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Power of Two–Classes in *k*–Generalized Fibonacci Sequences

Clases de potencias de dos en sucesiones *k*–generalizadas de Fibonacci

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Abstract. The k-generalized Fibonacci sequence $\left(F_n^{(k)}\right)_{n\geq 2-k}$ is the linear recurrent sequence of order k, whose first k terms are $0,\ldots,0,1$ and each term afterwards is the sum of the preceding k terms. Two or more terms of a k-generalized Fibonacci sequence are said to be in the same *power of two*-class if the largest odd factors of the terms are identical. In this paper, we show that for each $k\geq 2$, there are only two kinds of *power of two*-classes in a k-generalized Fibonacci sequence: one, whose terms are all the powers of two in the sequence and the other, with a single term.

Key words and phrases. k–Generalized Fibonacci numbers, Lower bounds for nonzero linear forms in logarithms of algebraic numbers.

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Resumen. La sucesión k–generalizada de Fibonacci $\left(F_n^{(k)}\right)_{n\geq 2-k}$ es la sucesión lineal recurrente de orden k, cuyos primeros k términos son $0,\ldots,0,1$ y cada término posterior es la suma de los k términos precedentes. Se dice que dos o más términos de una sucesión k–generalizada de Fibonacci están en la misma *clase de potencia de dos* si los mayores factores impares de los términos son idénticos. En este trabajo, se muestra que para cada $k\geq 2$, sólo hay dos tipos de clases de potencias de dos en una secuencia k–generalizada de Fibonacci: una, cuyos términos son todas las potencias de dos en la sucesión y la otra, con un único término.

Palabras y frases clave. Números de Fibonacci k-generalizados, cotas inferiores para formas lineales en logaritmos de números algebraicos.

1. Introduction

Let $k \geq 2$ be an integer. One generalization of the Fibonacci sequence, which is sometimes called the *k-generalized Fibonacci sequence* $\left(F_n^{(k)}\right)_{n \geq -(k-2)}$, is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \qquad \text{for all} \quad n \geq 2,$$

with the initial conditions $F_{2-k}^{(k)} = F_{3-k}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We refer to $F_n^{(k)}$ as the n^{th} *k-generalized Fibonacci number* or *k-Fibonacci number*. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, the familiar n^{th} Fibonacci number. For $k = 3$ such numbers are called *Tribonacci* numbers. They are followed by the *Tetranacci* numbers for $k = 4$, and so on. An interesting fact about the k–generalized Fibonacci sequence is that the k values after the k initial values are powers of two. Indeed,

$$F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \ldots, F_{k+1}^{(k)} = 2^{k-1}. \tag{1}$$

This is, $F_n^{(k)} = 2^{n-2}$, for all $2 \leq n \leq k+1$. Furthermore, Bravo and Luca showed in [1] that $F_n^{(k)} < 2^{n-2}$ for all $n \geq k+2$. They also showed that except for the trivial cases, there are no powers of two in any k–generalized Fibonacci sequence for any $k \geq 3$, and that the only nontrivial power of two in the Fibonacci sequence is $F_6 = 8$.

For $k \geq 2$, we say that distinct k–Fibonacci numbers $F_m^{(k)}$ and $F_n^{(k)}$ are in the same *power of two-class* if there exist positive integers x and y such that $2^x F_m^{(k)} = 2^y F_n^{(k)}$. That is to say that the largest odd factors are identical. The sequence $\left(F_n^{(k)}\right)_{n \geq 1}$ is partitioned into disjoint classes by means of the above equivalence relation. A *power of two*-class containing more than one term of the sequence is called *non–trivial*. This definition is an analogy to the one of square-class in Fibonacci and Lucas numbers given by Ribenboim [9].

In this paper, we characterize the *power of two*–class of k–generalized Fibonacci numbers for each k. This leads to analyzing the Diophantine equation

$$F_m^{(k)} = 2^s F_n^{(k)}, \qquad \text{with} \quad n, m \geq 1, \quad k \geq 2 \quad \text{and} \quad s \geq 1. \tag{2}$$

Equations analogous to (2) have been studied for the case of Fibonacci numbers:

$$F_m = 2x^2 F_n, \qquad F_m = 3x^2 F_n, \qquad F_m = 6x^2 F_n.$$

For more details, see [7].

Before getting to the details, we give a brief description of our method. We first use lower bounds for linear forms in logarithms of algebraic numbers to bound n, m and s polynomially in terms of k. When k is small, we use the theory of continued fractions by means of a variation of a result of Dujella and Pethö to lower such bounds to cases that allow us to treat our problem computationally. When k is large, we use the fact that the dominant root of the k-generalized Fibonacci sequence is exponentially close to 2, to substitute this root by 2 in our calculations with linear form in logarithms obtaining in this way a simpler linear form in logarithms which allows us to bound k and then complete the calculations.

2. Some Results on $k-$Fibonacci Numbers

The characteristic polynomial of the k-generalized Fibonacci sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The above polynomial has just one root $\alpha(k)$ outside the unit circle. It is real and positive so it satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. In particular, $\Psi_k(x)$ is irreducible in $\mathbb{Q}[x]$. Lemma 2.3 in [6] shows that

$$2\left(1 - 2^{-k}\right) < \alpha(k) < 2, \qquad \text{for all} \qquad k \geq 2. \tag{3}$$

This inequality was rediscovered by Wolfram [10].

We put $\alpha := \alpha(k)$. This is called the *dominant root* of $\Psi_k(x)$ for reasons that we present below. Dresden and Du [3], gave the following Binet-like formula for $F_n^{(k)}$

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha^{(i)} - 1}{2 + (k+1)(\alpha^{(i)} - 2)} {\alpha^{(i)}}^{n-1}, \tag{4}$$

where $\alpha = \alpha^{(1)}, \ldots, \alpha^{(k)}$ are the roots of $\Psi_k(x)$. Dresden and Du also showed that the contribution of the roots which are inside the unit circle to the right–hand side of (4) is very small. More precisely, he proved that

$$\left| F_n^{(k)} - \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2}, \qquad \text{for all} \quad n \geq 1. \tag{5}$$

Moreover, Bravo and Luca (see [1]) extended a well known property of the Fibonacci numbers, by proving that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}, \tag{6}$$

for all $n \geq 1$ and $k \geq 2$. Further, the sequences

$$\left(F_n^{(k)}\right)_{n\geq 1}, \qquad \left(F_n^{(k)}\right)_{k\geq 2} \qquad \text{and} \qquad \left(\alpha(k)\right)_{k\geq 2} \tag{7}$$

are non decreasing. Particularly, $\alpha \geq 2\left(1 - 2^{-3}\right) = 1.75$ for all $k \geq 3$.

We consider the function

$$f_k(z) := \frac{z-1}{2+(k+1)(z-2)}, \qquad \text{for} \qquad k \geq 2.$$

If $z \in \big(2(1-2^{-k}),\, 2\big)$, a straightforward verification shows that $\partial_z f_k(z) < 0$. Indeed,

$$\partial_z f_k(z) = \frac{1-k}{\big(2+(k+2)(z-2)\big)^2} < 0, \qquad \text{for all} \qquad k \geq 2.$$

Thus, from (3), we conclude that

$$\frac{1}{2} = f_k(2) \leq f_k(\alpha) \leq f_k\Big(2\big(1-2^{-k}\big)\Big) = \frac{2^{k-1}-1}{2^k-k-1} \leq \frac{3}{4},$$

for all $k \geq 3$. Even more, since $f_2\big((1+\sqrt{5})/2\big) = 0.72360\ldots < 3/4$, we deduce that $f_k(\alpha) \leq 3/4$ holds for all $k \geq 2$. On the other hand, if $z = \alpha^{(i)}$ with $i = 2, \ldots, k$, then $\big|f_k\big(\alpha^{(i)}\big)\big| < 1$ for all $k \geq 2$. Indeed, as $\big|\alpha^{(i)}\big| < 1$, then $\big|\alpha^{(i)} - 1\big| < 2$ and $\big|2+(k+1)(\alpha^{(i)}-2)\big| > k-1$. Further, $f_2\big((1-\sqrt{5})/2\big) = 0.2763\ldots$

Finally, in order to replace α by 2, we use an argument that is due to Bravo and Luca (see [1]). If $1 \leq r < 2^{k/2}$, then

$$\alpha^r = 2^r + \delta \qquad \text{and} \qquad f_k(\alpha) = f_k(2) + \eta$$

with $|\delta| < 2^{r+1}/2^{k/2}$ and $|\eta| < 2k/2^k$. Thus,

$$\big|f_k(\alpha)\alpha^r - 2^{r-1}\big| < \frac{2^r}{2^{k/2}} + \frac{2^{r+1}k}{2^k} + \frac{2^{r+2}k}{2^{3k/2}}.$$

Furthermore, if $k > 10$ then $4k/2^k < 1/2^{k/2}$ and $8k/2^{3k/2} < 1/2^{k/2}$. Hence,

$$\big|f_k(\alpha)\alpha^r - 2^{r-1}\big| < \frac{2^{r+1}}{2^{k/2}}. \tag{8}$$

3. Preliminary Considerations

We completely solve (2), which in turn solves the main problem of this paper: characterize the *power of two*–classes of k–generalized Fibonacci numbers. We suppose that (m, n, s, k) is a solution of (2) with $k \geq 2$, $m > n$ and s positive integers.

We first consider the Diophantine equation (2) with Fibonacci numbers. Carmichael's Primitive Divisor Theorem (see [2]) states that for $m \geq 13$, the m^{th} Fibonacci number F_m has at least one odd prime factor that is not a factor of any previous Fibonacci number. So, (2) is impossible whenever $m > 12$.

When $1 \leq n < m \leq 12$, a simple check of the first twelve terms of the Fibonacci sequence: **1**, **1**, **2**, 3, 5, **8**, 13, 21, 34, 55, 89, 144 shows that (2) has only the following solutions.

$$(m, n, s, k) \in \{(6,1,3,2), (6,2,3,2), (6,3,2,2), (3,1,1,2), (3,2,1,2)\}.$$

We assume $k \geq 3$ and consider the following cases which determine all solutions of (2) for $n \leq k+1$:

(i) $n = 1$ and $m \leq k+1$. The solutions of (2) are given by

$$(m, n, s, k) = (t+2, 1, t, k), \quad \text{with} \quad 1 \leq t \leq k-1.$$

(ii) $2 \leq n < m \leq k+1$. From (1), the possible solutions of (2) are

$$(m, n, s, k) = (v+t, v, t, k), \quad \text{with} \quad 2 \leq v \leq k \quad \text{and} \quad 1 \leq t \leq k-1.$$

(iii) $2 \leq n \leq k+1 < m$. (2) has no solutions. Indeed, we have that $F_m^{(k)} = 2^{n+s-2}$. However, it is known from [1] that when $m > k+1$, $F_m^{(k)}$ is not a power of 2.

In the remaining of this article, we prove the following theorem.

Theorem 3.1. *The Diophantine equation* (2) *has no positive integer solutions* (m, n, s, k) *with* $k \geq 3$, $m > n \geq k+2$ *and* $s \geq 1$.

To conclude this section, we present an inequality relating to m, n and s. By equations (2), (3) and (6), we have that

$$\alpha^{n+s-2} < 2^s \alpha^{n-2} \leq 2^s F_n^{(k)} = F_m^{(k)} \leq \alpha^{m-1}$$

and

$$\alpha^{m-2} \leq F_m^{(k)} = 2^s F_n^{(k)} \leq 2^s \alpha^{n-1}.$$

Thus,

$$s \leq m - n \leq 1.3s + 1, \tag{9}$$

where we used the fact that $\log 2/\log \alpha < \log 2/\log 1.75 < 1.3$. Estimate (9) is essential for our purpose.

4. A Inequality for m and s in Terms of k

From now on, $k \geq 3$, $m > n \geq k+2$ and $s \geq 1$ are integers satisfying (2), so $n \geq 5$ and $m \geq 6$. In order to find upper bounds for m and s, we use a result of E. M. Matveev on lower bound for nonzero linear forms in logarithms algebraic numbers.

Let γ be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} \left(X - \gamma^{(i)}\right) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The logarithmic height of γ is given by

$$h(\gamma) := \frac{1}{d}\Bigg(\log a_0 + \sum_{i=1}^{d} \log \max \left\{ \left|\gamma^{(i)}\right|, 1 \right\} \Bigg).$$

One of the most cited results today when it comes to the effective resolution of exponential Diophantine equations is the following theorem of Matveev [8].

Theorem 4.1. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \qquad \text{and} \qquad B \geq \max\left\{|b_1|, \ldots, |b_t|\right\}.$$

Let $A_i \geq \max\left\{Dh(\gamma_i), |\log \gamma_i|, 0.16\right\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right).$$

By using formula (4) and estimate (5), we can write

$$F_m^{(k)} = f_k(\alpha)\alpha^{m-1} + e_k(m), \qquad \text{where} \qquad |e_k(m)| < 1/2. \tag{10}$$

Hence, equation (2) can be rewritten as

$$f_k(\alpha)\alpha^{m-1} - 2^s f_k(\alpha)\alpha^{n-1} = 2^s e_k(n) - e_k(m). \tag{11}$$

Dividing both sides of (11) by $2^s f_k(\alpha)\alpha^{n-1}$ and taking absolute values, we get

$$\left|2^{-s}\alpha^{m-n} - 1\right| < \frac{2^s + 1}{2^{s+1} f_k(\alpha)\alpha^{n-1}} < \frac{1.5}{1.75^{n-1}}, \tag{12}$$

where we have used the facts: $f_k(\alpha) > 1/2$, $\alpha > 1.75$ for all $k \geq 3$ and $s \geq 1$.

We apply Theorem 4.1 with the parameters $t := 2$, $\gamma_1 := 2$, $\gamma_2 := \alpha$, $b_1 := -s$, $b_2 := m - n$. Hence, $\Lambda_1 := 2^{-s}\alpha^{m-n} - 1$ and from (12) we have that

$$|\Lambda_1| < \frac{1.5}{1.75^{n-1}}. \tag{13}$$

The algebraic number field $\mathbb{K} := \mathbb{Q}(\alpha)$ contains γ_1 and γ_2 and has degree k over $\mathbb{Q}$; i.e., $D = k$. To see that $\Lambda_1 \neq 0$, we note that otherwise we would get the relation $\alpha^{m-n} = 2^s$. Conjugating this relation by an automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ with $\sigma(\alpha) = \alpha^{(i)}$ for some $i > 1$, we get that $\left(\alpha^{(i)}\right)^{m-n} = 2^s$. Then $2^s = \left|\alpha^{(i)}\right|^{m-n} < 1$, which is impossible. Thus, $\Lambda_1 \neq 0$.

Since $h(\gamma_1) = \log 2$, and by the properties of the roots of $\Psi_k(x)$, $h(\gamma_2) = (\log\alpha)/k < (\log 2)/k$. We can take $A_1 := 0.7k$ and $A_2 := 0.7$. Finally, from (9), we can take $B := 1.3s + 1$.

Theorem 4.1 gives the following lower bound for $|\Lambda_1|$

$$\exp\Big(-1.4\times 30^5\times 2^{4.5}k^2(1+\log k)\big(1+\log(1.3s+1)\big)(0.7k)(0.7)\Big),$$

which is smaller than $1.5/1.75^{n-1}$ by inequality (13). Taking logarithms in both sides and performing the respective calculations, we get that

$$\begin{aligned} n &< 1 + \frac{\log 1.5}{\log 1.75} + \frac{1.4\times 30^5\times 2^{4.5}\times 0.7^2\times 6}{\log 1.75}k^3\log k\log(2s) \\ &< 4.1\times 10^9 k^3\log k\log(2s), \end{aligned} \tag{14}$$

where we used that $1 + \log k < 2\log k$ and $1 + \log(1.3s + 1) < 3\log(2s)$, for all $k \geq 3$ and $s \geq 1$.

Going back to equation (2), we rewrite it as

$$2^s F_n^{(k)} - f_k(\alpha)\alpha^{m-1} = e_k(m). \tag{15}$$

Dividing both sides of (15) by $f_k(\alpha)\alpha^{m-1}$ and taking into account identity (10) and the fact that $f_k(\alpha) > 1/2$, we get

$$\left|2^s F_n^{(k)} f_k(\alpha)^{-1}\alpha^{-(m-1)} - 1\right| < \frac{1}{2f_k(\alpha)\alpha^{m-1}} < \frac{1}{1.75^{m-1}}. \tag{16}$$

We apply again Theorem 4.1 with the parameters $t := 4$, $\gamma_1 := 2$, $\gamma_2 := F_n^{(k)}$, $\gamma_3 := f_k(\alpha)$, $\gamma_4 := \alpha$, $b_1 := s$, $b_2 := 1$, $b_3 := -1$, $b_4 := -(m-1)$. So, $\Lambda_2 := 2^s F_n^{(k)} f_k(\alpha)^{-1}\alpha^{-(m-1)} - 1$, and from (16)

$$|\Lambda_2| < \frac{1}{1.75^{m-1}}. \tag{17}$$

As in the previous application of Theorem 4.1, we have $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$, $A_1 := 0.7k$ and $A_4 := 0.7$. Moreover, we can take $B := m - 1$, since $s \leq m - n$ by inequality (9).

We are left to determine A_2 and A_3. From inequality (6), we obtain that $h(\gamma_2) = \log(F_n^{(k)}) < n\log 2$, so we can take $A_2 := 0.7nk$. Now, knowing that $\mathbb{Q}(\alpha) = \mathbb{Q}\big(f_k(\alpha)\big)$ and $\left|f_k\big(\alpha^{(i)}\big)\right| < 1$, for $i = 1, \ldots, k$ and all $k \geq 3$, we conclude that $h(\gamma_3) = (\log a_0)/k$, where a_0 is the leading coefficient of minimal primitive polynomial over the integers of γ_3. Putting

$$g_k(x) = \prod_{i=1}^{k} \Big(x - f_k\big(\alpha^{(i)}\big)\Big) \in \mathbb{Q}[x]$$

and $\mathcal{N} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}\big(2 + (k+1)(\alpha - 2)\big) \in \mathbb{Z}$, we conclude that $\mathcal{N} g_k(x) \in \mathbb{Z}[x]$ vanishes at $f_k(\alpha)$. Thus, a_0 divides $|\mathcal{N}|$. But

$$\begin{aligned} |\mathcal{N}| = \left|\prod_{i=1}^{k} \Big(2 + (k+1)\big(\alpha^{(i)} - 2\big)\Big)\right| &= (k+1)^k \left|\prod_{i=1}^{k}\Big(2 - \frac{2}{k+1} - \alpha^{(i)}\Big)\right| \\ &= (k+1)^k \left|\Psi_k\Big(2 - \frac{2}{k+1}\Big)\right| \\ &= \frac{2^{k+1}k^k - (k+1)^{k+1}}{k-1} < 2^k k^k. \end{aligned}$$

Therefore, $h(\gamma_3) < \log(2k) < 2\log k$ for all $k \geq 3$. Hence, we can take $A_3 \coloneqq 2k\log k$.

Let us see that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then $2^s F_n^{(k)} = f_k(\alpha)\alpha^{m-1}$, and from here, applying $\mathrm{N}_{\mathbb{K}/\mathbb{Q}}$ and taking value absolutes, we obtain that $\big|\mathrm{N}_{\mathbb{K}/\mathbb{Q}}\big(f_k(\alpha)\big)\big|$ is integer. However

$$\big|\mathrm{N}_{\mathbb{K}/\mathbb{Q}}\big(f_k(\alpha)\big)\big| = f_k(\alpha) \prod_{i=2}^{k} \Big|f_k\big(\alpha^{(i)}\big)\Big| < 1.$$

Therefore, $\Lambda_2 \neq 0$.

The conclusion of Theorem 4.1 and the inequality (17) yield, after taking logarithms, the following upper bound for $m-1$

$$\begin{aligned} m - 1 &< \frac{1.4 \times 30^7 \times 4^{4.5} \times 0.7^3 \times 2}{\log 1.75} k^5 n \log k (1 + \log k)\big(1 + \log(m-1)\big) \\ &< \frac{1.4 \times 30^7 \times 4^{4.5} \times 0.7^3 \times 2 \times 4}{\log 1.75} k^5 n (\log k)^2 \log(m-1), \end{aligned}$$

where we used that $1 + \log(m-1) < 2\log(m-1)$ holds for all $m \geq 6$. The last inequality leads to

$$m - 1 < 7.7 \times 10^{13}\, k^5 n (\log k)^2 \log(m-1). \tag{18}$$

Using inequality (14) to replace n in Inequality (18), we obtain

$$\begin{aligned} \frac{m-1}{\log(m-1)} &< 7.7 \times 10^{13}\, k^5 \big(4.1 \times 10^9 k^3 \log k \log(2s)\big)(\log k)^2 \\ &< 3.2 \times 10^{23}\, k^8 (\log k)^3 \log(2s). \end{aligned} \tag{19}$$

We next present an analytical argument that allows us to extract from (19) an upper bound for m depending on k and s. This argument will also be used later.

Let $h \geq 1$ be an integer. Whenever $A \geq 2(h+1)\log(h+1)$,

$$\frac{x}{(\log x)^h} < A \qquad \text{yields} \qquad x < (h+1)^h A(\log A)^h. \tag{20}$$

Indeed, we note that the function $x \mapsto x/(\log x)^h$ is increasing for all $x > e^h$. The case $h = 1$ was proved by Bravo and Luca [1], so we assume that $h \geq 2$. Arguing by contradiction, say that $x \geq (h+1)^h A(\log A)^h$, then $x > e^h$ because $A > e$. Hence,

$$A > \frac{x}{(\log x)^h} \geq \frac{(h+1)^h A(\log A)^h}{\Big(\log\big((h+1)^h A(\log A)^h\big)\Big)^h}.$$

After performing the respective simplifications, we get that $A/\log A < h+1$ and applying the argument with $h = 1$, we obtain that $A < 2(h+1)\log(h+1)$, which is false.

Applying the argument (20) in inequality (19) with $h := 1$, $x := m-1$ and $A := 3.2 \times 10^{23}\, k^8(\log k)^3\log(2s)$, we obtain

$$\begin{aligned} m-1 &< 2\big(3.2\times 10^{23}\, k^8(\log k)^3\log(2s)\big)\log\big(3.2\times 10^{23}\, k^8(\log k)^3\log(2s)\big) \\ &< 5.9\times 10^{25}\, k^8(\log k)^3\log(2s)\log \ell. \end{aligned} \tag{21}$$

Here, we used the fact that $\log\big(3.2\times 10^{23}\, k^8(\log k)^3\log(2s)\big) < 92\log\ell$, where $\ell := \max\{k, 2s\}$.

We record what we have just proved in inequalities (14) and (21).

Lemma 4.2. *If (m,n,s,k) is a solution of (2) with $k \geq 3$ and $m > n \geq k+2$, then both inequalities*

$$\begin{aligned} n &< 4.1\times 10^9 k^3\log k\log(2s), \\ m &< 6\times 10^{25}\, k^8(\log k)^3\log(2s)\log\ell \end{aligned} \tag{22}$$

hold with $\ell := \max\{k, 2s\}$.

In order to find an upper bound for m on k only, we look at ℓ. If $\ell = k$, then from (22), we conclude that

$$n < m < 6\times 10^{25}\, k^8(\log k)^5. \tag{23}$$

If $\ell = 2s$, then from (22), we get

$$n < m < 6\times 10^{25}\, k^8(\log k)^3\big(\log(2s)\big)^2. \tag{24}$$

We return to the inequality (15) and divide both sides by $2^sF_n^{(k)}$. From identity (10), we have

$$\left|2^{-s}(F_n^{(k)})^{-1}f_k(\alpha)\alpha^{m-1}-1\right| < \frac{1}{\left(2F_n^{(k)}\right)2^s} < \frac{1}{2^s}. \tag{25}$$

One more time, we apply Theorem 4.1 taking the parameters $t := 4$, $\gamma_1 := 2$, $\gamma_2 := F_n^{(k)}$, $\gamma_3 := f_k(\alpha)$, $\gamma_4 := \alpha$, $b_1 := -s$, $b_2 := -1$, $b_3 := 1$, $b_4 := m-1$. In this instance, $\Lambda_3 := 2^{-s}\left(F_n^{(k)}\right)^{-1}f_k(\alpha)\alpha^{m-1}-1$ and from (25)

$$|\Lambda_3| < \frac{1}{2^s}. \tag{26}$$

Also, as before, we have $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$, $A_1 := 0.7k$, $A_2 := 0.7nk$, $A_3 := 2k\log k$, $A_4 := 0.7$, $B := m$, and $\Lambda_3 \neq 0$.

Combining the conclusion of Theorem 4.1 with inequality (26), we get, after taking logarithms, the following upper bound for s

$$\begin{aligned} s &< \frac{1.4\times 30^7\times 4^{4.5}\times 0.7^3\times 2}{\log 2}k^5(1+\log k)(1+\log m)n\log k \\ &< \frac{1.4\times 30^7\times 4^{4.5}\times 0.7^3\times 2\times 4}{\log 2}k^5(\log k)^2 n\log m \\ &< 6.3\times 10^{13}k^5(\log k)^2 n\log m. \end{aligned} \tag{27}$$

Thus, given that $k \le 2s$, by (22), we obtain that $n < 4.1\times 10^9k^3\log k\log(2s)$, $\log m < 99\log(2s)$, and by substituting these in the previous bound (27) on s, we conclude that

$$\frac{2s}{\left(\log(2s)\right)^2} < 6\times 10^{25}k^8(\log k)^3.$$

Taking $h := 2$, $x := 2s$ and $A := 6\times 10^{25}k^8(\log k)^3$, we have from (20) an upper bound on $2s$ depending only on k

$$2s < 3.4\times 10^{28}k^8(\log k)^5, \tag{28}$$

where we used the fact that inequality $\log\left(6\times 10^{25}k^8(\log k)^3\right) < 66\log k$ holds for all $k \ge 3$.

Hence, $\log(2s) < 72\log k$ for all $k \ge 3$, and returning to inequality (24), we get

$$n < m < 3.2\times 10^{29}\, k^8(\log k)^5. \tag{29}$$

Combining inequalities (23), (28) and (29), we get the following result.

Lemma 4.3. *If* (m, n, s, k) *is a solution of* (2) *with* $k \geq 3$ *and* $m > n \geq k+2$, *then both inequalities*

$$n < m < 3.2 \times 10^{29}\, k^8(\log k)^5 \qquad \text{and} \qquad s < 1.7 \times 10^{28} k^8 (\log k)^5 \tag{30}$$

hold.

5. The Case of Small k

We next treat the case $k \in [3, 360]$ showing that in such range the equation (2) has no nontrivial solutions.

We make use several times of the following result, which is a slight variation of a result due to Dujella and Pethö which itself is a generalization of a result of Baker and Davenport (see [1] and [4]). For a real number x, we put $\|x\| = \min\left\{|x-n| \,:\, n \in \mathbb{Z}\right\}$ for the distance from x to the nearest integer.

Lemma 5.1. *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational* γ *such that* $q > 6M$, *and let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let* $\epsilon := \|\mu q\| - M\|\gamma q\|$. *If* $\epsilon > 0$, *then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-l},$$

in positive integers m, n *and* l *with*

$$m \leq M \qquad \text{and} \qquad l \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

Before continuing, we find a absolute bound for n by arguments of Diophantine approximation. Returning to inequality (12), we take

$$\Gamma_1 := (m-n)\log(\alpha) - s\log 2,$$

and conclude that

$$|\Lambda_1| = \left|e^{\Gamma_1} - 1\right| < \frac{1.5}{1.75^{n-1}} < \frac{1}{3}, \tag{31}$$

because $n \geq 4$. Thus, $e^{|\Gamma_1|} < 3/2$ and from (13), given that $\Lambda_1 \neq 0$,

$$0 < |\Gamma_1| \leq e^{|\Gamma_1|}\left|e^{\Gamma_1} - 1\right| < \frac{4}{1.75^n}.$$

Dividing the above inequality by $s\log\alpha$, we obtain

$$\left|\frac{\log 2}{\log\alpha} - \frac{m-n}{s}\right| < \frac{4}{1.75^n s\log\alpha} < \frac{7.2}{1.75^n s}. \tag{32}$$

Now, for $3 \leq k \leq 360$, we put $\gamma_k := \log 2/\log\alpha$, compute its continued fraction $\left[a_0^{(k)}, a_1^{(k)}, a_2^{(k)}, \ldots\right]$ and its convergents $p_1^{(k)}/q_1^{(k)}$, $p_2^{(k)}/q_2^{(k)}, \ldots$ In each case we find an integer t_k such that $q_{t_k}^{(k)} > 1.7 \times 10^{28} k^8 (\log k)^5 > s$ and take

$$a_M := \max_{3\leq k\leq 360}\{a_i^{(k)} \,:\, 0 \leq i \leq t_k\}.$$

Then, from the known properties of continued fractions, we have that

$$\left|\gamma_k - \frac{m-n}{s}\right| > \frac{1}{(a_M+2)s^2}. \tag{33}$$

Hence, combining the inequalities (32) and (33) and taking into account that $a_M + 2 < 3.3 \times 10^{108}$ (confirmed by Mathematica) and $s < 3.4 \times 10^{52}$ by (30), we obtain

$$1.75^n < 8.1 \times 10^{161},$$

so $n \leq 667$.

As noted above, $s < 3.4 \times 10^{52}$. In order to reduce this bound, we apply Lemma 5.1. Put

$$\Gamma_3 := m \log\alpha - s\log 2 + (\log f_k(\alpha) - \log\alpha - \log F_n^{(k)}).$$

Returning to Λ_3 given by the expression (25), we have that $e^{\Gamma_3} - 1 = \Lambda_3$ and $\Gamma_3 \neq 0$ since $\Lambda_3 \neq 0$, so we distinguish the following cases. If $\Gamma_3 > 0$, then $e^{\Gamma_3} - 1 > 0$ and

$$0 < \Gamma_3 < e^{\Gamma_3} - 1 < \frac{1}{2^s}.$$

Replacing Γ_3 and dividing both sides by $\log 2$, we get

$$0 < m\left(\frac{\log\alpha}{\log 2}\right) - s + \frac{\log f_k(\alpha) - \log\alpha - \log F_n^{(k)}}{\log 2} < \frac{1.5}{2^s}. \tag{34}$$

We put

$$\gamma := \frac{\log\alpha}{\log 2}, \qquad \mu := \frac{\log f_k(\alpha) - \log\alpha - \log F_n^{(k)}}{\log 2},$$

and

$$A := 1.5, \qquad B := 2.$$

The fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$ ensures that γ is an irrational number. Even more, γ_k is transcendental by the Gelfond-Schneider theorem. Inequality (34) can be rewritten as

$$0 < m\gamma - s + \mu < AB^{-s}. \tag{35}$$

Now, we take $M := \lfloor 3.2 \times 10^{29} k^8 (\log k)^5 \rfloor$ which is an upper bound on m by (30), and apply Lemma 5.1 for each $k \in [3, 360]$ and $n \in [k+2, 667]$ to inequality (35). A computer search with Mathematica showed that the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 982, which is an upper bound on s, according to Lemma 5.1.

Continuing with the case $\Gamma_3 < 0$, from (25), we have that $\left|e^{\Gamma_3} - 1\right| < 1/2$ and therefore $e^{|\Gamma_3|} < 2$. Moreover,

$$0 < |\Gamma_3| < e^{|\Gamma_3|} - 1 < e^{|\Gamma_3|}\left|e^{\Gamma_3} - 1\right| < \frac{2}{2^s}.$$

As in the case $\Gamma_3 > 0$, after replacing $|\Gamma_3|$ and divide by $\log \alpha$ we obtain

$$0 < s\gamma - m + \mu < AB^{-s}, \tag{36}$$

where now

$$\gamma := \frac{\log 2}{\log \alpha}, \qquad \mu := \frac{\log F_n^{(k)} + \log \alpha - \log f_k(\alpha)}{\log \alpha},$$

and

$$A := 3.6, \qquad B := 2.$$

Lastly, we take $M := \lfloor 1.7 \times 10^{28} k^8 (\log k)^5 \rfloor$, which is an upper bound on s by (30), and apply again Lemma 5.1 for each $k \in [3, 360]$ and $n \in [k+2, 667]$ to inequality (36). With the help of Mathematica, we found that the maximum value of $\lfloor \log(Aq/\epsilon)/\log B \rfloor$ is 984, which is an upper bound on s, according to Lemma 5.1.

Thus, gathering all the information obtained above and considering the inequality (9), our problem is reduced to search solutions for (2) in the following range

$$k \in [3, 360], \quad n \in [k+2, 667], \quad s \in [1, 984], \quad m \in [n+1, n+1.3s+1]. \tag{37}$$

A computer search with Mathematica revealed that there are no solutions to the equation (2) in the ranges given in (37). With this, we completed the analysis of the case when k is small.

6. The Case of Large k

In this section, we assume that $k > 360$ and show that the Equation (2) has no nontrivial solutions. We have, from (30), that

$$n < m < 3.2 \times 10^{29} k^8 (\log k)^5 < 2^{k/2}.$$

Then, using inequality (8), with $r = m - 1$ and $r = n - 1$, and inequality (11), we conclude that

$$\begin{aligned}&\left|2^{m-2}-2^{n-2+s}\right| < \\ &\left|2^{m-2}-f_k(\alpha)\alpha^{m-1}\right| + \left|f_k(\alpha)\alpha^{m-1}-2^s f_k(\alpha)\alpha^{n-1}\right| + 2^s\left|f_k(\alpha)\alpha^{n-1}-2^{n-2}\right| \\ &\qquad < \frac{2^m}{2^{k/2}} + \frac{2^s+1}{2} + \frac{2^{n+s}}{2^{k/2}}.\end{aligned}$$

Now, dividing both sides by 2^{m-2}, we get

$$|1-2^{n+s-m}| < \frac{4}{2^{k/2}} + \frac{1}{2^{m-1-s}} + \frac{1}{2^{m-1}} + \frac{4}{2^{m-n-s}2^{k/2}}. \tag{38}$$

On the other hand, by (9), the left–hand side in (38) is greater than or equal to 1/2 unless $m = n + s$ in which case it is zero. However, the equality $m = n + s$ is not possible, otherwise, since $F_{m+1}^{(k)} = 2F_m^{(k)} - F_{m-k}^{(k)}$ (see [5]), we would get that $F_m^{(k)} = F_{n+s}^{(k)} < 2^s F_n^{(k)}$, which is a contradiction.

So, in summary, from (38) and the previous observation, we have that

$$\frac{4}{2^{k/2}} + \frac{1}{2^{m-1-s}} + \frac{1}{2^{m-1}} + \frac{4}{2^{m-n-s}2^{k/2}} > \frac{1}{2}. \tag{39}$$

Inequality (39) is a fact impossible, given that:

(i) $k > 360$ and $m \geq 6$;

(ii) $m - n - s \geq 1$ and $m - 1 - s \geq n \geq 4$.

Thus, we have in fact showed that there are no solutions (m, n, s, k) to (2) with $k > 360$ which completes the proof of Theorem 3.1.

7. Conclusions

We note that according to the observations from Section 3 and Theorem 3.1, it follows that there are only two types of *power of two*–classes in k–generalized Fibonacci numbers, namely, one corresponding to all powers of two and the other with a single term. Or equivalently, there are no k–generalized Fibonacci numbers having the same largest odd factor greater than one.

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