

# On Upper and Lower Faintly $\omega$ -Continuous Multifunctions

Multifunciones débilmente  $\omega$ -continua superior y multifunciones débilmente  $\omega$ -continua inferior

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**Abstract.** The aim of this paper is to introduce and study upper and lower faintly  $\omega$ -continuous multifunctions as a generalization of upper and lower  $\omega$ -continuous multifunctions, respectively due to Zorlutuna [7].

**Keywords:** Topological spaces,  $\omega$ -open sets,  $\omega$ -closed sets, faintly  $\omega$ -continuous multifunctions.

**Resumen.** El propósito de este artículo es introducir y estudiar multifunciones débilmente  $\omega$ -continua superior y multifunciones débilmente  $\omega$ -continua inferior, como una generalización de las multifunciones  $\omega$ -continua superior y las multifunciones  $\omega$ -continua inferior, respectivamente, dadas por Zorlutuna [7].

**Palabras claves:** Espacios topológicos, conjuntos  $\omega$ -abiertos, conjuntos  $\omega$ -cerrados, multifunciones débilmente  $\omega$ -continuas.

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## 1. Introduction

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunctions. This implies that both, functions and multifunctions are important tools for studying other properties of spaces and for constructing new spaces from previously existing ones. Recently, Zorlutuna [7] introduced and studied the concept of  $\omega$ -continuous multifunctions in topological spaces. In this paper, we introduce and study upper and lower faintly  $\omega$ -continuous multifunctions in topological spaces and we obtain some characterizations of those continuous multifunctions and also some properties are given.

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## 2. Preliminaries

Throughout this paper  $(X, \tau)$  and  $(Y, \sigma)$  (or simply  $X$  and  $Y$ ) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let  $A$  be a subset of a space  $X$ . For a subset  $A$  of  $(X, \tau)$ ,  $Cl(A)$  and  $Int(A)$  denote the closure of  $A$  with respect to  $\tau$  and the interior of  $A$  with respect to  $\tau$ , respectively. Recently, as generalization of closed sets, the notion of  $\omega$ -closed sets were introduced and studied by Hdeib [3]. A point  $x \in X$  is called a condensation point of  $A$  if for each  $U \in \tau$  with  $x \in U$ , the set  $U \cap A$  is uncountable.  $A$  is said to be  $\omega$ -closed [3] if it contains all its condensation points. The complement of an  $\omega$ -closed set is said to be  $\omega$ -open. It is well known that a subset  $W$  of a space  $(X, \tau)$  is  $\omega$ -open if and only if for each  $x \in W$ , there exists  $U \in \tau$  such that  $x \in U$  and  $U \setminus W$  is countable. The family of all  $\omega$ -open subsets of a topological space  $(X, \tau)$  forms a topology on  $X$  finer than  $\tau$ . The  $\omega$ -closure and the  $\omega$ -interior, that can be defined in the same way as  $Cl(A)$  and  $Int(A)$ , respectively, will be denoted by  $\omega Cl(A)$  and  $\omega Int(A)$ , respectively. The family of all  $\omega$ -open subsets of a topological space  $(X, \tau)$ , denoted by  $\omega O(X, \tau)$ , forms a topology on  $X$  finer than  $\tau$ . We set  $\omega O(X, x) = \{A : A \in \omega O(X, \tau) \text{ and } x \in A\}$ . A subset  $N$  of a topological space  $(X, \tau)$  is said to be  $\omega$ -neighborhood of a point  $x \in X$ , if there exists an  $\omega$ -open set  $V$  such that  $x \in V \subset N$ . A point  $x \in X$  is called a  $\theta$ -cluster point of  $A$  [6] if  $Cl(V) \cap A \neq \emptyset$  for every open set  $V$  of  $X$  containing  $x$ . The set of all  $\theta$ -cluster points of  $A$  is called the  $\theta$ -closure of  $A$  and is denoted by  $Cl_\theta(A)$ . If  $A = Cl_\theta(A)$ , then  $A$  is said to be  $\theta$ -closed [6]. The complement of  $\theta$ -closed set is said to be  $\theta$ -open [6]. The union of all  $\theta$ -open sets contained in a subset  $A$  is called the  $\theta$ -interior of  $A$  and is denoted by  $Int_\theta(A)$ . It follows from [6] that the collection of  $\theta$ -open sets in a topological space  $(X, \tau)$  forms a topology  $\tau_\theta$  on  $X$ . By a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$ , we mean a point-to-set correspondence from  $X$  into  $Y$ , also we always assume that  $F(x) \neq \emptyset$  for all  $x \in X$ . For a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$ , the upper and lower inverse of any subset  $A$  of  $Y$ , denoted by  $F^+(A)$  and  $F^-(A)$ , respectively, are the subsets  $F^+(A) = \{x \in X : F(x) \subseteq A\}$  and  $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$ . In particular,  $F^-(y) = \{x \in X : y \in F(x)\}$  for each point  $y \in Y$ . A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be surjective if  $F(X) = Y$ . A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be lower  $\omega$ -continuous [7] (resp. upper  $\omega$ -continuous) multifunction if  $F^-(V) \in \omega O(X, \tau)$  (resp.  $F^+(V) \in \omega O(X, \tau)$ ) for every  $V \in \sigma$ .

## 3. Faintly $\omega$ -continuous multifunctions

**Definition 3.1.** A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is said to be

- (i) Upper faintly  $\omega$ -continuous at  $x \in X$  if for each  $\theta$ -open set  $V$  of  $Y$  containing  $F(x)$ , there exists  $U \in \omega O(X)$  containing  $x$  such that  $F(U) \subset V$ .
- (ii) Lower faintly  $\omega$ -continuous at  $x \in X$  if for each  $\theta$ -open set  $V$  of  $Y$  such that  $F(x) \cap V \neq \emptyset$ , there exists  $U \in \omega O(X)$  containing  $x$  such that  $F(u) \cap V \neq \emptyset$  for every  $u \in U$ .

- (iii) Upper (lower) faintly  $\omega$ -continuous if it has this property at each point of  $X$ .

*Remark 3.2.* It is clear that every upper  $\omega$ -continuous multifunction is upper faintly  $\omega$ -continuous. But the converse is not true in general, as the following example shows.

**Example 3.3.** Let  $X = \mathfrak{R}$  with the topology  $\tau = \{\emptyset, \mathfrak{R}, \mathfrak{R} - Q\}$ . Define a multifunction  $F : (\mathfrak{R}, \tau) \rightarrow (\mathfrak{R}, \tau)$  as follows:

$$F(x) = \begin{cases} Q & \text{if } x \in \mathfrak{R} - Q, \\ \mathfrak{R} - Q & \text{if } x \in Q. \end{cases}$$

Then  $F$  is upper faintly  $\omega$ -continuous but is not upper  $\omega$ -continuous. In a similar form, we can find a multifunction  $G$  that is lower faintly  $\omega$ -continuous but is not lower  $\omega$ -continuous.

**Definition 3.4.** [7] A sequence  $(x_\alpha)$  is said to  $\omega$ -converge to a point  $x$  if for every  $\omega$ -open set  $V$  containing  $x$ , there exists an index  $\alpha_0$  such that for  $\alpha \geq \alpha_0$ ,  $x_\alpha \in V$ . This is denoted by  $x_\alpha \xrightarrow{\omega} x$ .

**Theorem 3.5.** For a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$ , the following statements are equivalent.

- (i)  $F$  is upper faintly  $\omega$ -continuous.
- (ii) For each  $x \in X$  and for each  $\theta$ -open set  $V$  such that  $x \in F^+(V)$ , there exists an  $\omega$ -open set  $U$  containing  $x$  such that  $U \subset F^+(V)$ .
- (iii) For each  $x \in X$  and for each  $\theta$ -closed set  $V$  such that  $x \in F^+(Y \setminus V)$ , there exists an  $\omega$ -closed set  $H$  such that  $x \in X \setminus H$  and  $F^-(V) \subset H$ .
- (iv)  $F^+(V)$  is an  $\omega$ -open set for any  $\theta$ -open set  $V$  of  $Y$ .
- (v)  $F^-(V)$  is an  $\omega$ -closed set for any  $\theta$ -closed set  $V$  of  $Y$ .
- (vi)  $F^-(Y \setminus V)$  is an  $\omega$ -closed set for any  $\theta$ -open set  $V$  of  $Y$ .
- (vii)  $F^+(Y \setminus V)$  is an  $\omega$ -open set for any  $\theta$ -closed set  $V$  of  $Y$ .
- (viii) For each  $x \in X$  and for each net  $(x_\alpha)$  which is  $\omega$ -converges to  $x \in X$  and for each  $\theta$ -open set  $V$  of  $Y$  such that  $x \in F^+(V)$ , the net  $(x_\alpha)$  is eventually in  $F^+(V)$ .

**Proof.** (i) $\Leftrightarrow$ (ii) Clear.

(ii) $\Leftrightarrow$ (iii) Let  $x \in X$  and  $V$  be a  $\theta$ -open set of  $Y$  such that  $x \in F^+(Y \setminus V)$ . By (ii), there exists an  $\omega$ -open set  $U$  containing  $x$  such that  $U \subset F^+(Y \setminus V)$ . Then  $F^-(V) \subset X \setminus U$ . Take  $H = X \setminus U$ . We have  $x \in X \setminus H$  and  $H$  is  $\omega$ -closed. The converse is similar.

(i) $\Leftrightarrow$ (iv) Let  $x \in F^+(V)$  and  $V$  be a  $\theta$ -open set of  $Y$ . By (i), there exists an  $\omega$ -open set  $U_x$  containing  $x$  such that  $U_x \subset F^+(V)$ . It follows that  $F^+(V) = \bigcup_{x \in F^+(V)} U_x$ . Since any union of  $\omega$ -open sets is  $\omega$ -open,  $F^+(V)$  is  $\omega$ -open. The converse can be shown similarly.

(iv) $\Leftrightarrow$ (v)  $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) Clear.

(i) $\Rightarrow$ (viii) Let  $(x_\alpha)$  be a net which is  $\omega$ -convergent to  $x$  in  $X$  and let  $V$  be any  $\theta$ -open set of  $Y$  such that  $x \in F^+(V)$ . Since  $F$  is an upper faintly  $\omega$ -continuous multifunction, it follows that there exists an  $\omega$ -open set  $U$  of  $X$  containing  $x$  such that  $U \subset F^+(V)$ . Since  $(x_\alpha)$   $\omega$ -converges to  $x$ , it follows that there exists an index  $\alpha_0 \in J$  such that  $x_\alpha \in U$  for all  $\alpha \geq \alpha_0$ . From here, we obtain that  $x_\alpha \in U \subset F^+(V)$  for all  $\alpha \geq \alpha_0$ . Thus, the net  $(x_\alpha)$  is eventually in  $F^+(V)$ .

(viii)  $\Rightarrow$  (i) Suppose that (i) is not true. There exists a point  $x$  and a  $\theta$ -open set  $V$  with  $x \in F^+(V)$  such that  $U \not\subset F^+(V)$  for each  $\omega$ -open set  $U$  of  $X$  containing  $x$ . Let  $x_U \in U$  and  $x_U \notin F^+(V)$  for each  $\omega$ -open set  $U$  of  $X$  containing  $x$ . Then for each  $\omega$ -neighborhood net  $(x_U)$ ,  $x_U \xrightarrow{\omega} x$ , but  $(x_U)$  is not eventually in  $F^+(V)$ . This is a contradiction. Thus,  $F$  is an upper faintly  $\omega$ -continuous multifunction.  $\square$

**Theorem 3.6.** For a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$ , the following statements are equivalent.

- (i)  $F$  is lower faintly  $\omega$ -continuous.
- (ii) For each  $x \in X$  and for each  $\theta$ -open set  $V$  such that  $x \in F^-(V)$ , there exists an  $\omega$ -open set  $U$  containing  $x$  such that  $U \subset F^-(V)$ .
- (iii) For each  $x \in X$  and for each  $\theta$ -closed set  $V$  such that  $x \in F^-(Y \setminus V)$ , there exists an  $\omega$ -closed set  $H$  such that  $x \in X \setminus H$  and  $F^+(V) \subset H$ .
- (iv)  $F^-(V)$  is an  $\omega$ -open set for any  $\theta$ -open set  $V$  of  $Y$ .
- (v)  $F^+(V)$  is an  $\omega$ -closed set for any  $\theta$ -closed set  $V$  of  $Y$ .
- (vi)  $F^+(Y \setminus V)$  is an  $\omega$ -closed set for any  $\theta$ -open set  $V$  of  $Y$ .
- (vii)  $F^-(Y \setminus V)$  is an  $\omega$ -open set for any  $\theta$ -closed set  $V$  of  $Y$ .
- (viii) For each  $x \in X$  and for each net  $(x_\alpha)$  which is  $\omega$ -converges to  $x \in X$  and for each  $\theta$ -open set  $V$  of  $Y$  such that  $x \in F^-(V)$  the net  $(x_\alpha)$  is eventually in  $F^-(V)$ .

**Proof.** The proof is similar to the one given by Theorem 3.5.  $\square$

**Lemma 3.7.** [2] Let  $A$  and  $B$  be subsets of a topological space  $(X, \tau)$ . If  $A \in \omega O(X)$  and  $B \in \tau$ , then  $A \cap B \in \omega O(B)$ .

**Theorem 3.8.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction and  $U \in \tau$ . If  $F$  is a lower (upper) faintly  $\omega$ -continuous multifunction, then multifunction  $F|_U : U \rightarrow Y$  is a lower (upper) faintly  $\omega$ -continuous multifunction.

**Proof.** Let  $V$  be any  $\theta$ -open set of  $Y$ ,  $x \in U$  and  $x \in F|_U^-(V)$ . Since  $F$  is a lower faintly  $\omega$ -continuous multifunction, there exists an  $\omega$ -open set  $G$  containing  $x$  such that  $G \subset F^-(V)$ . By Lemma 3.7, we obtain that  $x \in G \cap U \in \omega O(A)$  and  $G \cap U \subset F|_U^-(V)$ . This shows that the restriction multifunction  $F|_U$  is a lower faintly  $\omega$ -continuous.

The proof of the upper faintly  $\omega$ -continuity of  $F|_U$  can be done by the similar manner.  $\square$

**Lemma 3.9.** [4] *The following hold for a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$ .*

$$(i) G_F^+(A \times B) = A \cap F^+(B) \text{ and}$$

$$(ii) G_F^-(A \times B) = A \cap F^-(B)$$

for each subsets  $A \subset X$  and  $B \subset Y$ .

**Theorem 3.10.** *Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction. If the graph multifunction of  $F$  is an upper faintly  $\omega$ -continuous, then  $F$  is upper faintly  $\omega$ -continuous.*

**Proof.** Let  $x \in X$  and  $V$  be any  $\theta$ -open subset of  $Y$  such that  $x \in F^+(V)$ . We obtain that  $x \in G_F^+(X \times V)$  and that  $X \times V$  is a  $\theta$ -open set. Since the graph multifunction  $G_F$  is upper faintly  $\omega$ -continuous, it follows that there exists an  $\omega$ -open set  $U$  of  $X$  containing  $x$  such that  $U \subset G_F^+(X \times V)$ . Since  $U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$ . We obtain that  $U \subset F^+(V)$ . Thus,  $F$  is upper faintly  $\omega$ -continuous.  $\square$

**Theorem 3.11.** *A multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is lower faintly  $\omega$ -continuous if  $G_F : (X, \tau) \rightarrow (X \times Y, \tau \times \sigma)$  is lower faintly  $\omega$ -continuous.*

**Proof.** Suppose that  $G_F$  is lower faintly  $\omega$ -continuous. Let  $x \in X$  and  $V$  be any  $\theta$ -open set of  $Y$  such that  $x \in F^-(V)$ . Then  $X \times V$  is  $\theta$ -open in  $X \times Y$  and  $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$ . Since  $G_F$  is lower faintly  $\omega$ -continuous, there exists an  $\omega$ -open  $U$  containing  $x$  such that  $U \subset G_F^-(X \times V)$ ; hence  $U \subset F^-(V)$ . This shows that  $F$  is lower faintly  $\omega$ -continuous.  $\square$

**Theorem 3.12.** *Suppose that  $(X, \tau)$  and  $(X_\alpha, \tau_\alpha)$  are topological spaces where  $\alpha \in J$ . Let  $F : X \rightarrow \prod_{\alpha \in J} X_\alpha$  be a multifunction from  $X$  to the product space  $\prod_{\alpha \in J} X_\alpha$  and let  $P_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha$  be the projection multifunction for each  $\alpha \in J$  which is defined by  $P_\alpha((x_\alpha)) = \{x_\alpha\}$ . If  $F$  is an upper (lower) faintly  $\omega$ -continuous multifunction, then  $P_\alpha \circ F$  is an upper (lower) faintly  $\omega$ -continuous multifunction for each  $\alpha \in J$ .*

**Proof.** Take any  $\alpha_0 \in J$ . Let  $V_{\alpha_0}$  be a  $\theta$ -open set in  $(X_{\alpha_0}, \tau_{\alpha_0})$ . Then  $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P_{\alpha_0}^+(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$  (resp.  $(P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P_{\alpha_0}^-(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ ). Since  $F$  is an upper (lower) faintly  $\omega$ -continuous multifunction and since  $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$  is a  $\theta$ -open set, it follows that  $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$  (resp.  $F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ ) is an  $\omega$ -open set in  $(X, \tau)$ . This shows that  $P_\alpha \circ F$  is an upper (lower) faintly  $\omega$ -continuous multifunction. Hence, we obtain that  $P_{\alpha_0} \circ F$  is an upper (lower) faintly  $\omega$ -continuous multifunction for each  $\alpha \in J$ .  $\square$

**Theorem 3.13.** *Suppose that for each  $\alpha \in J$ ,  $(X_\alpha, \tau_\alpha)$ ,  $(Y_\alpha, \sigma_\alpha)$  are topological spaces. Let  $F_\alpha : X_\alpha \rightarrow Y_\alpha$  be a multifunction for each  $\alpha \in J$  and let  $F : \prod_{\alpha \in J} X_\alpha \rightarrow \prod_{\alpha \in J} Y_\alpha$  be defined by  $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$  from the product space  $\prod_{\alpha \in J} X_\alpha$*

to the product space  $\prod_{\alpha \in J} Y_\alpha$ . If  $F$  is an upper (lower) faintly  $\omega$ -continuous multifunction, then each  $F_\alpha$  is an upper (lower) faintly  $\omega$ -continuous multifunction for each  $\alpha \in J$ .

**Proof.** Let  $V_\alpha$  be a  $\theta$ -open set of  $Y_\alpha$ . Then  $V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$  is a  $\theta$ -open set. Since  $F$  is an upper (lower) faintly  $\omega$ -continuous multifunction, it follows that  $F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^+(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$  (resp.  $F^-(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F_\alpha^-(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$ ) is an  $\omega$ -open set. Consequently, we obtain that  $F_\alpha^+(V_\alpha)$  (resp.  $F_\alpha^-(V_\alpha)$ ) is an  $\omega$ -open set. Thus, we show that  $F_\alpha$  is an upper (lower) faintly  $\omega$ -continuous multifunction.  $\square$

**Theorem 3.14.** Let  $F$  and  $G$  be upper faintly  $\omega$ -continuous and punctually closed multifunctions from a topological space  $(X, \tau)$  to a  $\theta$ -normal space  $(Y, \sigma)$ . Then the set  $K = \{x : F(x) \cap G(x) \neq \emptyset\}$  is  $\omega$ -closed in  $X$ .

**Proof.** Let  $x \in X \setminus K$ . Then  $F(x) \cap G(x) = \emptyset$ . Since  $F$  and  $G$  are punctually closed multifunctions and  $Y$  is a  $\theta$ -normal space, it follows that there exist disjoint  $\theta$ -open sets  $U$  and  $V$  containing  $F(x)$  and  $G(x)$ , respectively. Since  $F$  and  $G$  are upper faintly  $\omega$ -continuous multifunctions, then the sets  $F^+(U)$  and  $G^+(V)$  are  $\omega$ -open sets containing  $x$ . Let  $H = F^+(U) \cap G^+(V)$ . Then  $H$  is an  $\omega$ -open set containing  $x$  and  $H \cap K = \emptyset$ ; hence  $K$  is  $\omega$ -closed in  $X$ .  $\square$

**Definition 3.15.** A topological space  $(X, \tau)$  is said to be  $\omega$ - $T_2$  [1] (resp.  $\theta$ - $T_2$  [5]) if for each pair of distinct points  $x$  and  $y$  in  $X$ , there exist disjoint  $\omega$ -open (resp.  $\theta$ -open) sets  $U$  and  $V$  in  $X$  such that  $x \in U$  and  $y \in V$ .

**Theorem 3.16.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be an upper faintly  $\omega$ -continuous multifunction and punctually closed from a topological space  $X$  to a  $\theta$ -normal space  $Y$  and let  $F(x) \cap F(y) = \emptyset$  for each pair of distinct points  $x$  and  $y$  of  $X$ . Then  $X$  is an  $\omega$ - $T_2$  space.

**Proof.** Let  $x$  and  $y$  be any two distinct points in  $X$ . Then we have  $F(x) \cap F(y) = \emptyset$ . Since  $Y$  is  $\theta$ -normal, it follows that there exist disjoint  $\theta$ -open sets  $U$  and  $V$  containing  $F(x)$  and  $F(y)$ , respectively. Thus  $F^+(U)$  and  $F^+(V)$  are disjoint  $\omega$ -open sets containing  $x$  and  $y$ , respectively and hence  $(X, \tau)$  is  $\omega$ - $T_2$ .  $\square$

**Definition 3.17.** A topological space  $(X, \tau)$  is said to be  $\theta$ -compact [5] (resp.  $\omega$ -compact [1]) if every  $\theta$ -open (resp.  $\omega$ -open) cover of  $X$  has a finite subcover. A subset  $A$  of a topological space  $X$  is said to be  $\theta$ -compact relative to  $X$  if every cover of  $A$  by  $\theta$ -open sets of  $X$  has a finite subcover.

**Theorem 3.18.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be an upper faintly  $\omega$ -continuous surjective multifunction such that  $F(x)$  is  $\theta$ -compact for each  $x \in X$ . If  $X$  is an  $\omega$ -compact space, then  $Y$  is  $\theta$ -compact.

**Proof.** Let  $\{V_\alpha : \alpha \in \Lambda\}$  be a  $\theta$ -open cover of  $Y$ . Since  $F(x)$  is  $\theta$ -compact for each  $x \in X$ , there exists a finite subset  $\Lambda(x)$  of  $\Lambda$  such that  $F(x) \subset \cup\{V_\alpha : \alpha \in \Lambda(x)\}$ . Put  $V(x) = \cup\{V_\alpha : \alpha \in \Lambda(x)\}$ . Since  $F$  is an upper faintly  $\omega$ -continuous, there exists an  $\omega$ -open set  $U(x)$  of  $X$  containing  $x$  such that  $F(U(x)) \subset V(x)$ . Then the family  $\{U(x) : x \in X\}$  is an  $\omega$ -open cover of  $X$  and since  $X$  is  $\omega$ -compact, there exists a finite number of points, say,  $x_1, x_2, x_3, \dots, x_n$  in  $X$  such

that  $X = \cup\{U(x_i): i = 1, 2, \dots, n\}$ . Hence we have  $Y = F(X) = F(\bigcup_{i=1}^n U(x_i)) = \bigcup_{i=1}^n F(U(x_i)) \subset \bigcup_{i=1}^n V(x_i) = \bigcup_{i=1}^n \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha$ . This shows that  $Y$  is  $\theta$ -compact.  $\square$

**Definition 3.19.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction. The multigraph  $G(F)$  is said to be  $\omega$ - $\theta$ -closed if for each  $(x, y) \notin G(F)$ , there exist an  $\omega$ -open set  $U$  and a  $\theta$ -open set  $V$  containing  $x$  and  $y$ , respectively, such that  $(U \times V) \cap G(F) = \emptyset$ .

**Theorem 3.20.** If a multifunction  $F : (X, \tau) \rightarrow (Y, \sigma)$  is an upper faintly  $\omega$ -continuous such that  $F(x)$  is  $\theta$ -compact relative to  $Y$  for each  $x \in X$  and  $Y$  is a  $\theta$ - $T_2$  space, then the multigraph  $G(F)$  of  $F$  is  $\omega$ - $\theta$ -closed in  $X \times Y$ .

**Proof.** Let  $(x, y) \in (X \times Y) \setminus G(F)$ . That is  $y \notin F(x)$ . Since  $Y$  is  $\theta$ - $T_2$  for each  $z \in F(x)$ , there exist disjoint  $\theta$ -open sets  $V(z)$  and  $U(z)$  of  $Y$  such that  $z \in U(z)$  and  $y \in V(z)$ . Then  $\{U(z) : z \in F(x)\}$  is a  $\theta$ -open cover of  $F(x)$  and since  $F(x)$  is  $\theta$ -compact subset relative to  $Y$ , there exists a finite number of points, say,  $z_1, z_2, \dots, z_n$  in  $F(x)$  such that  $F(x) \subset \cup\{U(z_i) : i = 1, 2, \dots, n\}$ . Put  $U = \cup\{U(z_i) : i = 1, 2, \dots, n\}$  and  $V = \cap\{V(z_i) : i = 1, 2, \dots, n\}$ . Then  $U$  and  $V$  are  $\theta$ -open sets in  $Y$  such that  $F(x) \subset U$ ,  $y \in V$  and  $U \cap V = \emptyset$ . Since  $F$  is upper faintly  $\omega$ -continuous multifunction, there exists an  $\omega$ -open set  $W$  of  $X$  containing  $x$  such that  $F(W) \subset U$ . We have  $(x, y) \in W \times V \subset (X \times Y) \setminus G(F)$ . We obtain that  $(W \times V) \cap G(F) = \emptyset$  and hence  $G(F)$  is  $\omega$ - $\theta$ -closed in  $X \times Y$ .  $\square$

**Theorem 3.21.** Let  $F : (X, \tau) \rightarrow (Y, \sigma)$  be a multifunction having  $\omega$ - $\theta$ -closed multigraph  $G(F)$ . If  $B$  is a  $\theta$ -compact subset relative to  $Y$ , then  $F^{-}(B)$  is  $\omega$ -closed in  $X$ .

**Proof.** Let  $x \in X \setminus F^{-}(B)$ . For each  $y \in B$ ,  $(x, y) \notin G(F)$  and there exist an  $\omega$ -open set  $U(y) \subset X$  and a  $\theta$ -open set  $V(y) \subset Y$ , containing  $x$  and  $y$ , respectively, such that  $F(U(y)) \cap V(y) = \emptyset$ . That is,  $U(y) \cap F^{-}(V(y)) = \emptyset$ . Then  $\{V(y) : y \in B\}$  is a  $\theta$ -open cover of  $B$  and since  $B$  is  $\theta$ -compact relative to  $Y$ , there exists a finite subset  $B_0$  of  $B$  such that  $B \subset \cup\{V(y) : y \in B_0\}$ . Put  $U = \cap\{U(y) : y \in B_0\}$ . Then  $U$  is  $\omega$ -open in  $X$ ,  $x \in U$  and  $U \cap F^{-}(B) = \emptyset$ ; that is,  $x \in U \subset X \setminus F^{-}(B)$ . This shows that  $F^{-}(B)$  is  $\omega$ -closed in  $X$ .  $\square$

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