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THE GEOMETRIC LORENZ ATTRACTOR IS A HOMOCLINIC CLASS

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To Jairo Charris in memoriam.

Abstract. An *attractor* is a transitive set to which all nearby positive orbits converge. An example of an attractor is the geometric Lorenz attractor [GH]. In this paper we prove that the geometric Lorenz attractor is a homoclinic class.

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1. Introduction

The geometric Lorenz attractor and the horseshoe are two basic examples in the modern theory of dynamical systems [PT]. The latter is a classical example of a hyperbolic set and it was a main motivation to build up the hyperbolic theory [PT]. The former was introduced in an attempt to model the Lorenz equation [GW]. These two examples are different from the hyperbolic viewpoint: the horseshoe is a hyperbolic set while the geometric Lorenz attractor is not. Still, the geometric Lorenz attractor resembles the horseshoe in some aspects: each one of them is the closure of its periodic orbits, transitive (see Corollary 1) and has sensitivity with respect to initial conditions.

A homoclinic class is the closure of the transverse intersection points of the stable and unstable manifold of a hyperbolic periodic orbit. In this paper we show that the geometric Lorenz attractor is a homoclinic class.

2. Basic definitions and Theorem 1

Hereafter M denotes a compact 3-manifold. Let X be a vector field of class C^r, $r \geq 2$. We denote by X_t, $t \in \mathbb{R}$ the flow generated by X. Recall that this flow is a C^r action of $\mathbb{R}$ into M, i.e., $X : \mathbb{R} \times M \to M$, where $X_0 = id_M$ and

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$X_s \circ X_t = X_{s+t}$ for all $s, t \in I\!R$. An *orbit* of X is the set $\mathcal{O} = \mathcal{O}_X(q) = \{X_t(q) : t \in I\!R\}$ for some $q \in M$. The *omega-limit set* of a point p is the set $\omega_X(p) = \{x \in M : x = \lim_{n\to\infty} X_{t_n}(p)$ for some sequence $t_n \to \infty\}$. A *singularity* of X is a point $\sigma \in M$ such that $X(\sigma) = 0$ (equivalently $\mathcal{O}_X(\sigma) = \{\sigma\}$). A *periodic orbit* of X is an orbit $\mathcal{O} = \mathcal{O}_X(p)$ such that $X_T(p) = p$ for some minimal $T > 0$ (equivalently $\mathcal{O}$ is compact and $\mathcal{O} \neq \{p\}$).

A compact set $\Lambda \subset M$ is said to be:

- *Invariant*, if $X_t(\Lambda) = \Lambda$, $\forall t \in I\!R$.
- *Transitive*, if $\Lambda = \omega_X(p)$ for some $p \in \Lambda$.
- *Isolated*, if there is a compact neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in I\!R} X_t(U)$$

  (U is called *isolating block*).
- *Attracting*, if it is isolated and has a positively invariant isolating block U, i.e.,

$$X_t(U) \subset U, \quad \forall t \geq 0.$$

- *Attractor*, if it is a transitive attracting set for X.

Attracting sets are isolated but not conversely. For example, a saddle-type singularity is isolated but not attracting. Many authors call attractors what we call attracting sets, see [Mi].

A compact invariant set H of X is *hyperbolic* if there is a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ over H such that E^s_H is contracting, E^u_H is expanding and E^X_H denotes the direction of X [PT]. A closed orbit of X is hyperbolic if it is hyperbolic as compact invariant set of X. A hyperbolic set is *saddle-type* if $E^s_x \neq 0$ and $E^u_x \neq 0$ for all $x \in H$. The most representative example of a saddle-type hyperbolic set for 3-dimensional flows is the suspension of the horseshoe map in S^2 [PT].

Recall some properties of hyperbolic sets [HPS]. By the Invariant Manifold Theory, we have that if $H \subset M$ is a hyperbolic set for X, then for every $p \in H$ the sets

$$W^{ss}(p, X) = \{q \in M : d(X_t(q), X_t(p)) \to 0, \quad \text{as} \quad t \to \infty\}$$

and

$$W^{uu}(p, X) = \{q \in M : d(X_t(q), X_t(p)) \to 0, \quad \text{as} \quad t \to -\infty\}$$

are injectively immersed C^r submanifolds tangent at p to E^s_p and E^u_p respectively. These manifolds are called strong stable and unstable manifolds of the point p, and are invariant, i.e., $X_t(W^{ss}(p, X)) = W^{ss}(X_t(p), X)$ and $X_t(W^{uu}(p, X)) = W^{uu}(X_t(p), X)$, $\forall t \in I\!R$. The local strong stable and unstable manifolds of size $\epsilon > 0$ for p are determined as follows:

$$W^{ss}_\epsilon(p, X) = \{q \in M : d(X_t(q), X_t(p)) \leqslant \epsilon, \quad \forall t \geq 0\}$$

and

$$W^{uu}_{\epsilon}(p,X) = \{q \in M : d(X_t(q), X_t(p)) \leqslant \epsilon, \quad \forall t \leqslant 0\}.$$

Therefore, an other way of obtaining the strong stable and unstable manifolds of point p is

$$W^{ss}(p,X) = \bigcup_{t\geq 0} X_{-t}\left(W^{ss}_{\epsilon}(X_t(p),X)\right)$$

and

$$W^{uu}(p,X) = \bigcup_{t\geq 0} X_{t}\left(W^{uu}_{\epsilon}(X_{-t}(p),X)\right).$$

If $p, p' \in H$, we have that $W^{ss}(p,X)$ and $W^{ss}(p',X)$ either coincide or are disjoint (similarly for W^{uu}). The maps $p \in H \to W^{ss}(p,X)$ and $p \in H \to W^{uu}(p,X)$ are continuous on compact subsets. For all $p \in H$ we define

$$W^{s}(p,X) = \bigcup_{t\in I\!R} W^{ss}(X_t(p),X) \quad \text{and} \quad W^{u}(p,X) = \bigcup_{t\in I\!R} W^{uu}(X_t(p),X)$$

the stable and unstable sets of the point p. Then $W^s(p,X)$ and $W^u(p,X)$ are tangent at p to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ respectively, and depending continuously on p. Since M is 3-dimensional, both $W^s(p,X)$ and $W^u(p,X)$ are 2-dimensional if H is of saddle-type and $X(p) \neq 0$. If $p, p' \in H$, we have that $W^s(p,X)$ and $W^s(p',X)$ either coincide or are disjoint (similarly for W^u). When the orbit of p is compact (periodic orbit or singularity), then $W^s(p,X)$ and $W^u(p,X)$ represent the stable and unstable subminfolds of the orbit of p.

The *homoclinic class* associated to a hyperbolic periodic orbit $O = O_X(p)$ of X, denoted by $H_X(p)$, is the closure of the set of points of transverse intersection between $W^s(p,X)$ and $W^u(p,X)$,

$$H_X(p) = \mathrm{CL}(W^u(p,X) \pitchfork W^s(p,X)).$$

In Section 3 we describe the geometric Lorenz attractor. Our main result is the following:

Theorem 1. *The geometric Lorenz attractor is a homoclinic class.*

The proof of this theorem uses arguments established in [BMP] and is based on the existence of a return map F for the flow geometric Lorenz which preserves a stable foliation (whose leaves are vertical lines) of a cross-section of the flow. The map f induced in the space of leaves by F is differentiable and expanding. Then the dynamics is reduced to one-dimensional dynamics.

Corollary 1. *The geometric Lorenz attractor is the closure of its periodic orbits and is transitive.*

Proof. The result follows from fact that every homoclinic class is the closure of its periodic orbits and is transitive (Birkhoff-Smale Theorem). □

3. The geometric Lorenz attractor and Theorem 2

To start, denote by $S^3 = \mathbb{R}^3 \bigcup \{\infty\}$ the 3-sphere. The geometric Lorenz attractor is an attractor in S^3 of a flow that we will denote by Y and that we will describe next. This attractor has isolating block a solid bi-torus U in $\mathbb{R}^3$ such that the flow Y is transversal and points inward along its boundary. In $S^3 \setminus U$ the flow Y has three hyperbolic singularities, two saddle-type in $\mathbb{R}^3$ with complex stable eigenvalues, and one source in $\{\infty\}$. We define by

$$\Lambda = \bigcap_{t \geq 0} Y_t(U)$$

the invariant maximal set of Y in U. The set Λ is called geometric Lorenz attractor. See Figure 1.

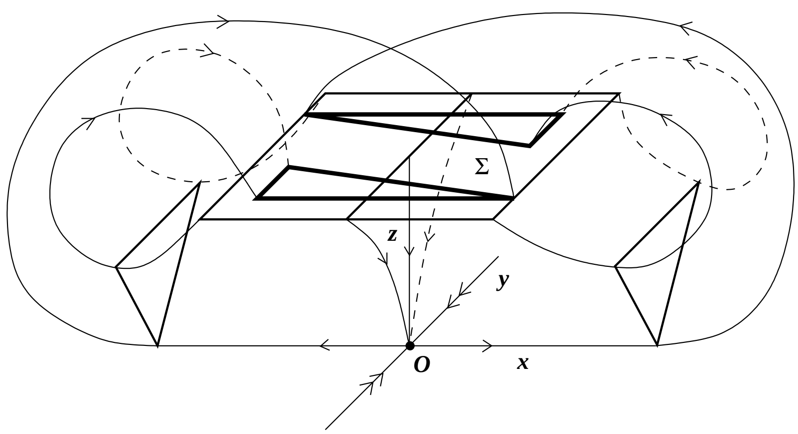

Figure 1. The geometric Lorenz attractor.

This geometric model is motivated by the Lorenz field

$$X(x, y, z) = (-ax + ay, rx - y - xz, xy - bz);\ a, r, b > 0 \tag{3.1}$$

that resulted out of a tentative to model the weather forecast in the sixties (1963). When the parameters in 3.1 are $a = 10, r = 28$ and $b = 8/3$, the numeric simulation of this field exhibits a similar behavior to a field Y called the geometric Lorenz model, which was introduced by Guckenheimer (1976).

To understand this geometric model, first consider the flow associated to the Lorenz field near the origin O. Similarly the field Y has in $O = (0, 0, 0)$ a hyperbolic singularity and by Hartman-Grobman Theorem is conjugate to the linearized equations in a neighborhood of the origin,

$$\begin{cases} x' &= \lambda_1 x \\ y' &= \lambda_2 y \\ z' &= \lambda_3 z \end{cases}$$

Resolving this system with the initial conditions $(x(0), y(0), z(0)) = (x_0, y_0, 1)$ we have:

$$\begin{cases} x(t) &= x_0(e^t)^{\lambda_1} \\ y(t) &= y_0(e^t)^{\lambda_2} \\ z(t) &= (e^t)^{\lambda_3} \end{cases}$$

Let $x_0 > 0$ and let T be the first time the orbit intersects the plane $x = 1$, i.e., $x(T) = 1$. Then, $e^T = (x_0)^{-1/\lambda_1}$ and therefore

$$\begin{cases} x(T) &= 1 \\ y(T) &= y_0(x_0)^{-\lambda_2/\lambda_1} \\ z(T) &= (x_0)^{-\lambda_3/\lambda_1} \end{cases} \tag{3.2}$$

Let $\Sigma = \{(x, y, 1) : |x| \leqslant 1/2, |y| \leqslant 1/2\}$ be a cross-section of field Y such that the map F of first return is well defined in $\Sigma^* = \Sigma \setminus \{x = 0\}$. The line $x = 0$ in Σ is contained in the intersection of $W^s(0, Y)$ with Σ. Let

$$F : \Sigma^* \to int(\Sigma) : \quad p \mapsto F(p)$$

defined for $F(p) = Y_\tau(p)$, where τ is the first positive time such that $Y_\tau(p) \in \Sigma$. Assume the following hypotheses about the field Y (for more details to see [GH], p. 273):

(H1) The point $O = (0, 0, 0)$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ that satisfy the condition $0 < -\lambda_3 < \lambda_1 < -\lambda_2$, where λ_3 is the eigenvalue of the $z-$axis, which is assumed to be invariant under the flow generated by Y.

(H2) There is a foliation $\mathcal{F}^s$ of Σ whose leaves are vertical lines such that if $L \in \mathcal{F}^s$ and F is defined in L, then $F(L)$ is contained in a leaf of $\mathcal{F}^s$. The foliation $\mathcal{F}^s$ is part of a strong stable manifold for the flow which is defined in a neighborhood of the attractor (Robinson [R]).

(H3) All point of Σ^* return to Σ, and the return map F is "sufficiently" expanding in the direction transverse to the leaves of $\mathcal{F}^s$.

(H4) The flow is symmetric with respect to rotation $\theta = \pi$ around the $z-$axis.

This four hypotheses define the geometric Lorenz flow. Analytically these hypotheses can be reformulated by means of the system of coordinates (x, y) on Σ such that F has the following properties:

(P1) The leaves of $\mathcal{F}^s$ are given by $x = c$, with $-1/2 \leqslant x \leqslant 1/2$.

(P2) There are maps f and g such that F has the form

$$F(x, y) = (f(x), g(x, y))$$

for $x \neq 0$ and $F(-x, -y) = -F(x, y)$.

(P3) $f'(x) \geq \lambda > \sqrt{2}$, for all $x \neq 0$ and $\lim_{x \to 0} f'(x) = \infty$.

(P4) $0 < \dfrac{\partial g}{\partial y} < \delta < 1$, for all $x \neq 0$ and $\lim_{x\to 0} \dfrac{\partial g}{\partial y} = 0$.

Remark 1. *Observe that:*

1. *Except from the fact that F is not defined on $x = 0$, properties* **(P3)** *and* **(P4)** *on F imply that there is a hyperbolic structure in the orbits of F in Σ.*
2. *By hypotheses* **(H1)** *and Equation 3.2 we have that*
$$z(T) = (x_0)^{-\lambda_3/\lambda_1} > x_0.$$
Therefore, near the origin O there is an expansion transverse to the foliation $\mathcal{F}^s$.
3. *We can think that each leave $L \in \mathcal{F}^s$ is contained in the intersection of Σ with $W^s(q, Y)$ for some $q \in L$ and $L \subset W^{ss}(q, Y)$.*

Let $B = [-1/2, 1/2]$ be the space of leaves of the foliation $\mathcal{F}^s$, i.e., B is the quotient of Σ by $\mathcal{F}^s$ with $\pi : \Sigma \to B : (x, y) \to x$ being the projection map. Regarding property **(P2)**, let $f : B \setminus \{0\} \to B$ be the quotient map induced for F, where $f \circ \pi = \pi \circ F$. See Figure 2. Note that $f(B \setminus \{0\}) = (-1/2, 1/2)$.

If $V \subset \Sigma$ and $J \subset B$, we have the following conventions: $F(V) := F(V \setminus \{x = 0\})$, $f(J) := f(J \setminus \{0\})$ and $B^0 = (-1/2, 1/2)$. Therefore we can write $f : B \to B$, $f : B^0 \to B^0$ and $F : \Sigma \to \Sigma$.

We define

$$A = \mathrm{CL}\left(\bigcap_{n\geq 0} F^n(\Sigma)\right). \tag{3.3}$$

Here $\mathrm{CL}(S)$ denotes the closure of S. Clearly, all points of Σ either tend to A or have trajectories that end on the leaf $x = 0$, where F is not defined.

Theorem 2. *The compact invariant subset A of F is a homoclinic class for F.*

4. Proofs of Theorems 1 and 2

Theorem 1 is direct consequence of Theorem 2. Therefore, first we will prove Theorem 1 using Theorem 2, and at the end of this section we will prove Theorem 2.

Proof. **Of Theorem 1.**
Since

$$\Lambda = \mathrm{CL}\left(\bigcup_{t\geq 0} Y_t(A)\right),$$

then by Theorem 2, we have that Λ is a homoclinic class of flow Y. ∎

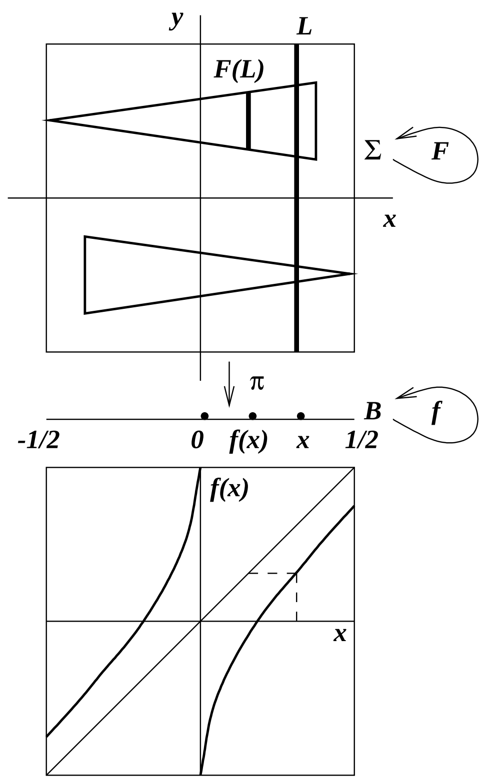

FIGURE 2. Maps of the geometric Lorenz model.

Lemma 1. *The map $f : B^0 \to B^0$ is LEO (locally eventually onto), i.e., for all open interval I of B^0 there is a integer $n \geq 0$ such that $f^n(I) = (-1/2, 1/2)$.*

Proof. Let $l(I)$ denote the length of the interval I and let $I_0 \subset B^0$ be an open interval.

(1) If $0 \notin I_0$, then $I_1 = f(I_0)$, $l(I_1) \geq \lambda l(I_0)$, replace I_0 for I_1 and continue the algorithm.

(2) If $0 \in I_0$ pick I_0^+ = the longest connected component of $I_0 \setminus \{0\}$, then $l(I_0^+) \geq (1/2)l(I_0)$. Now we analyze $f(I_0^+)$:

  (2.1) In the case $0 \notin f(I_0^+)$, then $I_1 = f^2(I_0^+)$, $l(I_1) \geq (\lambda^2/2)l(I_0)$, replace I_0 for I_1 and continue the algorithm.

  (2.2) In the case $0 \in f(I_0^+)$, then $f^2(I_0^+)$ contain $(-1/2, 0]$ or $[0, 1/2)$ and in this case we have that $f^4(I_0^+) = (-1/2, 1/2)$ and the algorithm finishes.

Now, let $\beta = \min\{\lambda, \lambda^2/2\} > 1$. Then in any case (1) or (2.1) we have that $l(I_1) \geq \beta l(I_0)$. Since $(-1/2, 1/2)$ is of finite length, there is $n \geq 0$ such that $f^n(I_0) = (-1/2, 1/2)$. □

Lemma 2. *Let $f : B^0 \to B^0$ be such as above. Then for all $x \in B^0$, it holds that*

$$B = CL\left(\bigcup_{k\geq 0} f^{-k}(\{x\})\right). \tag{4.1}$$

Proof. Let $I \subset B^0$ be any small open interval and $x \in B^0$. Since f is LEO, there is $n \geq 0$ such that $f^n(I) = (-1/2, 1/2)$. Then $x \in f^n(I)$ and so $f^{-n}(\{x\}) \bigcap I \neq \emptyset$. This proves the Lemma. □

Observe that Lemma 2 implies that for each leaf $L_x = \pi^{-1}(x)$ with $x \neq \pm 1/2$ we have

$$\Sigma = \mathrm{CL}\left(\bigcup_{k\geq 0} F^{-k}(L_x)\right). \tag{4.2}$$

Lemma 3. *The map $f : B \to B$ has dense periodic points.*

Proof. Let $I \subset B^0$ be any small open interval. By Lemma 1 there is $n \geq 0$ such that $f^n(I) = (-1/2, 1/2) \supset I$. Then f^n has a fixed point in I which implies that there is a periodic point of f in I, proving the Lemma. □

Observe that if b is a periodic point for f then the leaf $L_b = \pi^{-1}(b)$ is a periodic leaf for F. Thus, Lemma 3 together the fact that the leaves of $\mathcal{F}^s$ are contracted by F (see property **(P4)**) implies that F also has periodic points.

Lemma 4. *For all periodic points p of F, it holds that*

$$\Sigma \subset CL\left(W^s(p, F)\right). \tag{4.3}$$

Proof. Let $p = (c, d)$ be a periodic point of F with $\pi(p) = c$. Then $c \neq \pm 1/2$, $p \in A$ and the leaf $L_c = \{(c, y) : -1/2 \leqslant y \leqslant 1/2\} \in \mathcal{F}^s$ is contained in the stable manifold of the point p, $W^s(p, F)$. By Lemma 2, $B = \mathrm{CL}\left(\bigcup_{k\geq 0} f^{-k}(\{c\})\right)$, which implies that

$$\Sigma = \mathrm{CL}\left(\bigcup_{k\geq 0} F^{-k}(L_c)\right) \subset \mathrm{CL}\left(W^s(p, F)\right),$$

proving the Lemma. □

Lemma 5. *For all periodic point p of F, it holds*

$$\Sigma \setminus (L_{-1/2} \bigcup L_{1/2}) = \bigcup_{(x',y') \in W^u(p,F)} L_{x'}.$$

Proof. Since p is periodic point of F, then $O_F(p) \subset \bigcap_{n\geq 0} F^n(\Sigma)$ and by Remark 1 (1), there is $W^u(p,F)$. Let $I = W^u_\epsilon(p,F)$ for some small $\epsilon > 0$. Since f is LEO, there is $n \geq 0$ such that $f^n(\pi(I)) = (-1/2, 1/2)$. On the other hand, $f^n(\pi(I)) = \pi(F^n(I))$. If $(x,y) \in \Sigma \setminus (L_{-1/2} \cup L_{1/2})$ then $\pi(x,y) = x \in (-1/2, 1/2) = \pi(F^n(I))$. Therefore $L_x \cap F^n(I) \neq \emptyset$ and as $F^n(I) \subset W^u(p,F)$ then there is $(x', y') \in W^u(p,F) \cap L_x$ with $L_x = L_{\pi(x',y')} = L_{x'}$. The other inclusion is trivial. □

Lemma 6. *For all periodic points p of the map F, it holds that*

$$CL\,(W^u(p,F)) = A. \tag{4.4}$$

Proof. First we prove the inclusion $\mathrm{CL}(W^u(p,F)) \subset A$. To prove this, observe that $F^k(p) \notin L_{-1/2} \cup L_{1/2}$ for all $k \in \mathbb{Z}$, then $O_F(p) \subset \mathrm{int}(\Sigma)$ so there is a $\epsilon > 0$ small such that $W^u_\epsilon(F^{-k}(p), F) \subset \Sigma$ for all $k \in \mathbb{Z}$. Therefore for any $k \geq 0$,

$$W^u(F^{-k}(p), F) = \bigcup_{j\geq 0} F^j\left(W^u_\epsilon(F^{-j}(F^{-k}(p)), F)\right)$$

and

$$W^u(p,F) = F^k\left(W^u(F^{-k}(p), F)\right) \subset F^k(\Sigma).$$

Then

$$\mathrm{CL}(W^u(p,F)) \subset \mathrm{CL}(\bigcap_{k\geq 0} F^k(\Sigma)) = A.$$

Now we will prove the other inclusion. For this it is sufficient to prove that $\bigcap_{n\geq 0} F^n(\Sigma) \subset \mathrm{CL}(W^u(p,F))$. If $(x,y) \in \bigcap_{n\geq 0} F^n(\Sigma)$, since F contracts leaves of $\mathcal{F}^s$ (see property **(P4)**), we have that for all $\epsilon > 0$ there is $n_\epsilon \geq 0$ such that $d((x,y),(x,w)) < \epsilon$ for all $(x,w) \in F^{n_\epsilon}(L_{x'})$, where

$$L_{x'} \in F^{-n_\epsilon}(L_x) = \{L' \in \mathcal{F}^s : F^{n_\epsilon}(L') \subset L_x\} \ \text{ and } \ (x,y) \in F^{n_\epsilon}(L_{x'}).$$

By Lemma 5, we have that

$$\Sigma \setminus (L_{-1/2} \bigcup L_{1/2}) \subset \bigcup_{(x',y') \in W^u(p,F)} L_{x'},$$

then there is $(x', y') \in W^u(p,F) \cap L_{x'}$. Hence $d((x,y), F^{n_\epsilon}(x',y')) < \epsilon$. Since ϵ is arbitrary and $F^{n_\epsilon}(x',y') \in W^u(p,F)$ we conclude that $(x,y) \in \mathrm{CL}(W^u(p,F))$. □

Proof. **Of Theorem 2.**
Since $H_F(p) = \mathrm{CL}(W^u(p,F) \pitchfork W^s(p,F)) \subset \mathrm{CL}(W^u(p,F))$, then by Lemma 6 we have that $H_F(p) \subset A$.

Now we will prove that $A \subset H_F(p)$. By Lemma 6 it is sufficient to prove that $W^u(p,F) \subset H_p(F)$. Let $q \in W^u(p,F) \subset A$ and pick as $I \subset W^u(p,F)$ any small "horizontal" open interval such that $q \in I$. Let L be the stable leaf that contains the point p. Since f is LEO, there is $n \geq 0$ such that $f^n(\pi(I)) = (-1/2, 1/2)$. Then $F^n(I) \cap L \neq \emptyset$ and therefore, $I \cap F^{-n}(L) \neq \emptyset$. Then there is $s \in I \cap F^{-n}(L) \subset W^u(p,F) \pitchfork W^s(p,F))$ close to q. This proves Theorem 2. □

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