbf{x} \geq -\|\nabla c^0\|_{L^\infty(\Omega_T)} \|\nabla \rho^0\|_{L^2(\Omega)} \|\rho\|_{L^2(\Omega)}$$

$$\geq -\frac{1}{2}\|\nabla c^0\|_{L^\infty(\Omega)} \left[\varepsilon \|\nabla \rho^0\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\rho^0\|^2_{L^2(\Omega)}\right] \quad (22)$$

and

$$\int_\Omega \triangle c^0 \; (\rho^0)^2 \; d\mathbf{x} \geq -\|\triangle c^0\|_{L^\infty(\Omega_T)} \|\rho^0\|^2_{L^2(\Omega)}.$$

Thus, for all $\varepsilon \in (0, 2\theta/\|\nabla c^0\|_{L^\infty(\Omega_T)})$, the inequality (20) holds, with

$$\kappa = \|\triangle c^0\|_{L^\infty(\Omega_T)} + \frac{1}{2\varepsilon}\|\nabla c^0\|_{L^\infty(\Omega_T)} \qquad \text{and} \qquad \mu = \theta - \frac{\varepsilon}{2}\|\nabla c^0\|_{L^\infty(\Omega_T)}. \quad (23)$$

Applying, the theory of linear parabolic equations in [4], we get the existence and uniqueness of the weak solution ρ^0. In addition, since the intial data ρ_0 is in $H^3(\Omega)$, theorem 7.16 in [4] implies that ρ^0 satisfies (11), (12) and (14).

The task is now to show that $\rho^0 \geq 0$. We test with $(\rho^0)^- := \min(\rho^0, 0)$ the variational formulation of (7), then

$$\frac{d}{dt}(\rho^0, (\rho^0)^-) + B[\rho^0, (\rho^0)^-; t] = 0. \quad (24)$$

After adding $\kappa\|(\rho^0)^-\|^2_{L^2(\Omega)}$ to both sides of (24), and applying property (20), we get

$$\frac{d}{dt}\|(\rho^0)^-\|^2_{L^2(\Omega)} \leq \kappa \|(\rho^0)^-\|^2_{L^2(\Omega)}. \quad (25)$$

By Gronwall's lemma, we can now deduce that

$$\|(\rho^0)^-(t)\|^2_{L^2(\Omega)} \leq \|(\rho^0)^-(0)\|^2_{L^2(\Omega)} = 0 \quad (26)$$

since $(\rho^0)^-(0) = \rho_0 \geq 0$ by assumption. Then $(\rho^0)^-(t) = 0$ almost everywhere in $\Omega \times (0, T)$, and therefore $\rho^0 \geq 0$ almost everywhere in $\Omega \times (0, T)$.

To show the upper bound of ρ^0, we use the same trick but test now with $(\rho^0 - \rho_\infty)^+ := \max(\rho^0 - \rho_\infty, 0)$. As ρ_∞ is a constant we have $\partial_t \rho_\infty = \nabla \rho_\infty = \Delta \rho_\infty = 0$ and therefore

$$\frac{d}{dt}(\rho^0, (\rho^0 - \rho_\infty)^+) + B[\rho^0, (\rho^0 - \rho_\infty)^+; t] = 0$$

is equivalent to

$$\frac{1}{2}\frac{d}{dt}\|(\rho^0 - \rho_\infty)^+\|^2_{L^2(\Omega)} + B[(\rho^0 - \rho_\infty)^+, (\rho^0 - \rho_\infty)^+; t] = 0.$$

Property (20) of B implies

$$\frac{d}{dt}\|(\rho^0-\rho_\infty)^+\|^2_{L^2(\Omega)} \le \kappa\|(\rho^0-\rho_\infty)^+\|^2_{L^2(\Omega)}.$$

Now, we use Gronwall's lemma and the fact that $\rho_0 \le \rho_\infty$ to deduce

$$\|(\rho^0-\rho_\infty)^+\|^2_{L^2(\Omega)} \le \|(\rho^0(0)-\rho_\infty)^+\|^2_{L^2(\Omega)} = 0.$$

Therefore $(\rho^0-\rho_\infty)^+ = 0$ almost everywhere in $\Omega\times(0,T)$, which yields $\rho^0 \le \rho_\infty$ almost everywhere in $\Omega\times(0,T)$.

Induction hypothesis: Assume the lemma holds for k.

Induction step $(k \to k+1)$*:* By induction hypothesis $0 \le \rho^k(\mathbf{x},t) \le \rho_\infty$ for a.e $\mathbf{x}\in\Omega,\ t\in[0,T]$, then it is easy to see that the right hand sides

$$f(\rho^k(\mathbf{x},t)) := \frac{s\rho^k}{\beta+\rho^k} \quad\text{and}\quad g(\rho^k(\mathbf{x},t)) := r\rho^k(\rho_\infty-\rho^k) \tag{27}$$

of equations (9) and (10) belong to the space $L^2(0,T;L^2(\Omega))$. Indeed

$$\int_0^T \|f(\rho^k)\|^2_{L^2(\Omega)}\,dt = \int_0^T \left\|\frac{s\rho^k}{(\beta+\rho^k)}\right\|^2_{L^2(\Omega)} dt \tag{28}$$

$$= \int_0^T\int_\Omega \Big(\frac{s\rho^k}{(\beta+\rho^k)}\Big)^2 dx\,dt \tag{29}$$

$$\le \int_0^T\int_\Omega s^2\,dx\,dt \tag{30}$$

$$\le s^2|\Omega|T \tag{31}$$

and

$$\int_0^T \|g(\rho^k)\|^2_{L^2(\Omega)}\,dt = \int_0^T \left\|r\rho^k(\rho_\infty-\rho^k)\right\|^2_{L^2(\Omega)} dt \tag{32}$$

$$= \int_0^T\int_\Omega (r\rho^k(\rho_\infty-\rho^k))^2 dx\,dt \tag{33}$$

$$\le \int_0^T\int_\Omega \frac{r^2\rho_\infty^4}{16} dx\,dt \tag{34}$$

$$\le \frac{r^2\rho_\infty^4}{16}|\Omega|T. \tag{35}$$

Now the linear theory yields the existence of a unique weak solution of (9) and (10) with initial data (4) and boundary conditions (3). The solution $(c^{k+1},\rho^{k+1)})$ satisfies

$$c^{k+1},\rho^{k+1} \in L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega)).$$

In order to see that c^{k+1} and ρ^{k+1} satisfy the regularity properties (13) and estimate (14), we apply theorem 7.1.6 in [4]. Then, it is sufficient to prove that $f(\rho^k)$ and $g(\rho^k)$ belongs to the space $L^2(0,T;H^2(\Omega))$ and $\partial_t f(\rho^k), \partial_t g(\rho^k) \in L^2(0,T;L^2(\Omega))$. To this end, we observe that:

- The functions $f(x)$ and $g(x)$ in (27) are continuous differentiable functions for all $x \in \mathbb{R}_+$.
- By induction hypothesis c^k and ρ^k belong to $H^4(\Omega)$ and the Sobolev embedding $H^4(\Omega) \subset C^2(\bar{\Omega})$, we have that c^k and ρ^k are $C^2(\bar{\Omega})$ functions. Further, $\rho_\infty \geq \rho^k \geq 0$ almost everywhere in $\Omega \times (0,T)$.

Hence $f(\rho^k(\mathbf{x},t))$ and $g(\rho^k(\mathbf{x},t))$ belong to $H^2(\Omega)$ a.e. $t \in [0,T]$ and

$$\int_0^T \|f(\rho^k)\|^2_{H^2(\Omega)}\, dt < \infty \quad \text{and} \int_0^T \|g(\rho^k)\|^2_{H^2(\Omega)}\, dt < \infty$$

i.e., $f(\rho^k(\mathbf{x},t)), g(\rho^k(\mathbf{x},t)) \in L^2(0,T;H^2(\Omega))$.

In addition, $\partial_t f(\rho^k) \in L^2(0,T;L^2(\Omega))$ since

$$\int_0^T \|\partial_t f(\rho^k)\|^2_{L^2(\Omega)}\, dt = \int_0^T \left\| \frac{s\beta}{(\beta+\rho^k)^2}\, \partial_t \rho^k \right\|^2_{L^2(\Omega)} dt \tag{36}$$

$$= \int_0^T \int_\Omega \Big(\frac{s\beta}{(\beta+\rho^k)^2}\, \partial_t \rho^k \Big)^2 dx\, dt \tag{37}$$

$$\leq \int_0^T \Big(\frac{s}{\beta}\Big)^2 \int_\Omega (\partial_t \rho^k)^2 dx\, dt \tag{38}$$

$$= \int_0^T \Big(\frac{s}{\beta}\Big)^2 \|\partial_t \rho^k\|^2_{L^2(\Omega)} dt \tag{39}$$

$$= \Big(\frac{s}{\beta}\Big)^2 \|\partial_t \rho^k\|^2_{L(0,T;L^2(\Omega))} \tag{40}$$

$$\leq \mathcal{C}(\Omega,T)\|\rho_0\|^2_{H^3(\Omega)}. \tag{41}$$

We next show that $\partial_t g(\rho^k)$ belongs to $L^2(0,T;L^2(\Omega))$:

$$\int_0^T \|\partial_t g(\rho^k)\|^2_{L^2(\Omega)}\, dt = \int_0^T \left\| r(\rho_\infty - 2\rho^k)\, \partial_t \rho^k \right\|^2_{L^2(\Omega)} dt \tag{42}$$

$$= \int_0^T \int_\Omega \Big(r(\rho_\infty - 2\rho^k)\, \partial_t \rho^k \Big)^2 dx\, dt \tag{43}$$

$$\tag{44}$$

$$\leq \int_0^T (r\rho_\infty)^2 \int_\Omega (\partial_t \rho^k)^2 dx\ dt \tag{45}$$

$$= \int_0^T (r\rho_\infty)^2 \|\partial_t \rho^k\|^2_{L^2(\Omega)} dt \tag{46}$$

$$= (r\rho_\infty)^2 \|\partial_t \rho^k\|^2_{L(0,T;L^2(\Omega))} \tag{47}$$

$$\leq \mathcal{C}(\Omega, T)\|\rho_0\|^2_{H^3(\Omega)}. \tag{48}$$

We now turn to show that $c^{k+1}(\mathbf{x}, t) \geq 0$. Consider the weak formulation of (9) and test with $(c^{k+1})^- :=\min(c^{k+1}, 0)$, then

$$\int_\Omega \partial_t c^{k+1}\ (c^{k+1})^-\ dx + \int_\Omega D_c \nabla c^{k+1} \nabla (c^{k+1})^-\ dx + \int_\Omega \gamma c^{k+1}\ (c^{k+1})^-\ dx = \int_\Omega \Big(\frac{s\rho^k}{\beta + \rho^k}\Big)(c^{k+1})^-\ dx.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(c^{k+1})^-|^2\ dx + D_c \int_\Omega |\nabla (c^{k+1})^-|^2\ dx + \gamma \int_\Omega |(c^{k+1})^-|^2\ dx = \int_\Omega \Big(\frac{s\rho^k}{\beta + \rho^k}\Big)(c^{k+1})^-\ dx.$$

which gives by integration in time

$$\frac{1}{2}\int_\Omega |(c^{k+1})^-(t)|^2\ dx + D_c \int_0^t \int_\Omega |\nabla (c^{k+1})^-|^2\ dx\ ds + \gamma \int_0^t \int_\Omega |(c^{k+1})^-|^2\ dx\ ds = \int_\Omega \int_0^t \Big(\frac{s\rho^k}{\beta + \rho^k}\Big)(c^{k+1})^-\ dx\ ds + \frac{1}{2}\int_\Omega |(c^{k+1})^-(0)|^2\ dx.$$

As $(c^{k+1})^-(0) = (c_0)^- = 0$ and $\rho^k \geq 0$ by induction hypothesis, we deduce

$$\frac{1}{2}\int_\Omega |(c^{k+1})^-(t)|^2\ dx + D_c \int_0^t \int_\Omega |\nabla (c^{k+1})^-|^2\ dx\ ds + \gamma \int_0^t \int_\Omega |(c^{k+1})^-|^2\ dx\ ds \leq 0,$$

that is to say that $(c^{k+1})^- = 0$ a.e in $(0, T) \times \Omega$ and therefore $c^{k+1} \geq 0$ a.e in $\Omega \times (0, T)$.

***Remark* 2.2.** If $\gamma \geq 1$ then $c^{k+1}(\mathbf{x}, t) \leq S$ a.e in $\Omega \times (0, T)$.

It remains to show that $0 \leq \rho^{k+1} \leq \rho_\infty$. For the positivity of ρ^{k+1}, we use the variational formulation of (10) and test with $(\rho^{k+1})^- := \min(\rho^{k+1}, 0)$, this yields

$$\frac{d}{dt}(\rho^{k+1}, (\rho^{k+1})^-) + B[\rho^{k+1}, (\rho^{k+1})^-; t] = (r\rho^k(\rho_\infty - \rho^k), (\rho^{k+1})^-). \tag{49}$$

By induction hypothesis $0 \leq \rho^k \leq \rho_\infty$, then from (49) we get that

$$\frac{1}{2}\frac{d}{dt}\|(\rho^{k+1})^-\|^2_{L^2(\Omega)} + B[(\rho^{k+1})^-, (\rho^{k+1})^-; t] \leq 0.$$

Adding to both sides $\kappa\|(\rho^{k+1})^-\|^2_{L^2(\Omega)}$ with κ as in (23), we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\rho^{k+1})^-\|^2_{L^2(\Omega)} \leq \kappa\|(\rho^{k+1})^-\|^2_{L^2(\Omega)}$$

and applying Gronwall's lemma, we can deduce that

$$\frac{1}{2}\|(\rho^{k+1}(t))^-\|^2_{L^2(\Omega)} \leq \|(\rho^{k+1}(0))^-\|^2_{L^2(\Omega)}e^{\kappa t} = 0$$

since $\rho^{k+1}(0) = \rho_0 \geq 0$ by assumption. This yields $(\rho^{k+1}(t))^- = 0$ a.e in $\Omega \times (0, T)$ and therefore $\rho^{k+1} \geq 0$ a.e in $\Omega \times (0, T)$.

Finally, we have to show that ρ^{k+1} is bounded from above by ρ_∞ a.e on $\Omega \times (0, T)$. Testing the variational formulation of (10) with $(\rho^{k+1} - \rho^\infty)^+$, we find by the rules of calculus Sobolev spaces that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\rho^{k+1} - \rho^\infty)^+\|^2_{L^2(\Omega)} + B[(\rho^{k+1} - \rho^\infty)^+, (\rho^{k+1} - \rho^\infty)^+; t]\\ = (r\rho^k(\rho_\infty - \rho^k), (\rho^{k+1} - \rho_\infty)^+).\end{aligned} \tag{50}$$

After adding $\kappa\|(\rho^{k+1} - \rho_\infty)^+\|^2_{L^2(\Omega)}$ to both sides of (50), taking in account the inequality (20) and that ρ^k is bounded, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(\rho^{k+1} - \rho^\infty)^+\|^2_{L^2(\Omega)} &\leq \kappa\|(\rho^{k+1} - \rho_\infty)^+\|^2_{L^2(\Omega)} + (r\rho^k(\rho_\infty - \rho^k), (\rho^{k+1} - \rho_\infty)^+)\\ \frac{d}{dt}\|(\rho^{k+1} - \rho^\infty)^+\|^2_{L^2(\Omega)} &\leq 2\kappa\|(\rho^{k+1} - \rho_\infty)^+\|^2_{L^2(\Omega)} + r^2\frac{\rho_\infty^4}{8}\|(\rho^{k+1} - \rho_\infty)^+\|^2_{L^2(\Omega)}\\ &\leq (2\kappa + r^2\frac{\rho_\infty^4}{8})\|(\rho^{k+1} - \rho_\infty)^+\|^2_{L^2(\Omega)}.\end{aligned} \tag{51}$$

Gronwall´s inequality and the fact that $\rho^{k+1}(0) \leq \rho_\infty$ imply

$$\|(\rho^{k+1}(t) - \rho^\infty)^+\|^2_{L^2(\Omega)} \leq \|(\rho^{k+1}(0) - \rho_\infty)^+\|^2_{L^2(\Omega)}e^{\int_0^t 2\kappa + r^2\frac{\rho_\infty^4}{8}\,ds} = 0.$$

Thus $\rho^{k+1} \leq \rho_\infty$ a.e in $\Omega \times (0, T)$.

This completes the induction proof. ☑

Proof. of Theorem 1.2

Existence: We show that the iterative sequence constructed above is a Cauchy sequence, which will lead to the existence of the solution (c, ρ) as its limit.

Let $k \in \mathbb{N}$ be arbitrary. Since c^k and c^{k+1} solve (9) with the same initial data and $c^k, c^{k+1} \in L^2(0,T,H^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega))$, (by Lemma 2.1), then theorem 7.1.5 in [4] implies

$$\begin{aligned}
\|c^{k+1} - c^k\|^2_{L^\infty(0,T,H^1(\Omega))} &+ \|c^{k+1} - c^k\|^2_{L^2(0,T,H^2(\Omega))} \\
&\le C(\Omega,T)\|f(\rho^k) - f(\rho^{k-1})\|^2_{L^2(0,T,L^2(\Omega))} \\
&= C(\Omega,T)\int_0^T \Big\|\frac{s\rho^k}{\beta+\rho^k} - \frac{s\rho^{k-1}}{\beta+\rho^{k-1}}\Big\|^2_{L^2(\Omega)} dt \\
&= C(\Omega,T)\int_0^T \Big\|\frac{s\beta(\rho^k - \rho^{k-1})}{(\beta+\rho^k)(\beta+\rho^{k-1})}\Big\|^2_{L^2(\Omega)} dt \\
&\le C(\Omega,T)\Big(\frac{s}{\beta}\Big)^2 \int_0^T \|\rho^k - \rho^{k-1}\|^2_{L^2(\Omega)} dt \qquad (52) \\
&\le C(\Omega,T)\Big(\frac{s}{\beta}\Big)^2 T\|\rho^k - \rho^{k-1}\|^2_{L^\infty(0,T;H^1(\Omega))} \qquad (53)
\end{aligned}$$

for $0 < T \le T_1$ with $T_1 = \min\{\frac{1}{8}, \frac{\beta^2}{C(\Omega,T)s^2}\}$.

Similarly, due to (10) and theorem 7.1.5 in [4], we estimate

$$\begin{aligned}
&\|\rho^{k+1} - \rho^k\|^2_{L^\infty(0,T,H^1(\Omega))} \\
&\le C(\Omega,T)\ \|\nabla(c^{k+1} - c^k)\nabla\rho^k + \Delta(c^{k+1} - c^k)\rho^k + g(\rho^k) - g(\rho^{k-1})\|^2_{L^2(0,T,L^2(\Omega))} \\
&\le 3C(\Omega,T)\int_0^T \Big\{\|\nabla(c^{k+1} - c^k)\nabla\rho^k\|^2_{L^2(\Omega)} + \|\Delta(c^{k+1} - c^k)\rho^k\|^2_{L^2(\Omega)} \\
&\qquad\qquad + \|g(\rho^k) - g(\rho^{k-1})\|^2_{L^2(\Omega)}\Big\} dt \\
&\le 3C(\Omega,T)\int_0^T \|\nabla(c^{k+1} - c^k)\nabla\rho^k\|^2_{L^2(\Omega)} dt + 3C(\Omega,T)\int_0^T \|\Delta(c^{k+1} - c^k)\rho^k\|^2_{L^2(\Omega)} dt \\
&\quad + 3C(\Omega,T)\int_0^T \|g(\rho^k) - g(\rho^{k-1})\|^2_{L^2(\Omega)} dt \\
&= I_1 + I_2 + I_3. \qquad (54)
\end{aligned}$$

Now we estimate each of the three terms separately

As $n \le 3$, by the Sobolev embedding there exist $C_1 > 0$ such that

$$\|w\|_{L^4(\Omega)} \le C_1\|w\|_{H^1(\Omega)} \qquad (55)$$

for all $w \in H^1(\Omega)$. Then

$$
\begin{aligned}
I_1 &\leq 3\ C(\Omega,T)\int_0^T \int_\Omega (\nabla(c^{k+1}-c^k))^2\ (\nabla\rho^k)^2 dxdt && (56)\\
&\leq 3\ C(\Omega,T)\int_0^T \Big(\int_\Omega (\nabla(c^{k+1}-c^k))^4 dx\Big)^{1/2}\ \Big(\int_\Omega (\nabla\rho^k)^4 dx\Big)^{1/2} dt && (57)\\
&\leq 3\ C(\Omega,T)\int_0^T \|\nabla(c^{k+1}-c^k)\|^2_{L^4(\Omega)}\ \|\nabla\rho^k\|^2_{L^4(\Omega)} dt\\
&\overset{(55)}{\leq} 3\ C(\Omega,T)^2 C_1^4 \int_0^T \|\nabla(c^{k+1}-c^k)\|^2_{H^1(\Omega)}\ \|\nabla\rho^k\|^2_{H^1(\Omega)} dt\\
&\overset{(14)}{\leq} 3\ C(\Omega,T)^2\ C_1^4\ \|\rho_0\|^2_{H^3(\Omega)} \int_0^T \|\nabla(c^{k+1}-c^k)\|^2_{H^1(\Omega)}\ dt\\
&\leq 3\ C(\Omega,T)^2\ C_1^4\ \|\rho_0\|^2_{H^3(\Omega)} \int_0^T \|c^{k+1}-c^k\|^2_{H^2(\Omega)}\ dt\\
&\overset{(53)}{\leq} 3\ C(\Omega,T)^4\ C_1^4\ \|\rho_0\|^2_{H^3(\Omega)} \frac{s^2}{\beta^2} T\ \|\rho^k-\rho^{k-1}\|^2_{L^\infty(0,T;H^1(\Omega))} && (58)
\end{aligned}
$$

for $0 < T \leq T_2$ with $T_2 = \min\{T_1, \frac{\beta^2}{3C(\Omega,T)^4 c_1^4 s^2 \|\rho_0\|^2_{H^3(\Omega)}}\}$.

Further, for $0 < T \leq T_3$ with $T_3 = \min\{T_1, \frac{\beta^2}{3C(\Omega,T)^2 s^2 \rho_\infty^2}\}$ we have

$$
\begin{aligned}
I_2 &\leq 3\ C(\Omega,T)\int_0^T \|\Delta(c^{k+1}-c^k)\|^2_{L^2(\Omega)} \|\rho^k\|^2_{L^\infty(\Omega)} dt\\
&\leq 3\ C(\Omega,T)\rho_\infty^2 \int_0^T \|c^{k+1}-c^k\|^2_{H^2(\Omega)} dt\\
&\overset{(53)}{\leq} 3\ C(\Omega,T)^2\ \rho_\infty^2 \frac{s^2}{\beta^2} T_1\ \|\rho^k-\rho^{k-1}\|^2_{L^\infty(0,T;H^1(\Omega))}, && (59)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= 3\ C(\Omega,T)\int_0^T \|r\rho^k(\rho_\infty-\rho^k) - r\rho^{k-1}(\rho_\infty-\rho^{k-1})\|^2_{L^2(\Omega)} dt\\
&= 3\ C(\Omega,T)\int_0^T \|r\rho_\infty(\rho^k-\rho^{k-1})(1-\frac{1}{\rho_\infty}(\rho^k+\rho^{k-1}))\|^2_{L^2(\Omega)} dt\\
&= 3\ C(\Omega,T)\int_0^T \int_\Omega |r^2\rho_\infty^2(\rho^k-\rho^{k-1})^2(1-\frac{1}{\rho_\infty}(\rho^k+\rho^{k-1}))^2| dx\ dt && (60)
\end{aligned}
$$

$$
\begin{aligned}
&\leq 3\ C(\Omega,T)\int_0^T\int_\Omega |r^2\rho_\infty^2(\rho^k-\rho^{k-1})^2(1+\frac{1}{\rho_\infty^2}(\rho^k+\rho^{k-1}))^2|dx\ dt\\
&\leq 15\ C(\Omega,T)r^2\rho_\infty^2\int_0^T\int_\Omega |(\rho^k-\rho^{k-1})^2|dx\ dt\\
&\leq 15\ C(\Omega,T)r^2\rho_\infty^2\int_0^T \|(\rho^k-\rho^{k-1})\|^2_{L^2(\Omega)}dt\\
&\leq 15\ C(\Omega,T)r^2\rho_\infty^2\int_0^T \|(\rho^k-\rho^{k-1})\|^2_{H^1(\Omega)}dt\\
&\leq 15\ C(\Omega,T)r^2\rho_\infty^2 T\ \|(\rho^k-\rho^{k-1})\|^2_{L^\infty(0,T;H^1(\Omega))} \qquad (61)
\end{aligned}
$$

for $0 < T \leq T_4$ with $T_4 = \min\{T_3, \frac{1}{15C(\Omega,T)^2r^2\rho_\infty^2}\}$.

Altogether, (53), (58), (59) and (61) yield

$$
\|c^{k+1}-c^k\|^2_{L^\infty(0,T,H^1(\Omega))}+\|\rho^{k+1}-\rho^k\|^2_{L^\infty(0,T,H^1(\Omega))} \leq \frac{1}{2}\|\rho^k-\rho^{k-1}\|^2_{L^\infty(0,T,H^1(\Omega))} \qquad (62)
$$

whenever $0 < T \leq T_4$. That is, for $T := T_4$ the sequences $\{c^k\}$ and $\{\rho^k\}$ are Cauchy sequences in $L^\infty(0,T;H^1(\Omega))$ and there are functions c and ρ in $L^\infty(0,T;H^1(\Omega))$ such that

$$
c^k \to c \ \text{ and } \ \rho^k \to \rho \ \text{ strongly in } L^\infty(0,T;H^1(\Omega)).
$$

Since $L^2(0,T;H^4(\Omega))$ and $L^2(0,T;H^2(\Omega))$ are Hilbert spaces, the uniform bounds (13) and (14) imply that for subsequences c^{k_l} and ρ^{k_l}

$$
\begin{aligned}
c^{k_l} \rightharpoonup c; \qquad \rho^{k_l} \rightharpoonup \rho \ \text{ weakly in } L^2(0;T;H^4(\Omega));\\
\partial_t c^{k_l} \rightharpoonup c; \qquad \partial_t\rho^{k_l} \rightharpoonup \rho \ \text{ weakly in } L^2(0,T;H^2(\Omega)).
\end{aligned}
$$

Using all these convergences in the weak formulation of (9), (10) and letting $l \to \infty$, we conclude that (c, ρ) is a weak solution to (1)-(4) and also satisfies (11) -(15).

Uniqueness if (c_1, ρ_1) and (c_2, ρ_2) are two weak solutions of (1)-(4), they satisfy (62). Then

$$
\|c_1-c_2\|^2_{L^\infty(0,T,H^1(\Omega))}+\|\rho_1-\rho_2\|^2_{L^\infty(0,T,H^1(\Omega))} \leq \frac{1}{2}\|\rho_1-\rho_2\|^2_{L^\infty(0,T,H^1(\Omega))}
$$

for $T := T_4$. Therefore, both solutions coincide.

Hence, the proof of theorem 1.2 is complete. ☑

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