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Analysis of a Fourier-Galerkin numerical scheme for a 1D Benney–Luke–Paumond equation

Análisis de un esquema numérico Fourier-Galerkin para una ecuación unidimensional Benney-Luke-Paumond

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Abstract. We study convergence of the semidiscrete and fully discrete formulations of a Fourier- Galerkin numerical scheme to approximate solutions of a nonlinear Benney-Luke-Paumond equation that models long water waves with small amplitude propagating over a shallow channel with flat bottom. The accuracy of the numerical solver is checked using some exact solitary wave solutions. In order to apply the Fourier-spectral scheme in a non periodic setting, we approximate the initial value problem with $x \in \mathbb{R}$ by the corresponding periodic Cauchy problem for $x \in [0, L]$, with a large spatial period L.

Key words and phrases. Solitary waves, water waves, spectral methods.

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Resumen. Estudiamos la convergencia de las formulaciones semidiscreta y completamente discreta de un método espectral Fourier-Galerkin para aproximar las soluciones de una ecuación no lineal Benney-Luke-Paumond que modela ondas largas con pequeña amplitud que se propagan sobre un canal raso con fondo plano. La precisión del método numérico se verifica usando algunas soluciones de onda solitaria exactas. A fin de aplicar el esquema Fourier-espectral en un contexto no periódio, aproximamos el problema de valor inicial con $x \in \mathbb{R}$ por el correspondiente problema de Cauchy periódico para $x \in [0, L]$, con un periodo espacial L grande.

Palabras y frases clave. Ondas solitarias, ondas acuáticas, métodos espectrales.

1. Introduction

In this paper we shall study numerically the nonlinear one-dimensional Boussinesq type equation (named as the Benney-Luke-Paumond equation (BLP))

$$\begin{aligned}\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) + \mu^2(B\Phi_{xxxxtt} - A\Phi_{xxxxxx}) \\ + \mu^2\big(k\Phi_t\Phi_x^{k-1}\Phi_{xx} + 2\Phi_x^k\Phi_{xt}\big) = 0,\end{aligned} \tag{1}$$

for $x \in (0, L), t > 0$ with periodic spatial boundary conditions. Here the constants a, b A, B are real positive numbers such that

$$a - b = \sigma - 1/3,\ A - B = \frac{1}{45} + \left(b - \frac{1}{3}\right)(a - b),$$

the parameter σ is related to surface tension effects, μ represents the long-wave parameter (dispersion coefficient) and k is a positive integer.

In the case that $k = 1$, the Benney-Luke-Paumond equation (1) is an asymptotically simplified model valid for μ small, derived by L. Paumond [11] to describe the evolution of two-dimensional small amplitude waves over a channel with flat bottom, and considering surface tension and gravity forces. It is also important to point out that equation (1) corresponds to a generalization for values of $\sigma \approx \frac{1}{3}$ of the Benney-Luke model

$$\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) + \mu^2(k\Phi_t\Phi_x^{k-1}\Phi_{xx} + 2\Phi_x^k\Phi_{xt}) = 0,$$

studied in [2, 16].

In this paper, we shall develop a rigorous analysis of the convergence and error of the semidiscrete and fully discrete formulations of a Fourier-Galerkin scheme to approximate solutions of the Benney-Luke-Paumond equation (1). To formulate the numerical scheme, the BLP equation is rewritten as a system of two coupled equations in terms of the variables $q = \Phi_x$, $r = \Phi_t$. The time-stepping method is of second order, where dispersive terms in the (q, r) system are approximated by means of an implicit strategy, in contrast to nonlinear terms which are treated in explicit form. Implicit-explicit schemes (IMEX) were already discussed in [1], [3] for spectral methods applied to scalar dispersive evolution equations. An important feature of the proposed scheme is that the time evolution can be performed explicitly without using any Newton-type iteration. The rates of convergence of the semidiscrete and fully discrete schemes are $O(N^{2-\alpha})$ and $O(N^{2-\alpha} + \Delta t^2)$, respectively, where $\alpha \geq 5$ depends only on the smoothness of the exact solution, Δt is the time step and N is the number of spatial Fourier modes. The strategy to obtain these rates of convergence follows the one used by Muñoz for other dispersive systems [6, 5, 7, 8]. This type of analysis has not been performed in previous works on the BLP equation (1) to the best knowledge of the author. We further show that the semidiscrete scheme conserves in time the Hamiltonian of the system. Through some numerical simulations we illustrate that this Hamiltonian is also approximately conserved by the fully discrete scheme. This is an important property for a numerical scheme since the failure of conservation of invariants of evolution can lead to blow up of the computed solution.

The accuracy and convergence rate of the Fourier-spectral scheme proposed in this paper are illustrated by using a family of exact solitary wave solutions of equation (1) derived in [15]. In order to apply this scheme in a non-periodic setting, we approximate the initial value problem for equation (1) with $x \in \mathbb{R}$, by the corresponding periodic Cauchy problem for $x \in [0, L]$, with a large spatial period L. This type of approximation can be justified by the decay of the solutions of the unrestricted problem as $|x| \to \infty$.

The numerical scheme presented was employed successfully by Muñoz in [15, 14] to analyze orbital stability under small initial perturbations of travelling wave solutions, and in [4, 13] for the case of equation (1) with $A = B = 0$ (generalized Benney-Luke equation). The Cauchy problem for the one-dimensional equation (1) has been considered in [15] and for the two-dimensional case in [12].

This paper is organized as follows. In section 2, we introduce notation and functional spaces which will be used in our work. In section 3, the analytical properties and convergence of the semidiscrete scheme to approximate solutions of the BLP equation (1) are investigated. Section 4 deals with the convergence of the fully discrete scheme that we propose for solving the BLP model. Finally in section 5, to validate the theoretical results, some numerical experiments using a family of analytical solutions of solitary-wave type of (1) are performed.

2. Preliminaries

We set

$$L^2(0, L) := \left\{ f : [0, L] \to \mathbb{C},\ \|f\|_0 = \left[\int_0^L |f(x)|^2 dx \right]^{1/2} < \infty \right\},$$

with the inner product

$$\langle f, g \rangle = \int_0^L f(x)\ \overline{g(x)}\ dx.$$

The space of all functions of class C^k that are L-periodic is denoted by $C^k_{per}(0, L)$, $k = 0, 1, 2, \ldots$. Further $C_{per} = C^0_{per}(0, L)$ is the space of all continuous functions of period L.

We will denote by $\mathcal{P}$ to the space of all infinitely differentiable functions that are L-periodic so as all their derivatives. We say that $T : \mathcal{P} \to \mathbb{C}$, defines a periodic distribution, i.e., $T \in \mathcal{P}'$ if T is linear and there exists a sequence $(\Psi_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ such that

$$\langle T, \varphi\rangle = \lim_{n\to\infty} \int_0^L \Psi_n(x)\varphi(x)dx, \quad \text{for all } \varphi \in \mathcal{P}.$$

Let $s \in \mathbb{R}$. The Sobolev space, denoted by $H^s_{per} = H^s_{per}(0, L)$, is defined as

$$H^s_{per}(0, L) = \left\{ f \in \mathcal{P}' : \|f\|_s^2 = L \sum_{n\in\mathbb{Z}} (1 + k^2)^s |\widehat{f}(n)|^2 < \infty \right\},$$

where $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ represents the Fourier transform of $f \in \mathcal{P}'$ defined by

$$\widehat{f}(n) = \frac{1}{L}\langle f, e^{-2\pi inx/L}\rangle, \; n \in \mathbb{Z}.$$

In case that $f \in C_{per}$, we can rewrite $\widehat{f}(n)$ as

$$\widehat{f}(n) = \frac{1}{L}\int_0^L f(x)e^{-2\pi inx/L}dx, \; n \in \mathbb{Z}.$$

It can be shown that for all $s \in \mathbb{R}$, H^s_{per} is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_s$ defined as follows:

$$\langle f, g\rangle_s = L \sum_{n\in\mathbb{Z}} (1 + n^2)^s \widehat{f}(n)\overline{\widehat{g}(n)}.$$

In particular, when $s = 0$, we get the Hilbert space denoted by $L^2_{per} = H^0_{per}$. It is important to note that this space is isometrically isomorphic to $L^2(0, L)$. Further we recall that Parseval's identity holds, i.e., for $f \in C_{per}$

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{1}{L}\|f\|_0^2,$$

or equivalently,

$$\langle f, g\rangle = L \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}}(n) = L\langle \hat{f}, \hat{g}\rangle.$$

Let N be an even integer and consider the finite dimensional space S_N defined by

$$S_N = \text{ span}\left\{ \frac{1}{\sqrt{L}}e^{\frac{2\pi inx}{L}} \;:\; -N/2 \le n \le N/2 \right\}.$$

Remember that the family $\{\frac{1}{\sqrt{L}}e^{\frac{2\pi inx}{L}} \;:\; n \in \mathbb{Z}\}$ is an orthonormal and complete system in $L^2_{per}(0, L)$. Let P_N be the orthogonal projection $P_N : L^2_{per}(0, L) \to S_N$ on the space S_N,

$$P_N g := \sum_{n=-N/2}^{N/2} \hat{g}_n \phi_n,$$

with

$$\phi_n(x) = e^{\frac{2\pi i n x}{L}}, \ \hat{g}_n = \frac{1}{L}\int_0^L g(x)\overline{\phi}_n(x)dx.$$

This operator has the following properties (see [9, 10]): For any $g \in L^2_{per}(0, L)$,

$$\langle P_N g - g, \phi\rangle = 0, \quad \text{for all } \phi \in S_N.$$

Furthermore, given integers $0 \le s \le \alpha$, there exists a constant C independent of N such that, for any $g \in H^\alpha_{per}(0, L)$

$$\|P_N g - g\|_s \le C N^{s-\alpha}\|g\|_\alpha. \tag{2}$$

Finally, for α, s positive real, with $\alpha \le s$ there exists a constant C_0 independent of N such that for all $\phi \in S_N$,

$$\|\phi\|_\alpha \le \|\phi\|_s \le C_0 N^{s-\alpha}\|\phi\|_\alpha. \tag{3}$$

This result is known as the *inverse inequality.*

It is convenient to rewrite 1D Benney-Luke-Paumond model, using the following change of variables:

$$q = \Phi_x, \quad r = \Phi_t,$$

and let us define the operators $\mathcal{A}, \mathcal{B}$ as

$$\mathcal{A} = I - \mu a\partial_x^2 + \mu^2 A\partial_x^4, \quad \mathcal{B} = I - \mu b\partial_x^2 + \mu^2 B\partial_x^4.$$

Therefore equation (1) formally becomes the following system

$$q_t = r_x \tag{4}$$

$$r_t = \mathcal{B}^{-1}\left(\mathcal{A}q_x - \mu^2\left((rq^k)_x + q^k r_x\right)\right). \tag{5}$$

In other words, in the variable $U = (q, r)^t$, the Benney-Luke-Paumond (1) can be viewed as the first order system

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = M\begin{pmatrix} q \\ r \end{pmatrix} + G\begin{pmatrix} q \\ r \end{pmatrix}, \tag{6}$$

where

$$M = \begin{pmatrix} 0 & \partial_x \\ \partial_x\mathcal{B}^{-1}\mathcal{A} & 0 \end{pmatrix} \text{ and } G\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathcal{B}^{-1}\left((rq^k)_x + q^k r_x\right) \end{pmatrix}.$$

We see directly that the following energy

$$\begin{aligned} \mathcal{H}\begin{pmatrix} q \\ r \end{pmatrix} &= \frac{1}{2}\int_0^L \left(r^2 + \mu b(r_x)^2 + \mu^2 B(r_{xx})^2 + q^2 + a\mu(q_x)^2 + A\mu^2(q_{xx})^2\right)dx \\ &= \frac{1}{2}\int_0^L \mathcal{B}(r)r + \mathcal{A}(q)q\,dx \end{aligned} \tag{7}$$

is conserved in time for classical solutions of the one-dimensional Boussinesq system (6). In fact,

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}\begin{pmatrix} q \\ r \end{pmatrix} &= \int_0^L \mathcal{B}(r)r_t + \mathcal{A}(q)q_t\, dx \\
&= \int_0^L \mathcal{B}(r_t)r + \mathcal{A}(q)r_x\, dx \\
&= \int_0^L (\mathcal{A}q_x - \mu^2(rq^k)_x + q^k r_x)r + \mathcal{A}(q)r_x\, dx \\
&= \int_0^L (\mathcal{A}(q)r - \mu^2 r^2 q^k)_x\, dx = 0.
\end{aligned}$$

It is not difficult to see that $\mathcal{H}$ can be compared with the H^2_{per}-norm. More exactly, there are constants $C_1(a,b,A,B), C_2(a,b,A,B) > 1$ such that,

$$C_1 \left\| \begin{pmatrix} q \\ r \end{pmatrix} \right\|^2_{H^2_{per}\times H^2_{per}} \le \mathcal{H}\begin{pmatrix} q \\ r \end{pmatrix} \le C_2 \left\| \begin{pmatrix} q \\ r \end{pmatrix} \right\|^2_{H^2_{per}\times H^2_{per}}. \tag{8}$$

3. The semidiscrete scheme

Let us consider problem (4)-(5) which can be rewritten as

$$q_t = r_x,\ x \in (0,L), t > 0, \tag{9}$$

$$r_t = \mu b\partial_x^2 r_t - \mu^2 B\partial_x^4 r_t + q_x - \mu a\partial_x^3 q + \mu^2 A\partial_x^5 q - \mu^2\left((rq^k)_x + q^k r_x\right), \tag{10}$$

subject to the initial conditions $q(0,x) = q_0(x),\ r(0,x) = r_0(x)$, and q, r L-periodic real valued functions in the variable x.

The semidiscrete Fourier-Galerkin spectral scheme to solve problem (9)-(10) is to find $q_N, r_N \in C([0,T], S_N)$ such that

$$\begin{aligned}
&\langle q_{N,t}, \phi\rangle = \langle r_{N,x}, \phi\rangle, \\
&\langle \psi, r_{N,t} - \mu b\partial_x^2 r_{N,t} + \mu^2 B\partial_x^4 r_{N,t}\rangle \\
&\qquad = \langle \psi, q_{N,x} - \mu a\partial_x^3 q_N + \mu^2 A\partial_x^5 q_N - \mu^2((r_N q_N^k)_x + q_N^k r_{N,x}\rangle, \\
&q_N(0) = P_N(q_0),\ r_N(0) = P_N(r_0),
\end{aligned} \tag{11}$$

for any $\phi, \psi \in S_N$, and $0 \le t \le T$. Here we suppose that the initial data q_0 and r_0 are real valued functions.

Theorem 3.1. *The Hamiltonian $\mathcal{H}$ given in* (7) *is conserved in time by solutions* (q_N, r_N) *of the semidiscrete problem (11).*

Proof. We have that

$$\frac{d}{dt}\mathcal{H}\begin{pmatrix} q_N \\ r_N \end{pmatrix} = \frac{1}{2}\int_0^L (\mathcal{B}(r_{N,t})r_N + \mathcal{B}(r_N)r_{N,t} + \mathcal{A}(q_{N,t})q_N + \mathcal{A}(q_N)q_{N,t})\, dx.$$

Using equations (11) with $\phi = \mathcal{A}q_N$ and $\psi = r_N$ which belong to the space S_N, we have that

$$\int_0^L \mathcal{A}(q_N)q_{N,t}dx = \int_0^L \mathcal{A}(q_N)r_{N,x}\ dx,$$
$$\int_0^L r_N\mathcal{B}(r_{N,t}) = \int_0^L r_N(\mathcal{A}(q_{N,x}) - \mu^2((r_Nq_N^k)_x + q_N^k r_{N,x})\ dx.$$

Therefore

$$\frac{d}{dt}\mathcal{H}\begin{pmatrix} q_N \\ r_N \end{pmatrix} = \int_0^L r_N\Big(\mathcal{A}(q_{N,x}) - \mu^2((r_Nq_N^k)_x + q_N^k r_{N,x})\Big) + \mathcal{A}(q_N)r_{N,x}\ dx$$
$$= \int_0^L (\mathcal{A}(q_N)r_N - \mu^2 r_N^2 q_N^k)_x\ dx = 0.$$

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From the previous result we see that the numerical solution q_N, r_N has the same conserved quantity as the original problem (9)-(10).

Theorem 3.2. *Let $q, r \in C([0,T], H^{\alpha}_{per}(0,L))$ be the classical solution of problem (9)-(10), with $\alpha \geq 5$ integer. Then the semidiscrete problem (11) has a unique solution $q_N, r_N \in C([0,T], S_N)$ and for N sufficiently large there exists a constant $C > 0$, independent of N, t, such that*

$$\|q(t) - q_N(t)\|_2 + \|r(t) - r_N(t)\|_2 \leq CN^{2-\alpha},$$

for any $0 \leq t \leq T$.

Proof. The existence of a maximal time $0 < T_N \leq T$ such that there exists a unique real valued solution $q_N, r_N \in C([0,T_N], S_N)$ to the semidiscrete problem (11), follows from a classical result of the theory of ordinary differential equations. Furthermore, since the initial conditions q_0, r_0 are real then $q_N(0), r_N(0)$ are real, and thus we can deduce that $q_N(t), r_N(t)$ is real for any $0 \leq t \leq T_N$ from the uniqueness of the solution to problem (11).

Applying the orthogonal projection P_N to equations (9)-(10), and taking inner product in L^2 with $\phi, \psi \in S_N$, we obtain that

$$\begin{aligned} &\langle P_N q_t, \phi\rangle = \langle P_N r_x, \phi\rangle, \\ &\langle \psi, P_N r_t - \mu b P_N \partial_x^2 r_t + \mu^2 B P_N \partial_x^4 r_t\rangle \\ &\quad = \left\langle \psi, P_N q_x - \mu a P_N \partial_x^3 q + \mu^2 A P_N \partial_x^5 q - \mu^2 P_N\left((rq^k)_x + q^k r_x\right)\right\rangle, \end{aligned} \tag{12}$$

for any $\phi, \psi \in S_N$, and $0 \leq t \leq T$.

Let us define

$$\theta := P_N q - q_N,\ \rho := q - P_N q,\ \xi := P_N r - r_N,\ \sigma := r - P_N r.$$

Substraction of equations (11) from equations (12) gives that

$$\begin{aligned}
&\langle \theta_t, \phi\rangle = \langle \xi_x, \phi\rangle,\\
&\langle \psi, \xi_t\rangle - \mu b\langle \psi, \partial_x^2 \xi_t\rangle + \mu^2 B\langle \psi, \partial_x^4 \xi_t\rangle = \langle \psi, \theta_x\rangle - \mu a\langle \psi, \partial_x^3\theta\rangle +\\
&\mu^2 A\langle \psi, \partial_x^5\theta\rangle + \mu^2\langle \psi, (r_N q_N^k)_x + q_N^k r_{N,x} - P_N((rq^k)_x) - P_N(q^k r_x)\rangle.
\end{aligned} \tag{13}$$

Let $\phi = \theta$, $\psi = \xi$ in equations above. We obtain that

$$\begin{aligned}
&\langle \theta_t, \theta\rangle = \langle \xi_x, \theta\rangle,\\
&\langle \xi, \xi_t\rangle - \mu b\langle \xi, \partial_x^2 \xi_t\rangle + \mu^2 B\langle \xi, \partial_x^4 \xi_t\rangle = \langle \xi, \theta_x\rangle - \mu a\langle \xi, \partial_x^3\theta\rangle +\\
&\mu^2 A\langle \xi, \partial_x^5\theta\rangle + \mu^2\langle \xi, (r_N q_N^k)_x + q_N^k r_{N,x} - P_N((rq^k)_x) - P_N(q^k r_x)\rangle.
\end{aligned} \tag{14}$$

But using that $\partial_x^2\theta, \partial_x^4\theta \in S_N$, and the first equation in system (14), we have that

$$-\mu a\langle \xi, \partial_x^3\theta\rangle = \mu a\langle \xi_x, \partial_x^2\theta\rangle = \mu a\langle \theta_t, \partial_x^2\theta\rangle = -\mu a\langle \partial_x\theta_t, \partial_x\theta\rangle = -\mu a\frac{1}{2}\partial_t\|\theta_x\|_0^2.$$

$$\begin{aligned}
\mu^2 A\langle \xi, \partial_x^5\theta\rangle &= -\mu^2 A\langle \xi_x, \partial_x^4\theta\rangle = -\mu^2 A\langle \theta_t, \partial_x^4\theta\rangle = -\mu^2 A\langle \partial_x^2\theta_t, \partial_x^2\theta\rangle\\
&= -\mu^2 A\frac{1}{2}\partial_t\|\partial_x^2\theta\|_0^2.
\end{aligned}$$

Further, an application of integration by parts yields that

$$-\mu b\langle \xi, \partial_x^2\xi_t\rangle = \mu b\langle \xi_x, \xi_{xt}\rangle = \mu b\frac{1}{2}\partial_t\|\xi_x\|_0^2,$$

$$\mu^2 B\langle \xi, \partial_x^4\xi_t\rangle = \mu^2 B\langle \partial_x^2\xi, \partial_x^2\xi_t\rangle = \mu^2 B\frac{1}{2}\partial_t\|\partial_x^2\xi\|_0^2.$$

Replacing these results in equations (14) and adding them, we arrive at

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\theta\|_0^2 + \frac{1}{2}\partial_t\|\xi\|_0^2 + \frac{1}{2}\mu b\partial_t\|\xi_x\|_0^2 + \frac{1}{2}\mu^2 B\partial_t\|\partial_x^2\xi\|_0^2 + \frac{1}{2}\mu a\partial_t\|\theta_x\|_0^2 +\\
&\frac{1}{2}\mu^2 A\partial_t\|\partial_x^2\theta\|_0^2 = \mu^2\langle \xi, (r_N q_N^k)_x - (rq^k)_x\rangle + \mu^2\langle \xi, q_N^k r_{N,x} - q^k r_x\rangle.
\end{aligned} \tag{15}$$

To estimate the nonlinear terms in the right hand side of the previous equation, let us observe that

$$\begin{aligned}
&\mu^2\langle \xi, (r_N q_N^k)_x - (rq^k)_x\rangle = -\mu^2\langle \xi_x, r_N q_N^k - rq^k\rangle\\
&= \mu^2\langle \xi_x, (r - r_N)q^k + r_N(q^k - q_N^k)\rangle\\
&= \mu^2\langle \xi_x, (r - P_N r + P_N r - r_N)q^k\\
&+ r_N(q - P_N q + P_N q - q_N)(q^{k-1} + kq^{k-2}q_N + \cdots + q_N^{k-1})\rangle\\
&= \mu^2\langle \xi_x, (\sigma + \xi)q^k + r_N(\rho + \theta)(q^{k-1} + kq^{k-2}q_N + \cdots + q_N^{k-1})\rangle,
\end{aligned}$$

and

$$\begin{aligned}\mu^2\langle\xi, q_N^k r_{N,x} - q^k r_x\rangle &= \mu^2\langle\xi, q_N^k r_{N,x} - q_N^k r_x + q_N^k r_x - q^k r_x\rangle \\ &= \mu^2\langle\xi, q_N^k((r_N - P_N r)_x + (P_N r - r)_x + \\ &\quad (q_N - P_N q + P_N q - q)(q_N^{k-1} + kq_N^{k-2}q + \cdots + q^{k-1})r_x\rangle \\ &= \mu^2\langle\xi, -q_N^k(\xi_x + \sigma_x) - (\theta + \rho)(q_N^{k-1} + kq_N^{k-2}q + \cdots + q^{k-1})r_x\rangle.\end{aligned}$$

Let $M > 0$ be denote a positive constant such that

$$\max_{t\in[0,T]}\{\|r\|_2, \|q\|_2\} < M.$$

Further suppose that $0 < T_N < T$ is the largest value such that

$$\max_{t\in[0,T_N]}\{\|r_N(t)\|_2, \|q_N(t)\|_2\} < 2M.$$

Introducing these estimates in inequality (15) one gets that

$$\begin{aligned}&\frac{1}{2}\partial_t\|\theta\|_0^2 + \frac{1}{2}\partial_t\|\xi\|_0^2 + \frac{1}{2}\mu b\partial_t\|\xi_x\|_0^2 + \frac{1}{2}\mu^2 B\partial_t\|\xi_{xx}\|_0^2 + \frac{1}{2}\mu a\partial_t\|\theta_x\|_0^2 \\ &+ \frac{1}{2}\mu^2 A\partial_t\|\theta_{xx}\|_0^2 \le C\Big(\|q\|_\infty^k\|\xi_x\|_0\|\sigma\|_0 \\ &+ \|q\|_\infty^k\|\xi_x\|_0\|\xi\|_0 + \|\xi_x\|_0\|\rho\|_0\|r_N\|_\infty\|q^{k-1} + \cdots + q_N^{k-1}\|_\infty \\ &+ \|r_N\|_\infty\|\xi_x\|_0\|\theta\|_0\|q^{k-1} + \cdots + q_N^{k-1}\|_\infty + \|q_N\|_\infty^k\|\xi\|_0\|\xi_x\|_0 \\ &+ \|q_N\|_\infty^k\|\xi\|_0\|\sigma_x\|_0 + \|q_N^{k-1} + \cdots + q^{k-1}\|_\infty\|r_x\|_\infty\|\xi\|_0\|\theta\|_0 \\ &+ \|q_N^{k-1} + \cdots + q^{k-1}\|_\infty\|r_x\|_\infty\|\xi\|_0\|\rho\|_0\Big) \\ &\le C(\|\theta\|_0^2 + \|\xi\|_0^2 + \mu b\|\xi_x\|_0^2 + \mu a\|\theta_x\|_0^2 + \mu^2 B\|\xi_{xx}\|_0^2 + \mu^2 A\|\theta_{xx}\|_0^2 + \\ &\quad \|\sigma\|_0^2 + \|\sigma_x\|_0^2 + \|\rho\|_0^2).\end{aligned}$$

Using equation (2), we have for $\alpha > 2$ that

$$\|\rho\|_0 \le CN^{-\alpha}\|q\|_\alpha, \qquad \|\sigma\|_1 \le CN^{1-\alpha}\|r\|_\alpha,$$

and

$$\|\rho\|_2 \le CN^{2-\alpha}\|q\|_\alpha, \qquad \|\sigma\|_2 \le CN^{2-\alpha}\|r\|_\alpha.$$

Therefore

$$\begin{aligned}&\partial_t\|\theta\|_0^2 + \partial_t\|\xi\|_0^2 + \mu b\partial_t\|\xi_x\|_0^2 + \partial_t\|\xi_{xx}\|_0^2 + \mu a\partial_t\|\theta_x\|_0^2 + \mu^2 A\partial_t\|\theta_{xx}\|_0^2 \le \\ &C(\|\theta\|_0^2 + \|\xi\|_0^2 + \mu b\|\xi_x\|_0^2 + \mu a\|\theta_x\|_0^2 + \mu^2 B\|\xi_{xx}\|_0^2 + \mu^2 A\|\theta_{xx}\|_0^2) + CN^{2(1-\alpha)}.\end{aligned}$$

Thus using Gronwall's lemma,

$$\|\theta\|_2 + \|\xi\|_2 \le CN^{1-\alpha}, \tag{16}$$

for $0 \leq t \leq T_N$. Observe that $r_N = r - (\sigma + \xi)$ and $q_N = q - (\theta + \rho)$. Therefore

$$\|r_N\|_2 \leq \|r\|_2 + \|\sigma\|_2 + \|\xi\|_2 < M + CN^{2-\alpha} + CN^{1-\alpha} < 2M,$$

and

$$\|q_N\|_2 \leq \|q\|_2 + \|\theta\|_2 + \|\rho\|_2 < M + CN^{2-\alpha} + CN^{1-\alpha} < 2M,$$

for N large enough and $\alpha > 2$. This fact contradicts the maximality of T_N. Thus the solution q_N, r_N of problem (11) can be extended for any $0 \leq t \leq T$ and inequality (16) is satisfied for any $0 \leq t \leq T$. Therefore

$$\|q(t) - q_N(t)\|_2 + \|r(t) - r_N(t)\|_2 \leq \|q(t) - P_N q(t)\|_2 + \|P_N q(t) - q_N(t)\|_2 + \|r(t) - P_N r(t)\|_2 + \|P_N r(t) - r_N(t)\|_2 \leq CN^{2-\alpha} + CN^{1-\alpha} \leq CN^{2-\alpha},$$

for $0 \leq t \leq T$. ☑

***Remark* 3.3.** Using an analogous technique as in [15], we can show that the initial value problem associated to system (9)-(10) has a unique global mild solution $(q,r)(t,.) \in H^\alpha_{per} \times H^\alpha_{per}$ for $\alpha \geq 2$. It is still unknown whether this problem is well-posed for $\alpha < 2$. We point out that the technique used in the proof of theorem 3.2 requires that $\alpha \geq 5$ due to the fifth-order derivative applied to the variable q.

4. The fully discrete scheme

Notice that the semidiscrete problem (11) can be rewritten as

$$\langle q_{N,t}, \phi\rangle = \langle r_{N,x}, \phi\rangle, \tag{17}$$

$$\langle \psi, \partial_t r_N - \mathcal{B}^{-1}\mathcal{A}\partial_x q_N + \mu^2 \mathcal{B}^{-1}\overline{\mathcal{P}}\rangle = 0, \tag{18}$$

subject to $q_N(0) = P_N(q_0), r_N(0) = P_N(r_0)$, for all $\phi, \psi \in S_N$, and where

$$\overline{\mathcal{P}} = (r_N(q_N)^k)_x + (q_N)^k \partial_x r_N.$$

The fully discrete scheme to discretize the system above, consists in finding a sequence $\{q_N^n, r_N^n\}$ of elements of $S_N \times S_N$ which satisfies for all $\phi, \psi \in S_N$ and $n = 1, 2, \ldots, M-1$ that

$$\left\langle \frac{q_N^{n+1} - q_N^n}{\Delta t}, \phi \right\rangle = \left\langle \frac{\partial_x r_N^{n+1} + \partial_x r_N^n}{2}, \phi \right\rangle, \tag{19}$$

$$\left\langle \psi, \frac{r_N^{n+1} - r_N^n}{\Delta t} \right\rangle = \left\langle \psi, \mathcal{B}^{-1}\mathcal{A}\partial_x \left(\frac{q_N^{n+1} + q_N^n}{2}\right) - \frac{3}{2}\mu^2 \mathcal{B}^{-1}\mathcal{P}^n + \frac{1}{2}\mu^2\mathcal{B}^{-1}\mathcal{P}^{n-1} \right\rangle, \tag{20}$$

where

$$\mathcal{P}^n = (r_N^n (q_N^n)^k)_x + (q_N^n)^k \partial_x r_N^n,$$

for all $\phi, \psi \in S_N$, and subject to $q_N^0(0) = P_N(q_0), r_N^0(0) = P_N(r_0)$.

Theorem 4.1. *Let $(q, r) \in C([0, T], H^{\alpha}_{per}(0, L))$ be a classical solution of system (9)-(10) with $\alpha \geq 5$ integer, let $(q_N, r_N) \in C([0, T], S_N)$ be the solution of the semidiscrete problem (17)-(18) and let $\{(q^n_N, r^n_N)\}$ be the solution of the fully-discrete scheme (19)-(20). If $q^0_N = q_N(0), r^0_N = r_N(0)$ and q^1_N, r^1_N satisfy that $\|q^1_N - q_N(\Delta t)\|_2 \leq C\Delta t^2$, $\|r^1_N - r_N(\Delta t)\|_2 \leq C\Delta t^2$, then with the assumption that $q_0, r_0 \in H^{\alpha}_{per}(0, L)$, there exists a constant C independent of N and Δt such that if $CN^2\Delta t < \frac{1}{4}$, N large enough and Δt sufficiently small, we have that*

$$\max_{0\leq n\leq M}\{\|q(t_n) - q^n_N\|_2 + \|r(t_n) - r^n_N\|_2\} \leq C(N^{2-\alpha} + \Delta t^2).$$

Proof. It can be shown that a solution (q^n_N, r^n_N) of (19)-(20) also satisfies that

$$\Big\langle \frac{q^{n+1}_N - q^n_N}{\Delta t}, \phi\Big\rangle_2 = \Big\langle \frac{\partial_x r^{n+1}_N + \partial_x r^n_N}{2}, \phi\Big\rangle_2, \tag{21}$$

$$\Big\langle \psi, \frac{r^{n+1}_N - r^n_N}{\Delta t}\Big\rangle_2 = \tag{22}$$
$$\Big\langle \psi, \mathcal{B}^{-1}\mathcal{A}\partial_x\Big(\frac{q^{n+1}_N + q^n_N}{2}\Big) - \frac{3}{2}\mu^2\mathcal{B}^{-1}\mathcal{P}^n + \frac{1}{2}\mu^2\mathcal{B}^{-1}\mathcal{P}^{n-1}\Big\rangle_2.$$

Using that

$$q_N(t_{n+1}) - q_N(t_n) = \Delta t\partial_t q_N(t_n) + \frac{1}{2}\partial^2_t q_N(t_n)\Delta t^2 + O(\Delta t^3),$$
$$\partial_x r_N(t_{n+1}) + \partial_x r_N(t_n) = 2\partial_x r_N(t_n) + \Delta t\partial_t\partial_x r_N(t_n) + O(\Delta t^2),$$

we obtain that

$$\Big\langle q_N(t_{n+1}) - q_N(t_n) - \Delta t\Big(\frac{\partial_x r_N(t_{n+1}) + \partial_x r_N(t_n)}{2}\Big), \phi\Big\rangle_2 := \langle\theta, \phi\rangle_2, \tag{23}$$

where due to equation (17), it follows that

$$\begin{aligned}\langle\theta, \phi\rangle_2 :=& \Big\langle \Delta t\partial_t q_N(t_n) + \frac{1}{2}\partial^2_t q_N(t_n)\Delta t^2 + O(\Delta t^3) \\ &- \Delta t\Big(\frac{\partial_x r_N(t_{n+1}) + \partial_x r_N(t_n)}{2}\Big), \phi\Big\rangle_2 \\ =& \Big\langle \Delta t\partial_t q_N(t_n) + \frac{1}{2}\partial^2_t q_N(t_n)\Delta t^2 + O(\Delta t^3) - \Delta t\Big(\partial_x r_N(t_n) \\ &+ \frac{\Delta t}{2}\partial_t\partial_x r_N(t_n) + O(\Delta t^2)\Big), \phi\Big\rangle_2 = O(\Delta t^3).\end{aligned}$$

On the other hand,

$$\Big\langle \psi, r_N(t_{n+1}) - r_N(t_n) - \Delta t\mathcal{B}^{-1}\mathcal{A}\Big(\frac{\partial_x q_N(t_{n+1}) + \partial_x q_N(t_n)}{2}\Big) +$$
$$\frac{3}{2}\Delta t\mu^2\mathcal{B}^{-1}\overline{\mathcal{P}}^n - \frac{1}{2}\Delta t\mu^2\mathcal{B}^{-1}\overline{\mathcal{P}}^{n-1}\Big\rangle_2 := \langle\psi, \alpha\rangle_2, \tag{24}$$

where

$$\begin{aligned}\langle\psi,\alpha\rangle_2 = \Big\langle \psi, \partial_t r_N(t_n)\Delta t + \frac{1}{2}\partial_t^2 r_N(t_n)\Delta t^2 + O(\Delta t^3) - \Delta t\mathcal{B}^{-1}\mathcal{A}(\partial_x q_N(t_n) \\ + \frac{\Delta t}{2}\partial_t\partial_x q_N(t_n) + O(\Delta t^2)) + \frac{3}{2}\Delta t(-\partial_t r_N(t_n) + \mathcal{B}^{-1}\mathcal{A}\partial_x q_N(t_n)) \\ - \frac{1}{2}\Delta t(-\partial_t r_N(t_{n-1}) + \mathcal{B}^{-1}\mathcal{A}\partial_x q_N(t_{n-1})\Big\rangle_2.\end{aligned}$$

Here we used that

$$\begin{aligned}r_N(t_{n+1}) - r_N(t_n) = \Delta t\partial_t r_N(t_n) + \frac{1}{2}\partial_t^2 r_N(t_n)\Delta t^2 + O(\Delta t^3), \\ \partial_x q_N(t_{n+1}) + \partial_x q_N(t_n) = 2\partial_x q_N(t_n) + \Delta t\partial_t\partial_x q_N(t_n) + O(\Delta t^2),\end{aligned}$$

and equation (18). Therefore using that

$$\begin{aligned}\partial_t r_N(t_{n-1}) = \partial_t r_N(t_n) - \Delta t\partial_t^2 r_N(t_n) + O(\Delta t^2), \\ \partial_x q_N(t_{n-1}) = \partial_x q_N(t_n) - \Delta t\partial_x\partial_t q_N(t_n) + O(\Delta t^2),\end{aligned}$$

we arrive at

$$\begin{aligned}\langle\psi,\alpha\rangle_2 = \Big\langle \psi, \partial_t r_N(t_n)\Delta t + \frac{1}{2}\partial_t^2 r_N(t_n)\Delta t^2 + O(\Delta t^3) \\ - \Delta t\mathcal{B}^{-1}\mathcal{A}\Big(\partial_x q_N(t_n) + \frac{\Delta t}{2}\partial_t\partial_x q_N(t_n) + O(\Delta t^2)\Big) \\ + \frac{\Delta t}{2}(\partial_t r_N(t_n) - \Delta t\partial_t^2 r_N(t_n) + O(\Delta t^2)) \\ - \frac{3}{2}\Delta t\partial_t r_N(t_n) + \frac{3}{2}\Delta t\mathcal{B}^{-1}\mathcal{A}\partial_x q_N(t_n) \\ - \frac{1}{2}\Delta t\mathcal{B}^{-1}\mathcal{A}(\partial_x q_N(t_n) - \Delta t\partial_x\partial_t q_N(t_n) + O(\Delta t^2))\Big\rangle_2 = O(\Delta t^3).\end{aligned}$$

Let us define

$$e^n := q_N^n - q_N(t_n), \quad f^n := r_N^n - r_N(t_n), \ n = 0, 1, 2, \dots$$

By subtracting equations (21) and (23),

$$\Big\langle e^{n+1} - e^n - \Delta t\Big(\frac{\partial_x f^{n+1} + \partial_x f^n}{2}\Big), \phi\Big\rangle_2 = -\langle\theta,\phi\rangle_2.$$

In particular for $\phi = e^{n+1}$, we obtain that

$$\begin{aligned}\|e^{n+1}\|_2^2 = \langle e^n, e^{n+1}\rangle_2 + \Delta t\Big\langle\frac{\partial_x f^{n+1} + \partial_x f^n}{2}, e^{n+1}\rangle_2 - \langle\theta, e^{n+1}\rangle_2 \\ \le \frac{1}{2}\|e^n\|_2^2 + \frac{1}{2}\|e^{n+1}\|_2^2 + \frac{\Delta t}{2}\langle\partial_x f^{n+1}, e^{n+1}\rangle_2 + \frac{\Delta t}{2}\langle\partial_x f^n, e^{n+1}\rangle_2 + \|\theta\|_2\|e^{n+1}\|_2 \\ \le \frac{1}{2}\|e^n\|_2^2 + \frac{1}{2}\|e^{n+1}\|_2^2 + \frac{\Delta t}{2}\|f^{n+1}\|_3\|e^{n+1}\|_2 + \frac{\Delta t}{2}\|f^n\|_3\|e^{n+1}\|_2 \\ + C\Delta t^{5/2}\Delta t^{1/2}\|e^{n+1}\|_2^2.\end{aligned}$$

Now using that

$$\|f^{n+1}\|_3\|e^{n+1}\|_2 \leq CN^{3-2}\|f^{n+1}\|_2\|e^{n+1}\|_2,$$

we deduce that

$$\begin{aligned}\|e^{n+1}\|_2^2 \leq \frac{1}{2}\|e^n\|_2^2 + \frac{1}{2}\|e^{n+1}\|_2^2 + CN^2\Delta t\|f^{n+1}\|_2^2 + CN^2\Delta t\|f^n\|_2^2 \\ + C\Delta t\|e^{n+1}\|_2^2 + C\Delta t^5.\end{aligned} \tag{25}$$

On the other hand, by subtracting equations (22) and (24), we obtain that

$$\begin{aligned}&\Big\langle \psi, f^{n+1} - f^n - \frac{\Delta t}{2}\mathcal{B}^{-1}\mathcal{A}(\partial_x e^{n+1} + \partial_x e^n) \\ &+ \frac{3}{2}\Delta t\mu^2\mathcal{B}^{-1}(\mathcal{P}^n - \overline{\mathcal{P}}^n) + \frac{1}{2}\Delta t\mu^2\mathcal{B}^{-1}(\overline{\mathcal{P}}^{n-1} - \mathcal{P}^{n-1})\Big\rangle_2 = -\langle\psi, \alpha\rangle_2.\end{aligned}$$

Letting $\psi = f^{n+1}$ in the previous equation, we arrive at

$$\begin{aligned}\|f^{n+1}\|_2^2 \leq \frac{1}{2}\|f^{n+1}\|_2^2 + \frac{1}{2}\|f^n\|_2^2 + \Delta t\|f^{n+1}\|_2^2 + C\Delta t N^2\|e^{n+1}\|_2^2 \\ + C\Delta t N^2\|e^n\|_2^2 + C\Delta t\|\mathcal{B}^{-1}(\mathcal{P}^n - \overline{\mathcal{P}}^n)\|_2^2 \\ + C\Delta t\|\mathcal{B}^{-1}(\mathcal{P}^{n-1} - \overline{\mathcal{P}}^{n-1})\|_2^2 + C\Delta t^5.\end{aligned}$$

To estimate the nonlinear terms on the righthand side of the previous inequality, observe that

$$\begin{aligned}&\|\mathcal{B}^{-1}(\mathcal{P}^n - \overline{\mathcal{P}}^n)\|_2^2 \\ &= \|\mathcal{B}^{-1}((r_N^n(q_N^n)^k)_x + (q_N^n)^k\partial_x r_N^n) - \mathcal{B}^{-1}((r_N(t_n)q_N(t_n)^k)_x \\ &+ q_N(t_n)^k\partial_x r_N(t_n))\|_2 \\ &\leq \|r_N^n(q_N^n)^k - (r_N(t_n)q_N(t_n)^k\|_2 + \|(q_N^n)^k\partial_x r_N^n - (q_N(t_n))^k\partial_x r_N(t_n)\|_2.\end{aligned} \tag{26}$$

Let us suppose that there exists a constant B independent of N such that

$$\max_{t\in[0,T]}\{\|q_N(t)\|_2, \|r_N(t)\|_2\} \leq B,$$

and let $0 \leq n^* < M$ be the greatest integer such that

$$\max_{0\leq n\leq n^*}\{\|q_N^n\|_2, \|r_N^n\|_2\} < 2B.$$

The existence of n^* can be ensured inasmuch as

$$\|q_N^0\|_2 = \|q_N(0)\|_2 = \|P_N q_0\|_2 \leq \|q_0\|_2 < 2B,$$

and analogously for r_N^0. Therefore for all $n \leq n^*$,

$$
\begin{aligned}
&\|r_N^n(q_N^n)^k - r_N(t_n)q_N(t_n)^k\|_2\\
&\leq \|(r_N^n - r_N(t_n))(q_N^n)^k + r_N(t_n)((q_N^n)^k - q_N(t_n)^k)\|_2\\
&\leq \|q_N^n\|_2^k\|f^n\|_2 + \|r_N(t_n)\|_2\|(q_N^n)^k - q_N(t_n)^k\|_2\\
&\leq \|q_N^n\|_2^k\|f^n\|_2\\
&+ \|r_N(t_n)\|_2\|q_N^n - q_N(t_n)\|_2\|(q_N^n)^{k-1} + k(q_N^n)^{k-2}q_N(t_n) + \cdots + q_N(t_n)^{k-1}\|_2\\
&\leq C(\|f^n\|_2 + \|e^n\|_2).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\|(q_N^n)^k\partial_x r_N^n - q_N(t_n)^k\partial_x r_N(t_n)\|_2\\
&\leq \|(q_N^n)^k\partial_x(r_N^n - r_N(t_n))\|_2 + \|\partial_x r_N(t_n)((q_N^n)^k - q_N(t_n)^k)\|_2\\
&\leq \|q_N^n\|_2^k\|\partial_x f^n\|_2\\
&+ \|\partial_x r_N(t_n)\|_2\|q_N^n - q_N(t_n)\|_2\|(q_N^n)^{k-1} + k(q_N^n)^{k-2}q_N(t_n) + \cdots q_N(t_n)^{k-1}\|_2\\
&\leq CN\|f^n\|_2 + CN\|e^n\|_2.
\end{aligned}
$$

Using the estimates above in inequality (26), we have that

$$\|\mathcal{B}^{-1}(\mathcal{P}^n - \overline{\mathcal{P}}^n)\|_2^2 \leq C(1+N^2)(\|f^n\|_2^2 + \|e^n\|_2^2).$$

Analogously, we can show that

$$\|\mathcal{B}^{-1}(\mathcal{P}^{n-1} - \overline{\mathcal{P}}^{n-1})\|_2^2 \leq C(1+N^2)(\|f^{n-1}\|_2^2 + \|e^{n-1}\|_2^2).$$

Therefore

$$
\begin{aligned}
&\|f^{n+1}\|_2^2 \leq \frac{1}{2}\|f^{n+1}\|_2^2 + \frac{1}{2}\|f^n\|_2^2 + C\Delta t\|f^{n+1}\|_2^2\\
&+ C\Delta t N^2\|e^{n+1}\|_2^2 + C\Delta t N^2\|e^n\|_2^2 + C\Delta t^5 + C\Delta t\|f^n\|_2^2 + C\Delta t\|e^n\|_2^2 \qquad (27)\\
&+ C\Delta t N^2\|f^n\|_2^2 + C\Delta t N^2\|e^n\|_2^2 + C\Delta t\|f^{n-1}\|_2^2 + C\Delta t\|e^{n-1}\|_2^2\\
&+ C\Delta t N^2\|f^{n-1}\|_2^2 + C\Delta t N^2\|e^{n-1}\|_2^2.
\end{aligned}
$$

From equations (25), (27), we deduce that

$$
\begin{aligned}
\|e^{n+1}\|_2^2 + \|f^{n+1}\|_2^2 &\leq \frac{1}{2}\|e^n\|_2^2 + \frac{1}{2}\|e^{n+1}\|_2^2 + CN^2\Delta t\|f^{n+1}\|_2^2\\
&+ CN^2\Delta t\|f^n\|_2^2 + C\Delta t\|e^{n+1}\|_2^2 + \frac{1}{2}\|f^{n+1}\|_2^2 + \frac{1}{2}\|f^n\|_2^2\\
&+ C\Delta t\|f^{n+1}\|_2^2 + C\Delta t N^2\|e^{n+1}\|_2^2 + C\Delta t N^2\|e^n\|_2^2 + C\Delta t\|f^n\|_2^2\\
&+ C\Delta t\|e^n\|_2^2 + C\Delta t\|f^{n-1}\|_2^2 + C\Delta t\|e^{n-1}\|_2^2 + C\Delta t N^2\|f^{n-1}\|_2^2\\
&+ C\Delta t N^2\|e^{n-1}\|_2^2 + C\Delta t^5.
\end{aligned}
$$

Therefore

$$\begin{aligned}\|e^{n+1}\|_2^2 + \|f^{n+1}\|_2^2 \leq\ & \|e^n\|_2^2(1 + CN^2\Delta t + C\Delta t) + \|f^n\|_2^2(1 + CN^2\Delta t + C\Delta t)\\ & + \|e^{n-1}\|_2^2(C\Delta t + C\Delta t N^2) + \|f^{n-1}\|_2^2(C\Delta t + C\Delta t N^2)\\ & + (\|e^{n+1}\|_2^2 + \|f^{n+1}\|_2^2)(\frac{1}{2} + C\Delta t + C\Delta t N^2) + C\Delta t^5.\end{aligned}$$

Now letting

$$A^n := \|e^n\|_2^2 + \|f^n\|_2^2,$$

and due to $CN^2\Delta t < \frac{1}{4}$, we obtain that

$$\left(\frac{1}{4} - C\Delta t\right) A^{n+1} \leq \left(\frac{5}{4} + C\Delta t\right) A^n + \left(\frac{1}{4} + C\Delta t\right) A^{n-1} + C\Delta^5,$$

which implies that

$$A^{n+1} \leq C(1 + \Delta t)A^n + C(1 + \Delta t)A^{n-1} + C\Delta t^5. \tag{28}$$

We recall that by hypothesis $A^0 = 0$ and $A^1 = O(\Delta t^2)$. Let us suppose that $A^n \leq C\Delta t^2$, for any $n \leq n^*$. Therefore by inequality (28), we conclude that $A^{n^*+1} = O(\Delta t^2)$. As a consequence for all $n \leq n^* + 1$, we have that

$$\|e^n\|_2 + \|f^n\|_2 \leq C\Delta t^2.$$

Thus for Δt small enough and N sufficiently large

$$\begin{aligned}\|q_N^{n^*+1}\|_2 &\leq \|q_N^{n^*+1} - q_N(t_{n^*+1})\|_2 + \|q_N(t_{n^*+1}) - q(t_{n^*+1})\|_2 + \|q(t_{n^*+1}\|_2\\ &\leq \|e^{n^*+1}\|_2 + \|q_N(t_{n^*+1}) - q(t_{n^*+1})\|_2 + \|q(t_{n^*+1}\|_2\\ &\leq C\Delta t^2 + CN^{2-\alpha} + B < 2B,\end{aligned}$$

which contradicts the maximality of n^* previously assumed. Therefore,

$$\max_{0\leq n\leq M} \|e^n\|_2 + \|f^n\|_2 \leq C\Delta t^2.$$

Finally using theorem 3.2 and triangle inequality, we can conclude that

$$\max_{0\leq n\leq M}\{\|q(t_n) - q_N^n\|_2 + \|r(t_n) - r_N^n\|_2\} \leq C(N^{2-\alpha} + \Delta t^2).$$

☑

5. Numerical experiments

In this section we present some numerical simulations using the numerical scheme described in the previous section with $\mu = 1$. In first place, note that, since any function in $u \in S_N$ can be written as

$$u = \sum_{j=-N/2}^{N/2} \hat{u}_j \phi_j,$$

with

$$\hat{u}_j = \int_0^L u(x)\overline{\phi_j(x)}dx,$$

we also have that scheme (19)-(20) is equivalent to the system

$$\frac{\hat{q}_j^{n+1} - \hat{q}_j^n}{\Delta t} = iw_j \left(\frac{\hat{r}_j^{n+1} + \hat{r}_j^n}{2} \right), \tag{29}$$

$$\frac{\hat{r}_j^{n+1} - \hat{r}_j^n}{\Delta t} = \left(\frac{iw_j(1 + aw_j^2 + Aw_j^4)}{1 + bw_j^2 + Bw_j^4} \right) \left(\frac{\hat{q}_j^{n+1} + \hat{q}_j^n}{2} \right) + \frac{3}{2} H_j[q^n, r^n] - \frac{1}{2} H_j[q^{n-1}, r^{n-1}], \tag{30}$$

subject to $\hat{q}_j^0 = \hat{q}_{0j}, \hat{r}_j^0 = \hat{r}_{0j}$, and where

$$w_j = \frac{2\pi j}{L}, \quad j = -N/2, \dots 0, \dots N/2,$$

and $H_j[\cdot, \cdot]$, $P_j[\cdot]$ denote the operators

$$H_j[q, r] = -\frac{P_j[r((q)^k)_x + 2r_x(q)^k]}{1 + bw_j^2 + Bw_j^4},$$

and

$$P_j[g] = \frac{1}{L} \int_0^L g(x) e^{-iw_j x} dx.$$

Furthermore, q^n, r^n denote the approximations of the unknowns $q(x,t)$, $r(x,t)$, respectively, at time $t = n\Delta t$, where Δt is the time step of the method. Similarly, $\hat{q}_j^n, \hat{r}_j^n$ denote the approximations to the Fourier transforms of the functions q and r, respectively, with respect to the variable x, evaluated at time $n\Delta t$. The numerical approach adopted for solving system (11) ensures that the scheme results to be linearly unconditionally stable which can be easily verified. Further, observe that the dispersive terms are approximated by using an implicit strategy, in contrast to the nonlinear terms of the model which are treated in explicit form. The main advantage of the numerical scheme described is that at each time step we can solve explicitly the approximations of the unknowns $q(x,t)$ and $r(x,t)$ without using implicit Newton-type iterations. This was possible due to the simpleness of the first equation of system (11). Thus, the scheme results to be cheap and its computer implementation is easier. We find explicitly the Fourier coefficients $\hat{q}_j^{n+1}$, $j = -N/2 + 1, \dots, N/2$ from equation (29). Then we substitute the result into equation (30) to reach the unknown Fourier coefficients $\hat{r}_j^{n+1}$. In the scheme, the spatial derivatives q_x and r_x are computed by exact differentiation of the truncated Fourier series. For instance,

$$q_x(x,t) = \sum_j iw_j P_j[q(.,t)] e^{iw_j x}.$$

The numerical calculations presented in this paper were carried out in double precision by using Matlab R2012b on a Mac platform. The Fourier-type integral appearing in the operator $P_j[.]$ is approximated by using the well-known Fast Fourier Transform (FFT) routine. We point out that scheme (29)-(30) was used by the author in [15] to solve system (4)-(5) and study the stability/instability of a family of solitary-wave solutions of the Benney-Luke-Paumond equation (1).

In order to check the accuracy of the numerical scheme described above, for $k = n_1/n_2$ such that $(n_1, n_2) = 1$ and n_1, n_2 odd integers, $\mu = 1$, $A/B \geq a/b > 1$, we will use the exact solitary wave solution of system (4)-(5) with wave speed c given by

$$q(x,t) = q_c(x - ct - a_0), \quad r(x,t) = -cq_c(x - ct - a_0), \tag{31}$$

where

$$q_c(x) = \alpha \operatorname{sech}^{\frac{4}{k}}\left(\frac{k\nu}{2}x\right),$$

with $\alpha = (c, a, b, A, B, k)$ and $\nu = \nu(c, a, b, A, B, k)$ defined as

$$\alpha = -\left(\frac{(k+1)(k+4)(3k+4)(a-bc^2)^2}{2c(4+(k+2)^2)^2(A-Bc^2)}\right)^{\frac{1}{k}}, \ \nu = \frac{1}{\sqrt{4+(k+2)^2}}\sqrt{\frac{a-bc^2}{A-Bc^2}}$$

$$\alpha = -\left(\frac{(k+1)(k+4)(3k+4)(a-bc^2)}{2c(4+(k+2)^2)}\right)^{\frac{1}{k}} \nu^{\frac{2}{k}}$$

This solution was derived by Muñoz and Quintero in [15]. Here the wave speed $0 < c < 1$ is computed as a real root of the polynomial

$$\begin{aligned} p(c) &= \left((4+(k+2)^2)^2 B - 4(k+2)^2 b^2\right) c^4 \\ &+ \left(8ab(k+2)^2 - (4+(k+2)^2)^2(A+B)\right) c^2 \\ &+ \left((4+(k+2)^2)^2 A - 4(k+2)^2 a^2\right) = 0. \end{aligned}$$

In this case, to enable the application of the Fourier-spectral scheme (29)-(30) to this non periodic setting, we approximate the initial value problem (9)-(10), with $x \in \mathbb{R}$, $t > 0$, by the periodic Cauchy problem for $x \in [0, L]$, $t > 0$ with large spatial period L. This type of approximation can be justified by the decay of the solutions of the unrestricted problem as $|x| \to \infty$.

In our experiments, given the parameters a, b, B, the parameter A is computed using the equation

$$A = B + \frac{1}{45} + \left(b - \frac{1}{3}\right)(a - b),$$

and using as initial conditions

$$q(0,x) = q_c(x - a_0), \ r(0,x) = -cq_c(x - a_0).$$

Here a_0 regulates the initial position of the solitary wave. In figure 1, we plot the numerical solution (q, r) computed at time $t = 100$ using numerical scheme (29)-(30) with numerical parameters $L = 600$, $N = 2^{12}$, $\Delta t = 1e-3$, $a_0 = 300$, $a = 4, b = 2, A = 4.3556, B = 1, k = 1$ and the resulting wave speed is $c \approx 0.7113$.

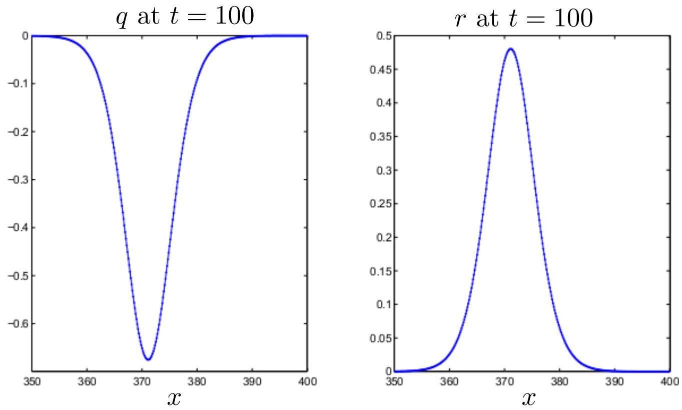

FIGURE 1. Solid line: numerical approximation using the numerical scheme (19)-(20). Pointed line: exact solitary wave (31) with speed $c \approx 0.7113$. The difference in the supremum norm between the two profiles is about $1e - 4$.

In figure 2 we repeat the previous experiment, but using $k = 3$, for which the wave speed is $c \approx 0.9$. All other parameters are left unchanged. We observe again a good accuracy of about $1e - 4$ of the numerical approximation.

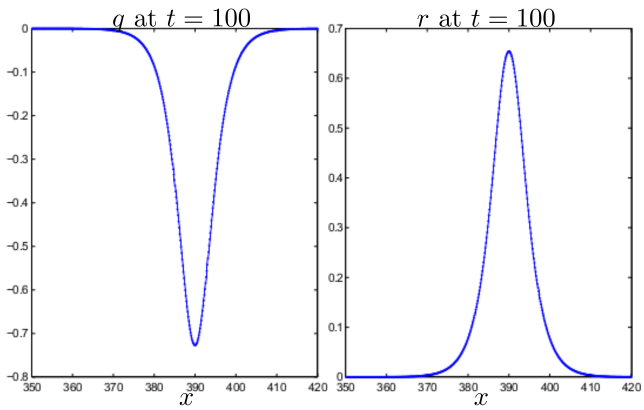

FIGURE 2. Solid line: numerical approximation using the numerical scheme (19)-(20). Pointed line: exact solitary wave (31) with speed $c \approx 0.9$. The difference in the supremum norm between the two profiles is about $1e - 4$.

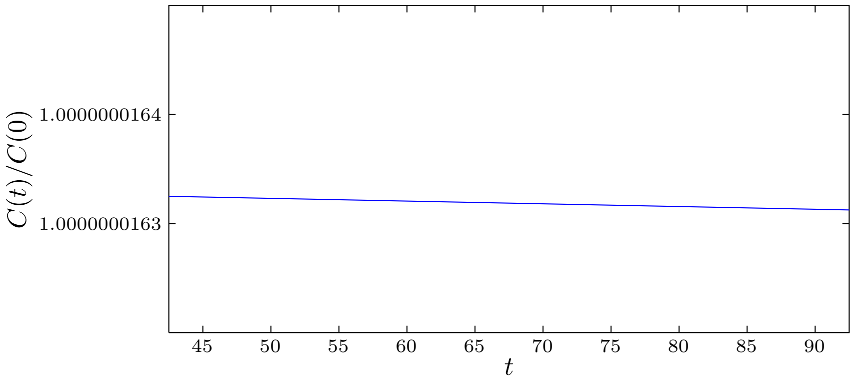

FIGURE 3. The time evolution of $C(t)/C(0)$ for the fully discrete method (19)-(20). Observe that this quantity remains near 1.

5.1. Checking conservation laws

In section 3, we established that the semidiscrete scheme given in (17)-(18) conserves in time the Hamiltonian

$$\mathcal{H}\begin{pmatrix} q \\ r \end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}} \left(r^2 + \mu b(r_x)^2 + \mu^2 B(r_{xx})^2 + q^2 + a\mu(q_x)^2 + A\mu^2(q_{xx})^2\right) dx. \tag{32}$$

In figure 3, we illustrate numerically that the fully discrete scheme (19)-(20) conserves approximately the discrete version of $\mathcal{H}(q(t), r(t))$ with $\mu = 1$, given by

$$\begin{aligned} C(t) = \frac{1}{2}\Delta x \sum_{i=0}^{N-1} \Big[&(r(i\Delta x, t))^2 + b(\partial_x^h r(i\Delta x, t))^2 + B(\partial_{xx}^h r(i\Delta x, t))^2 \\ &+ (q(i\Delta x, t))^2 + a(\partial_x^h q(i\Delta x, t))^2 + A(\partial_{xx}^h q(i\Delta x, t))^2 \Big], \end{aligned}$$

where

$$\begin{aligned} \partial_x^h q(x,t) &= \frac{q(x+\Delta x, t) - q(x,t)}{\Delta x}, \\ \partial_{xx}^h q(x,t) &= \frac{q(x+\Delta x) - 2q(x,t) + q(x-\Delta x, t)}{\Delta x^2}, \end{aligned}$$

and analogous expressions for $\partial_x^h r(x,t)$ and $\partial_{xx}^h r(x,t)$. In this experiment we used the same parameters as in the simulation displayed in figure 2.

5.2. Convergence rate in space

We want to validate the spectral order of convergence in space of the numerical scheme proposed in the present paper. In Figure 4, we fix a small time step $\Delta t = 1e - 5$, $L = 600$ and increase the number of points in space. We use the solitary wave solution given in (31) with the same model's parameters as in the experiment in the previous computer simulation. We start with $N = 2^9$ and the increase by 2 until we get $N = 2^{11}$. For every value of N we compute the numerical solution until time $t = 1$. We can check from Figure 4 that the fully discrete method (19)-(20) used in the present paper has spectral accuracy in space (as established in Theorem 4.1), and the error decreases very rapidly approximately as $N^{-7.03}$, in contrast with pure finite difference methods, for instance.

5.3. Convergence rate in time

We now validate numerically the order in time for the numerical scheme (19)-(20) proposed in the present paper. We use again the solitary wave solution (31) with the same model's parameters as in the previous numerical simulation, except that $L = 60$. We also choose $N = 2^{20}$, ($\Delta x = L/N \approx 5.7e - 5$) so that the error in space does not dominate the total error. By starting with $\Delta t = 1/2$ and decreasing the time step by 1/2 until $\Delta t = 1/8$, the numerical solution is computed until $t = 1$. We get Figure 5, from where we can see that the error in time of the numerical scheme is of order 2, in accordance with Theorem 4.1.

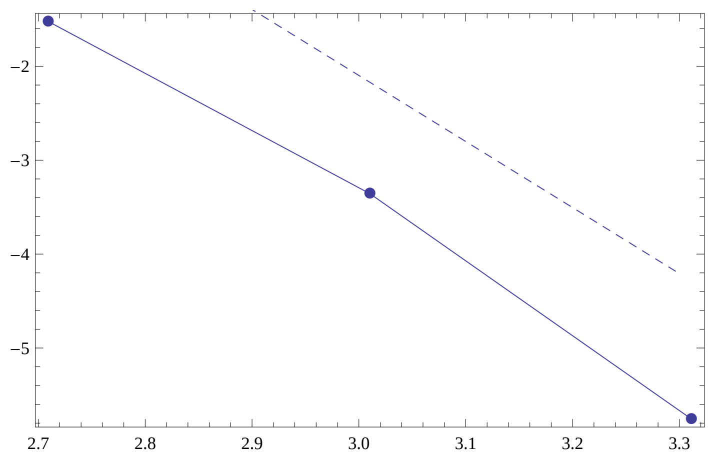

FIGURE 4. Plot of the decimal logarithm of the maximum error against $\log_{10} N$. The time step is fixed at $\Delta t = 1e - 5$. We see that the plot is approximately a line with slope -7.03.

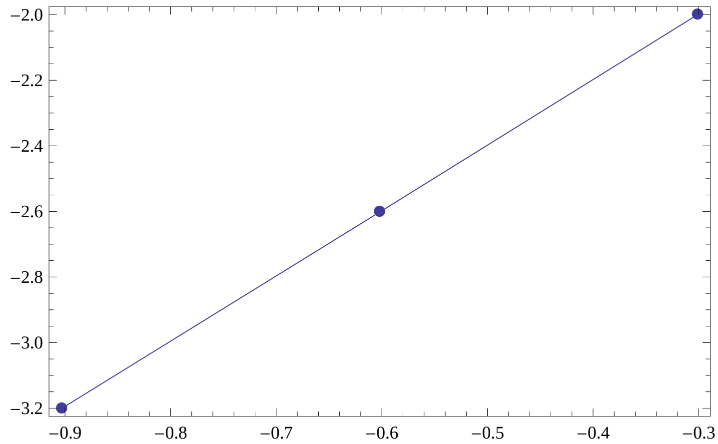

FIGURE 5. Plot of the decimal logarithm of the maximum error against $\log_{10} \Delta t$. The number of points in space is fixed at $N = 2^{20}$. We see that the plot is approximately a line with slope 2.

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