Maximality of noncommutative rings over orders

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Julio 2019

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Thesis Work to Obtain the Degree of
Master of Sciences in Mathematics

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Bogotá, D.C.
Julio 2019

Title in English

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Título en español

Maximalidad de anillos no conmutativos sobre órdenes.

Abstract: Order theory has been widely studied since the last part of the 20th century. In the noncommutative context, maximality of orders has been reviewed for classical objects of polynomial type such as Ore extensions and PBW extensions, among others, and more recently for Ore-Rees rings. In this work we extend some results found in the literature to skew PBW extensions.

Resumen: La teoría de órdenes ha sido ampliamente estudiada desde la última parte del siglo XX. En el contexto no conmutativo, la maximalidad de órdenes ha sido revisada para objetos clásicos de tipo polinomial como las extensiones de Ore y las extensiones PBW, entre otras, y más recientemente para anillos de Ore-Rees. En este trabajo extendemos algunos resultados encontrados en la literatura a las extensiones PBW torcidas.

Keywords: Order, maximal order, Ore extension, PBW extension, skew PBW extension, Ore-Rees ring.

Palabras clave: Orden, orden maximal, extensión de Ore, extensión PBW, extensión PBW torcida, anillo de Ore-Rees.

Dedicatory

Dedicated to life, love and its absence, whose actions on me are proof of my own existence.

Acknowledgements

I would like to thank the National University of Colombia for nurturing me as a person and as a mathematician.

I thank my family for their love and support. My mother, Clara, and my grandmother Maryluz for making me who I am and for teaching me to never give up. I also thank my dogs Blacky and Tami whose warmth and love gave me energy when I was tired.

I thank my friends, specially Arturo, Carolina, Deniz and Andrés for their company, support, and help.

I thank professors Lezama, O., Galeano, J., Rubio, M., Galvis, J., Ospina M. and others for teaching me, for inspiring me and for making me want to be a good student, a good mathematician and hopefully someday a good teacher.

I particularly thank professor Reyes, A. for his guidance, his patience and for believing in me.

Contents

Introduction

Given a quotient ring Q, a subring R of Q (which may not contain a unity) is called a *right order* in Q, if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$. Similarly, we define a *left order*. If R is both a right and a left order, we call it an *order*. In the commutative context there are many examples of orders. For instance, if R is an integral domain and L is a finite separable extension of the field of fractions K of R, then the integral closure S of R is an order in L and R itself is a maximal order in its quotient ring if and only if it is completely integrally closed, as it is shown in Proposition 1.1.10. The notion of a ring being an order has been widely studied for a long time in the context of noncommutative rings, classically over Dedekind domains, see for example Section 5.3 of [MR01]. As a matter of fact, in [MR01], McConnell and Robson compile several results regarding this notion and its relation with key notions in abstract algebra, such as Noetherian rings and uniform dimension. For us, in this work, the following concept is very important: we can define an equivalence relation on a quotient ring, saying $R \sim R'$, if there are units $a, b, a', b' \in Q$ such that $aRb \subseteq R'$ and $a'R'b' \subseteq R$. R and R' are called *equivalent* right orders. An order R is said to be *maximal*, if it is so within its equivalence class. In the study of orders, it is of special importance to determine whether one of such objects is maximal.

Alongside orders, noncommutative rings of polynomial type have been studied in diverse contexts of mathematics. Therefore, it has been important to investigate when one of these rings is an order, and furthermore, a maximal one. One of the most classical noncommutative type of rings of polynomial type are the *Ore extensions* (named after Ore, who first studied them in a systematic way in [Ore33]), although their study from point of view of maximal orders is rather recent, starting with [MU17]. Further examples of noncommutative rings include *PBW extensions* defined in [BG88] by Bell and Goodearl, objects studied in this context in [MZ96] by Marubayashi and Zhang, and *Ore-Rees rings* defined and studied in [HMU16] by Helmi, Marubayashi and Ueda as a generalization to Rees rings but not necessarily satisfying the PI condition, as it had been usually studied.

On the other hand, Gallego and Lezama introduced *skew Poincaré-Birkhoff-Witt extensions* in [GL11] as a generalization to the PBW extensions. In fact, skew PBW extensions include strictly Ore extensions of injective type, which is not possible for classic PBW extensions (see [RS18b], Example 1 for a list of noncommutative rings which are skew PBW extensions but not Ore extensions). It has been shown that skew PBW extensions include a wide variety of algebras, such as the following: (1) universal enveloping algebras of finite-dimensional Lie algebras, (2) almost normalizing extensions defined by McConnell, J. and Robson, J., (3) solvable polynomial rings, (4) diffusion algebras and (5) three-dimensional skew polynomial algebras, (6) some G-algebras, (7) Auslander-Gorenstein rings, (8) some Calabi-Yau and skew Calabi-yau algebras, (9) some Artin-Schelter regular algebras, (10) quantum polynomials, (11) some quantum universal enveloping algebras and (12) some Koszul algebras among others (see [LR14], [SLR17], [Rey13],

[JR18], [RS17a], [RS17c] for a detailed list of examples). Ever since their introduction, it has been a continual and ceaseless task to study properties that hold for Ore extensions or PBW extensions to check whether they hold as well for these new type of objects. With this in mind, in this work we study Marubayashi and Zhang's results regarding orders for PBW extensions in [MZ96], as well as results in [MU17] and [HMU16] and verify if they hold, under certain additional conditions, for skew PBW extensions. Thus, the importance of this work is to establish several properties from the study of orders for noncommutative rings that can not be expressed as PBW extensions, Ore extensions or Ore-Rees rings.

The document is organized as follows: In Section 1.1, we establish some facts and definitions of the order theory, including the relation with uniform dimension and Goldie rings. Highlights of order theory for Ore extensions are studied in Section 1.2. In Section 1.3 we consider the definition of PBW extension and some examples of these objects. Ore-Rees rings are introduced in 1.4 and we present the most important results from [HMU16] regarding maximality of orders for these objects.

The definition, basic properties and examples of skew PBW extensions are presented in Section 2.1. In Section 2.2 we present our results regarding maximality of orders in skew PBW extensions, generalizing results from [MZ96], [MU17] and [HMU16]. Some key facts about graded and filtered rings are presented in Appendix A.1, since they be useful throughout this work. Finally, in Appendix A.2 we recall definitions and central results of Goldie theory, since we make us of some key facts of this theory.

Notations and conventions. Throughout this document, all rings are supposed to be associative and unless stated otherwise, noncommutative. R denotes a ring, not necessarily with unity, $\mathbb{K}$ denotes a field and $M_n(R)$ denotes the set of matrices of size $n \times n$ with entries in R. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the usual numerical systems, assuming that $0 \in \mathbb{N}$.

CHAPTER 1

Maximality on some families of noncommutative rings

The main objective of this chapter is to establish the main results of order theory and maximality of orders for some families of noncommutative rings of polynomial type that have been studied in the literature. With this purpose, we present the basic notions and key results of order theory in Section 1.1. In Section 1.2 we recall the central results of the study of maximality of orders in Ore extensions. In Section 1.3 we give the definition of PBW extension and some examples of these objects following [BG88]. Finally, in Section 1.4 we start by defining Ore-Rees rings and then we include some important results about orders in relation with such rings.

1.1 Orders and maximal orders

In this section we recall the notion of *order* and show some properties satisfied by orders and criteria to determine whether a ring is an order or not. Among orders, *maximal orders* have been objects of particular interest by several authors and researchers. We prove some of the results to show the way one works with orders.

Definition 1.1.1 ([MR01], 2.1.2)**.** $x \in R$ is said to be **right regular**, if $xr = 0$ implies $r = 0$ for $r \in R$. Similarly we define **left regular**, and **regular** means both left and right regular.

Definition 1.1.2 ([MR01], 2.1.3)**.** Let $\mathscr{S}$ be a multiplicatively closed subset of a ring R (meaning $1 \in \mathscr{S}$ and $xy \in \mathscr{S}$ if $x, y \in \mathscr{S}$) and $\operatorname{ass}(\mathscr{S}) = \{r \in R \mid rs = 0, \text{for some } s \in \mathscr{S}\}$. Then a ring Q with a homomorphism $\theta : R \to Q$ is said to be a **right quotient ring**, if it satisfies:

(i) for all $s \in \mathscr{S}$, $\theta(s)$ is a unit in Q.

(ii) for all $q \in Q$, $q = \theta(r)\theta(s)^{-1}$, for some $r \in R$, $s \in \mathscr{S}$

(iii) $\ker \theta = \operatorname{ass}(\mathscr{S})$

A ring Q is called a **quotient ring**, if every regular element of Q is a unit. Given a quotient ring Q, a subring R (which may not contain 1) is called a **right order** in Q, if each $q \in Q$ has the form rs^{-1} for some $r, s \in R$. Similarly, we define a **left order**. If R is both a right and a left order, we call it an **order** (see [MR01], 3.1.1 and 3.1.2).

Proposition 1.1.3 ([MR01], 3.1.6)**.** *Let R be a right order in a quotient ring Q and let S be a subring of Q which may not contain* 1*. If there are units a, b of Q such that $aRb \subseteq S$, then S is also a right order in Q.*

Proof. Let q be an element of Q. Then, $a^{-1}qa \in Q$. Since R is a right order in Q, there exist $r_1, r_2 \in R$ such that $a^{-1}qa = r_1r_2^{-1}$. Then, $q = ar_1r_2^{-1}a^{-1} = ar_1bb^{-1}r_2^{-1}a^{-1} = ar_1b(ar_2b)^{-1}$. Given that $aRb \subseteq S$, there exist $s_1, s_2 \in S$ such that $ar_1b = s_1$ and $ar_2b = s_2$. Thus, $q = s_1s_2^{-1}$ and S is a right order in Q. □

Proposition 1.1.4 ([MR01], 3.1.9)**.** *The relation in the previous result defines an equivalence relation on right orders R, R' in a fixed quotient ring Q, as follows: $R \sim R'$ if there are units $a, b, a', b' \in Q$ such that $aRb \subseteq R'$ and $a'R'b' \subseteq R$. R and R' are called **equivalent** right orders.*

Proof.

- Reflexive: $R \sim R'$ since $1_QR1_Q \subseteq R$.
- Symmetric: $R \sim S$ implies the existence of units $a, b, a', b' \in Q$ such that $aRb \subseteq S$ and $a'Sb' \subseteq R$. Then, $S \sim R$.
- Transitive: If $R \sim S$ and $S \sim R'$, then there exist units $a, b, c, d, a', b', c', d' \in Q$ such that $aSb \subseteq R$, $cR'd \subseteq S$, $a'Rb' \subseteq S$ and $c'Sd' \subseteq R$. Then, we get $acR'db \subseteq R$ and $a'c'Rd'b' \subseteq R'$. Therefore, $R \sim R'$.

□

Among the list of orders that a quotient ring may possess, *maximal* ones have a particular importance throughout the literature as well as this work in particular. We define them as follows:

Proposition 1.1.5 ([MR01], 3.1.10)**.** *If R, S are equivalent right orders in Q with $R \subseteq S$, then there are equivalent right orders T, T' in Q with $R \subseteq T \subseteq S$, $R \subseteq T' \subseteq S$ and units r_1, r_2 of Q contained in R such that $r_1S \subseteq T$, $Tr_2 \subseteq R$ and $Sr_2 \subseteq T'$, $r_1T' \subseteq R$. In particular, $r_1Sr_2 \subseteq R$.*

Definition 1.1.6 ([MR01], 5.1.1)**.** Let Q be a quotient ring and R a right order in Q. Then R is called a **maximal** right order, if it is maximal within its equivalence class.

The reader may remember that a ring is said to be **simple** if it does not have a nonzero bilateral ideal. As an illustration for the notion of maximal order, we have the following example:

Proposition 1.1.7 ([MR01], 5.1.2)**.** *Any simple right Goldie ring R (with 1) is a maximal order (see Definition A.2.14 in Appendix A for the definition of Goldie ring).*

Example 1.1.8. If $\mathbb{K}$ is a field of characteristic zero, then the Weyl Algebra $A_n(\mathbb{K})$ is a maximal order (see Definition 1.2.2 for a detailed explanation of $A_n(\mathbb{K})$).

Definition 1.1.9 ([MR01], 5.1.3)**.** A ring R is said to be **completely integrally closed** in its quotient field Q, if for $a, q \in Q$ with $a \neq 0$, $aq^n \in R$, for all n, implies $q \in R$.

Proposition 1.1.10 ([MR01], 5.1.3)**.** *A commutative integral domain is a maximal order in its quotient field if and only if it is completely integrally closed.*

Proof.

- $(\Rightarrow)$: If $a, q \in Q$ with $a \neq 0$, $aq^n \in R$ for all n, let us consider the ring $R' = R[q] \subseteq Q$. Then $aR' \subseteq R$ and since $R \subseteq R'$, we get $R \sim R'$. Given that R is a maximal order, we get $R' \subseteq R$, which means $R = R'$. Thus $q \in R$.

- $(\Leftarrow)$: Let R' be another order such that $R' \supseteq R$ and $R' \sim R$. Then, $aR' \subseteq R$ for some $a \in R \setminus \{0\}$. Let q be any element in R'. Then $R[q] \subseteq R'$ since $q \in R'$ and $R \subseteq R'$, and so, $aq^n \in R$ for all n. By hypothesis, $q \in R$ and thus $R = R'$.

□

We now introduce some important notions in order theory which are the concepts of *fractional ideal* and *left and right orders of a fractional ideal.* Its usefulness is immediately seen in Proposition 1.1.14.

Definition 1.1.11 ([MR01], 3.1.11)**.** Let R be a right or left order in a quotient ring Q. A **fractional right R-ideal** is a submodule I of Q_R such that $aI \subseteq R$ and $bR \subseteq I$ for some units $a, b \in Q$. Similarly, we define **fractional left R-ideal** and for both left and right conditions, we define **fractional R-ideal**. If I is a fractional right R-ideal and a fractional left S-ideal for some other order S, then we call I a **fractional (S,R)-ideal**. If I is a fractional R-ideal and aditionally $I \subseteq R$, we call I an R-ideal.

Definition 1.1.12 ([MR01], 3.1.12)**.** The **right order** of a fractional right (or left) R-ideal I is defined as $O_r(I) = \{q \in Q \mid Iq \subseteq I\}$ and the **left order** of a fractional left R-ideal I is given by the set $O_l(I) = \{q \in Q \mid qI \subseteq I\}$.

Proposition 1.1.13 ([MR01], 3.1.12)**.** *Let R be a right order in Q and let I be a fractional right or left R-ideal. Then:*

(i) $O_r(I)$ and $O_l(I)$ are right orders in Q and are equivalent to R.

(ii) I is a fractional $(O_l(I), O_r(I))$-ideal.

Proof. (i) Let us suppose that I is a fractional right R-ideal. Then, by definition, there exist $a, b \in Q$ such that $aI \subseteq R$ and $bR \subseteq I$. Then $abO_r(I) \subseteq R$ since $abq = a(b.1)q \in aIq \subseteq aI \subseteq R$ for every $q \in O_r(I)$; We get $R \subseteq O_r(I)$ because $IR \subseteq I$, for I being an R-ideal. Also, we have $aO_l(I)b \subseteq aO_l(I)I$ since $b.1 \in I$, $aO_l(I)I \subseteq aI$ since $O_l(I)I \subseteq I$ and thus, we get $aO_l(I)b \subseteq R$. Finally we see that $bRa \subseteq O_l(I)$ given that $(bRa)I \subseteq bR \subseteq I$. Therefore, by 1.1.3, we get $O_r(I)$ and $O_l(I)$ are both right orders and they are equivalent to R. The result follows similarly if I is a fractional left R-ideal.

(ii) From the relations obtained before, we see that $aI \subseteq O_r(I)$ and $babO_r(I) \subseteq bR \subseteq I$ which makes I a fractional right $O_r(I)$-ideal. On the other hand, we obtain $Ia \subseteq O_l(I)$ and $O_l(I)b \subseteq O_l(I)I \subseteq I$ given that $b = b.1 \in bR \subseteq I$. Therefore, I is a fractional $(O_l(I), O_r(I))$-ideal.

□

The following proposition is one of the most useful and commonly used ways to check the maximality of an order.

Proposition 1.1.14 ([MR01], 5.1.4)**.** *If R is a right order in Q then the following conditions are equivalent:*

(i) *R is a maximal right order*

(ii) *$O_r(I) = O_l(I) = R$, for all fractional R-ideals I*

(iii) *$O_r(I) = O_l(I) = R$, for all R-ideals I*

Proof. $(i) \Rightarrow (ii)$ If I is a fractional R-ideal, then $O_r(I)$ and $O_l(I)$ are both right orders and they are equivalent to R by Proposition 1.1.13. Since I is a fractional R-ideal, $I \subseteq R$, which implies that both $O_l(I)$ and $O_r(I)$ contain R.

$(ii) \Rightarrow (iii)$ Given I an R-ideal, it is a fractional R-ideal which further satisfies $R \subseteq I$. Then the result follows.

$(iii) \Rightarrow (i)$ Let us suppose that $S \supseteq R$ and $S \sim R$ for a right order S. Then by proposition 1.1.5 there exists an order T such that $R \subseteq T \subseteq S$ and $r_1 S \subseteq T$, $Tr_2 \subseteq R$ for units $r_1, r_2 \in Q$. Then the set $I = \{x \in R \mid Tx \subseteq R\}$ is an R-ideal since $T(x+y) \subseteq R$, $Txr \subseteq R$ for $x, y, r \in R$ and $I \subseteq R$. Also, $T \subseteq O_l(I)$, because given $t \in T$, $tI \subseteq R$, meaning that $T(tI) \subseteq R$ which implies that $tI \subseteq I$ and thus $t \in O_l(I)$. According to the hypothesis, $O_l(I) = R$, and then $T \subseteq R$, which means $R = T$. Repeating this process for S, we obtain $R = S$ and thus R is a maximal order. □

It is known that R is a semiprime right Goldie ring with right quotient ring Q if and only if $M_n(R)$ is a semiprime right Goldie ring with right quotient ring $M_n(Q)$ (see [MR01], 3.1.5). Also, by A.2.17, Q is a semisimple ring by Goldie's Theorem (see Proposition A.2.17). Interestingly, the following result can be thought as a version of Goldie's Theorem for rings without 1 while giving us an example of order related to semiprime Goldie rings.

Proposition 1.1.15 ([MR01], 3.1.7)**.** *A ring R, not necessarily with unity, is a right order in a semisimple Artinian ring Q if and only if R is semiprime right Goldie.*

The following proposition provides us with an interesting way to view a semiprime right Goldie ring and its quotient ring.

Proposition 1.1.16 ([MR01], 3.2.4)**.** *Let R be semiprime right Goldie and Q be its right quotient ring which is semisimple. Therefore $Q = \bigoplus_{i=1}^{k} Q_i$ and each Q_i being a simple Artinian ring generated by a central idempotent, $e_i = 1_{Q_i}$. The following statements hold:*

(i) $R \subseteq R' = \bigoplus_{i=1}^{k} e_i R$

(ii) *R' is a semiprime right Goldie ring and $R \sim R'$*

(iii) *$e_i R$ is a prime right Goldie ring and $e_i R \sim (Q_i \cap R)$*

Proposition 1.1.17 ([MR01], 5.1.5)**.** *Let R be a semiprime right Goldie ring. The following are equivalent:*

(i) *R is a maximal right order.*

(ii) *$M_n(R)$ is a maximal right order.*

(iii) R is a direct sum of prime right Goldie rings, each of which is a maximal right order.

The following result has a great importance since the graduation-filtration technique is a powerful and useful tool in noncommutative algebra (see Section A.1 for highlights in this topic).

Proposition 1.1.18 ([MR01], 5.1.6)**.** *Let R be a filtered ring with associated ring $\mathrm{Gr}(R)$ (see Definition A.1.1). If $\mathrm{Gr}(R)$ is a Noetherian integral domain and it is a maximal order in its quotient ring then R has the same properties.*

Proof. Let $r,s \in R$ be nonzero elements of degree n and m, respectively. Then, $\overline{r},\overline{s}$ are nonzero elements of $\mathrm{Gr}(R)$ and so $\overline{rs} \neq 0$. By the definition of $\mathrm{Gr}(R)$, this means that $rs \notin F_{m+n-1}$ and so, $rs \neq 0$.
Let us recall that if I is a right ideal of R, then we define $\mathrm{Gr}(I) = \bigoplus_n (Gr(I))_n$, with $(\mathrm{Gr}(I))_n = (I + F_{n-1}) \cap F_n / F_{n-1}$, and $\mathrm{Gr}(I)$ is an ideal of $\mathrm{Gr}(R)$. Therefore, since $\mathrm{Gr}(R)$ has no infinite strictly increasing chain of right ideals, neither has R.
Let $0 \neq I$ be an ideal of R and $q \in O_r(I)$. So $q = rs^{-1}$ with $r,s \in R$, and so, $Irs^{-1} \subseteq I$, which means $Ir \subseteq Is$. Let us see, by induction on the degree of r that this implies $r \in Rs$. Since $\mathrm{Gr}(R)$ is an integral domain, $\overline{ac} = \overline{ac}$ for all $a,c \in R$. Therefore $(\mathrm{Gr}(I))\overline{r} \subseteq (\mathrm{Gr}(I))\overline{s}$. However, $\mathrm{Gr}(I)$ is a nonzero ideal of $\mathrm{Gr}(R)$, which is a maximal order. This implies that $O_r(\mathrm{Gr}(I)) = \mathrm{Gr}(R)$, and since $\mathrm{Gr}(I)\overline{rs}^{-1} \subseteq \mathrm{Gr}(I)$, we have $\overline{rs}^{-1} \in \mathrm{Gr}(R)$, and so $\overline{r} \in \mathrm{Gr}(R)\overline{s}$. It follows that $r = xs + y$ with $x,y \in R$ and $\deg(y) < \deg(r)$. Given that $y = xs - r$, we obtain $Iy \subseteq Is$. By the induction hypothesis, since $\deg(y) < \deg(r)$, we have $y \in Rs$. Therefore, since $xs, y \in Rs$, we get $r \in Rs$ as we wanted. this means that $rs^{-1} \in R$ and thus, $O_r(I) = R$. Similarly, $O_l(I) = R$ and we conclude that R is a maximal order.

□

Remark 1.1.19. This result already gives us away to obtain skew PBW extensions which are maximal orders. For example, we can consider the Weyl algebra $A = A_n(\mathbb{K})$ defined in Example 1.2.2, which can be seen as a skew PBW extension of $\mathbb{K}[x_1,\dots,x_n]$. By Proposition 2.1.17, we know that $Gr(A) \cong (\mathbb{K}[x_1,\dots,x_n])[z_1,\theta_1]\cdots[z_n,\theta_n]$, and since A is a skew PBW extension of bijective type, then θ_i is bijective, for every $1 \leq i \leq n$. In fact, since A is a PBW extension, we have that θ_i is the identity map, for every $1 \leq i \leq n$. By Hilbert's basis theorem, we know that $\mathbb{K}[x_1,\dots,x_n]$ is Noetherian, which makes $Gr(A)$ Noetherian as well. It is known that $\mathbb{K}[x_1,\dots,x_n]$ is also a prime ring, which makes $Gr(A)$ a prime ring. Also, we have that $\mathbb{K}[x_1,\dots,x_n]$ is an integral domain and then so is $Gr(A)$; and it is a maximal order in its quotient ring $\mathbb{K}(x_1,\dots,x_n)$, which makes $Gr(A)$ by Proposition 1.2.5 a maximal order as well. With this in mind, by Proposition 1.1.18, given that $Gr(A)$ is a Noetherian integral domain which is a maximal order in its quotient ring, then A has the same properties.

Example 1.1.20 ([MR01], 5.1.6)**.** Let $\mathscr{G}$ be a finite dimensional $\mathbb{K}$-Lie algebra (see Definition A.1.11 for the definition of each item).

(i) The universal enveloping algebra $U(\mathscr{G})$ is a maximal order.

(ii) If R is an integrally closed commutative Noetherian integral domain which is a $\mathbb{K}$-algebra, then any crossed product $R * U(\mathscr{G})$ is a maximal order.

(iii) The Weyl algebra $A_n(\mathbb{K})$ is a maximal order (see Definition 1.2.2).

The reader may remember that if M_R is a right R-module, the **dual** of M is $M^* = \mathrm{Hom}(M,R)$, which is a left R-module, and M_R is **reflexive** if it is torsionless (meaning that the canonical map $M \to \mathrm{Hom}(M,R)$, $m \mapsto (f \mapsto f(m))$ for $m \in M, f \in M^*$, is inyective) and $M = M^{**}$. An immediate consequence of this definition is that for any M_R, $M^* = M^{***}$, which means $_RM^*$ is always reflexive. Also, it is known that finitely generated projective modules are reflexive and if M is a torsionless module, then $\mathrm{End}_R(M) \subseteq \mathrm{End}_R(M^*) \subseteq \mathrm{End}_R(M^{**})$ (see [MR01], 5.1.7). Keeping in mind that we can regard an ideal of R as an R-module, we state the following two results:

Proposition 1.1.21 ([MR01], 5.1.8). *(a) If R is a right order in a quotient ring Q and I is a fractional right R-ideal, then $I^* \cong \{q \in Q \mid qI \subseteq R\}$ and End$(I) \cong O_l(I)$.*

*(b) Let R and R' be maximal right orders in a quotient ring Q, and I a fractional (R,R')-ideal. Then $\{q \in Q \mid qI \subseteq R\} = \{q \in Q \mid Iq \subseteq R'\}$. If $I = I^{**}$, there is no ambiguity in calling I a **reflexive fractional** R**-ideal**.*

Let us remember that if P is a prime ideal of a ring R, then the **height** of P is the largest length of a chain of prime ideals contained in P or ∞ if there is no bound (see [MR01], 4.1.11). Having said this, it is convenient to show a consequence of being an order in relation to the notion of height.

Proposition 1.1.22 ([MR01], 5.1.9). *Let R be a prime right Goldie ring which is a maximal right order and let P be a nonzero prime ideal which is reflexive. Then P has height one.*

The following proposition relates orders to some important notions of algebra, like the **ascending chain condition** (or a.c.c.), satisfied by a ring when every such chain stabilizes; the **center** $Z(R)$ of a ring R being the subring consisting of the elements that commute with every element of the ring; and the notion of **Krull domain**, which can be seen as a generalization of Dedekind domains (see [MR01], 5.1.10 for a detailed definition).

Proposition 1.1.23 ([MR01], 5.1.10). *The following statements hold:*

(a) If R is a semiprime, prime or simple ring then the center of R, $Z(R)$, is respectively semiprime, an integral domain or a field.

(b) (i) If R is a prime right Goldie ring and a maximal right order then $Z(R)$ is a completely integrally closed integral domain.

(ii) If R satisfies the a.c.c. for reflexive ideals then $Z(R)$ is a Krull domain.

Proposition 1.1.24 ([MR01], 5.1.11). *Let R be a maximal right order in a quotient ring Q and let I be a reflexive fractional right R-ideal. Then $O_l(I)$ is a maximal right order in Q.*

Proof. Let us suppose that T is a right order with $O_l(I) \subseteq T$ and $aT \subseteq O_l(I)$ for a unit $a \in Q$. Then:

$$I^*TII^*TI \subseteq I^*TO_l(I)TI = I^*aTI \subseteq II^* \subseteq R$$

From the previous calculation, we have $I^*TI \subseteq O_r(I^*aTI) = R$ and it follows that $TI \subseteq I^{**} = I$ given that I is reflexive. Therefore, $T \subseteq O_l(I)$ which implies $T = O_l(I)$. On the other hand, if we have $Ta \subseteq O_l(I)$ instead, considering I^*TII^*TaI implies as well that $T \subseteq O_l(I)$. Then, by Proposition 1.1.5, we conclude that $O_l(I)$ is a maximal right order as we wanted to show. □

As a consequence of the previous result, note that if R is a maximal (right and left) order in a quotient ring Q then each order S in Q equivalent to R is contained in a maximal order equivalent to R (see [MR01], 5.1.11).

The following definition and the next proposition provide us with a different way to interpret, under certain additional conditions, a ring that happens to be a maximal order.

Definition 1.1.25 ([MR01], 13.1.1, 13.3.6, 13.9.2, 13.7.6)**.** (a) Let $f(x_1,\ldots,x_n) \in \mathbb{Z}\langle x_1,x_2,\ldots\rangle$. Then a ring R **satisfies** f and f is a **polynomial identity** of R, if $f(r_1,\ldots,r_n)=0$, for all $r_i \in R$, $1 \le i \le n$. If R satisfies a polynomial identity, it is called a **PI** ring.

(b) If R is a central simple algebra, the **PI-degree of** R (PIdeg(R)) is the least $n \in \mathbb{N}$ such that R satisfies the polynomial identity s_{2n}, where $s_k = \sum_{\sigma\in S_n} (\text{sgn}(\sigma)) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$, which is called the kth standard identity.

(c) Let R be a prime PI ring and let Q be quotient ring of R which is a central simple algebra of dimension n^2 over its center $Z(Q)$, where $n = \text{PIdeg}(R)$. Given $r \in R$, let $f_r(x)$ be the characteristic polynomial of the representation of r as a linear transformation of Q over $Z(Q)$. If T is the subring of $Z(Q)$ generated over $Z(R)$ by the coefficients of $f_r(x)$ as r varies throughout R, then TR is a subring of Q which contains R and we call it the **trace ring** of **characteristic closure** or R.

(d) The ring R is an **Azumaya algebra** over its centre C if:

  (i) R_C is finitely generated and projective.

  (ii) $\theta : R\otimes_C R^{op} \to \text{End}(R_C)$, $a\otimes b \mapsto (r \mapsto arb)$ is an isomorphism.

Proposition 1.1.26 ([MR01], 13.9.7, 13.9.8)**.** *The following statements hold:*

(a) R and TR are equivalent orders in Q.

(b) There is a 1-1 correspondence between regular primes P of TR and regular primes of R given by $P \mapsto P\cap R$.

(c) If R is a maximal order in Q then $R = TR$.

(d) If R is an Azumaya algebra over its centre C, then $R = TR$.

Proof. We give the outline of the proof, following the ideas of 13.9.7 and 13.9.8 from [MR01]. This ideas work around a polynomial called **central polynomial** $g_n(R)$, whose definition we omit, but can be found in [MR01], 13.5.11.

(a) The equivalence of R and TR as orders in Q is proved using Proposition 1.1.3 and the central polynomial $g_n(R)$.

(b) The correspondence is obtained between the prime ideals of R and TR which do not contain $g_n(R)R$ and these primes turn out to be the regular primes by [MR01], 13.7.2.

(c) The reader may notice that part (c) follows immediately from part (a), which states that $R \sim TR$ since $R \subseteq TR$ and thus we have $R = TR$.

(d) According to [MR01], 13.7.14, if R is an Azumaya algebra, $g_n(R)R = R$ and by [MR01], 13.9.6(ii), $g_n(R)$ is a nonzero ideal of both R and TR. This implies that $R = TR$.

□

Let us remember that a module M_R is called a **generator** if $M^*M = R$, and if M_R is both finitely generated projective module and a generator, it is called a **progenerator** (see [MR01], 3.5.3, 3.5.4). These notions allow us to check the maximality of an order through a different way, in the context of a prime Goldie ring, as the next proposition shows.

Proposition 1.1.27 ([MR01], 5.2.6)**.** *Let R be a prime Goldie ring. The following conditions are equivalent:*

(a) Each nonzero submodule of a (left or right) progenerator is a generator.

(b) R is a maximal order and each ideal is finitely generated projective as a left or right module.

(c) R is a maximal order and each is reflexive.

(d) Each nonzero ideal of R is invertible.

A prime Goldie ring satisfying the previous conditions is called an **Asano prime ring** or **Asano order**.

1.2 Ore extensions

In this section we consider the definition of *Ore extension*, which has been an algebraic object of great interest for a long time and we state some facts about order theory on these rings (see [Ore33] for details).

The next result has a great importance for the study of maximal orders in the context of Ore extensions, since it gives us sufficient conditions to guarantee that an Ore extension is a maximal order when the ring of coefficients has such property.

Definition 1.2.1 ([MR01], 1.2.1)**.** If σ is an endomorphism of R (a ring with unity) and δ is a σ-derivation, meaning that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$, then $R[x;\sigma,\delta]$ is called an **Ore extension** in the indeterminate x, and it is a polynomial ring where every element can be uniquely written as $\sum_{i=0}^{t} x^i a_i$ for elements $a_i \in R$ and such that $ax = x\sigma(a) + \delta(a)$ for all $a \in R$ (note that $x^0 := 1$).

As an example of Ore extension, we present one of the most important and widely studied algebras, which is the *Weyl algebra*.

Definition 1.2.2. [[MR01], 1.3.1] $A_n(\mathbb{K})$ denotes the $\mathbb{K}$-algebra with $2n$ generators $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ and relations $x_i y_j - y_j x_i = \delta_{ij}$ (the Kronecker delta), and $x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$. $A_n(\mathbb{K})$ is called the nth **Weyl Algebra** over $\mathbb{K}$. We can also regard $A_n(\mathbb{K})$ as an iterated skew polynomial ring: $A_n(\mathbb{K}) \cong \mathbb{K}[x_1, \ldots, x_n][y_1; \frac{\partial}{\partial x_1}] \cdots [y_n; \frac{\partial}{\partial x_n}]$, as it was proved in [MR01], 1.3.3.

As it was announced back in Example 1.1.8, $A_n(\mathbb{K})$ is a maximal order.

The reader may remember that a Dedekind domain D is a Noetherian, integrally closed ring such that every nonzero prime ideal of D is maximal. As an example of Dedekind domain, we can take a principal ideal domain (PID). A less trivial example is the ring $A/\mathfrak{p}$ with $\mathbb{Z}[\sqrt{-5}]$ and the ideal $\mathfrak{p} = \langle 2, 1+\sqrt{-5}\rangle$, which is not a PID.

With this in mind, if D is a Dedekind domain, $R = D[x;\sigma,\delta]$ is an Ore extension, and σ is an automorphism of D, then R has a right quotient ring Q by [MR01] 2.1.14 and 2.1.15. According to [AAMAI11], Section 2.2, it follows that R is then a right order in Q. This is due to the fact that the construction of a quotient ring of a ring R through a multiplicative set $S \subseteq R$ immediately implies that R is an order in Q. In fact, [AAMAI11] studies the notion of maximality of orders for Ore extensions over Dedekind domains, and also studies the factor rings of such Ore extensions over some prime ideals.

The following proposition has been a key fact in the study of order theory for Ore extensions. Let us recall that according to Chamarie in [Cha81], a **Krull ring** is an order in a simple Artinian ring Q which is a maximal order in Q and also satisfies the a.c.c. for right closed ideals (see [Cha81], Definition 2.1).

Proposition 1.2.3 ([Cha81], Proposition 3.3)**.** *If R is a Krull ring and σ is an automorphism of R, then $R[x;\sigma]$ is a Krull ring as well.*

We now state the next proposition from [AAMAI11] whose authors intended to state and prove explicitly a particular case of the previous result.

Proposition 1.2.4 ([AAMAI11], Theorem 2.1)**.** *If D is a Dedekind domain and $R = D[x;\sigma,\delta]$ is an Ore extension, then R is a maximal order.*

Proof. If I is an ideal of R and $a(x), b(x) \in R$, then Lemma 2.4 and Lemma 2.5 of [AAMAI11] prove that:

(i) $I[a(x)b(x)^{-1}] \subseteq I \Longrightarrow a(x)b(x)^{-1} \in R$

(ii) $[a(x)b(x)^{-1}]I \subseteq I \Longrightarrow a(x)b(x)^{-1} \in R$

These two items are proved by induction on the degrees of polynomials $a(x), b(x)$ and making use of the division algorithm in R. Therefore, (i) implies that $O_r(I) = R$ and (ii) implies that $O_l(I) = R$. Thus, we conclude that R is a maximal order by Proposition 1.1.14. □

The following result is taken from the abstract of [MU17] and it is of great importance since it provides sufficient conditions for an Ore extension to be a maximal order.

Proposition 1.2.5. *If R is a prime Goldie ring which is a maximal order in Q then so is $R[x;\sigma,\delta]$ provided that σ is an automorphism.*

In the next example we show a ring R which is not a maximal order, but $R[x;\sigma,\delta]$ is. The reader may remember that an hereditary ring is such that every submodule of a projective R-module is again projective. According to [LR11], a **cycle** of maximal ideals is a sequence $\mathscr{M}_1,\ldots,\mathscr{M}_n$ of distinct nonzero maximal ideals such that $O_l(\mathscr{M}_i) = O_r(\mathscr{M}_i+1)$ and $O_l(\mathscr{M}_n) = O_r(\mathscr{M}_1)$.

Example 1.2.6 ([MU17], Section 1, Theorem 3.7.)**.** Let D be an hereditary Noetherian prime ring (HNP for short) which satisfies:

(a) There is a cycle $\mathscr{M}_1,\ldots,\mathscr{M}_n$ ($n \geq 2$) such that $\mathscr{M}_1 \cap \cdots \cap \mathscr{M}_n = aD = Da$, for some $a \in D$.

(b) Any maximal ideal $\mathscr{N}$ different from $\mathscr{M}_i$ ($1 \leq i \leq n$) is invertible.

If we consider $R = D[t]$ for an indeterminate t, $\sigma(t) = t$ and $\delta(t) = a$, then $S = R[t;\sigma,\delta]$ is a maximal order but R is not. The reader may see [MU17], Lemmas 3.5 and 3.6 for the calculations that show that $O_l(I) = O_r(I) = S$, for any ideal I of S and then, by Proposition 1.1.14, S is indeed a maximal order.

1.3 PBW extensions

In this section we recall the definition of *PBW extension*, since the results of our interest obtained in [MZ96] are stated for these objects. PBW extensions are a very important family of rings of polynomial type and have been widely studied ever since their introduction in 1988.

Definition 1.3.1 ([BG88], Section 5)**.** We say that an overring T of a ring R is a **(finite) Poincaré-Birkhoff-Witt extension** of R (hereafter called a PBW extension, for short), if there exist elements $x_1,\ldots,x_n \in T$ such that

(a) The ordered monomials $x_1^{i(1)}x_2^{i(2)}\cdots x_n^{i(n)}$ (for nonnegative integers $i(1),\cdots,i(n)$) form a basis for T as a free left R-module;

(b) $x_i r - r x_i \in R$, for all $r \in R$ and for each $i = 1,\ldots,n$;

(c) $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$, for all $i, j = 1,\ldots,n$.

(An infinite PBW extension would be formed in a similar manner, using a well-ordered set of elements x_i.) It follows from (a),(b) that the ordered monomials $x_1^{i(1)}x_2^{i(2)}\cdots x_n^{i(n)}$ also form a basis for T as a free right R-module, and it follows from (b) that $R+Rx_1+\cdots+Rx_n = R+x_1R+\cdots+x_nR$. Thus this definition is left-right symmetric. To abbreviate conditions (a),(b),(c), we shall just write "let $T = R[x_1,\ldots,x_n]$ be a PBW extension" (sometimes noted $R\langle x_1,\ldots,x_n\rangle$). Let us notice that since $x_i r - r x_i \in R$, there is a derivation δ_i on R (meaning that $\delta_i(a+b) = \delta_i(a)+\delta_i(b)$ and $\delta_i(ab) = a\delta_i(b)+\delta_i(a)b$) such that $x_i r - r x_i = \delta_i(r)$, for all $r \in R$.

As examples of PBW extensions we mention the following: (a) the enveloping algebra of any finite-dimensional Lie algebra; (b) any differential operator ring $R[x_1,\ldots,x_n;\delta_1,\ldots,\delta_n]$ formed from commuting derivations $\delta_1,\ldots,\delta_n$ on R; (c) any Weyl algebra $A_n(R)$ (viewed as an extension of R by $2n$ elements); (d) those differential operator rings $V(R,L)$ introduced by Rinehart ([Rin63], p. 197) where L is a Lie algebra which is also a (finitely generated) free R-module equipped with a suitable Lie algebra map to derivations on R (a more general PBW Theorem is obtained here after assuming that L is a projective R-module [Rin63], Theorem 3.1); (e) the twisted or smash product differential operator ring $R\#_\sigma U(\mathscr{G})$ studied by McConnell ([McC74], Theorem 2.8), where $\mathscr{G}$ is a finite-dimensional Lie algebra acting on R by derivations, and σ is a Lie 2-cocycle with values in R; (f) the universal enveloping rings $U(V,R,K)$ introduced by Passman

([Pas87]) where K is a field, R is a K-algebra and V is a K-vector space which is also a Lie ring containing R and K as Lie ideals with suitable relations. (The enveloping ring U(V,R,K) is a finite PBW extension of R when $\dim_K(V/R)$ is finite). Conversely, a PBW extension $R[x_1,\dots,x_n]$ is a universal enveloping ring provided R contains a central subfield K, invariant under each $[x_i,-]$ such that each $[x_i,x_j]$ lies in the K-vector space $V = R + Kx_1 + \cdots + Kx_n$. In fact, if we change Passman's definition to allow K to be any central subring of R while also requiring that V/R be a free left K-module, then one can show in general that $R[x_1,\dots,x_n] = U(R + Kx_1 + \cdots + Kx_n, R, K)$, where K is the center of R (see [BG88], Section 5 for more details).

Next we state some results about order theory for PBW extensions.

Proposition 1.3.2 ([MZ96], Lemma 3)**.** *If $S = R\langle x_1,\dots,x_n\rangle$ for a prime Goldie ring R and f is a regular element in S, then there exists a regular element $g \in S$ such that $g \in fS$ and the leading coefficient of g is a regular element in R*

Proof. It is shown that the set $I = \{a \in R \mid \exists\alpha \in fS, lc(a) = \alpha\} \cup \{0\}$ satisfies $Ir \subseteq I$, for $r \in R$, and is a right ideal of R. If J is a right ideal of R, then so is $J\langle x_1,\dots,x_n\rangle = \{\gamma \in S \mid$all the coefficients of γ belong to $J\}$. Given that S is a prime right Goldie ring, $J\langle x_1,\dots,x_n\rangle \cap fS \neq 0$ and there exists $0 \neq \alpha = ax_1^{m_1}\cdots x_n^{m_n}+$(terms of lower lex. order)$\in J\langle x_1,\dots,x_n\rangle \cap fS$. Then we obtain $0 \neq a \in I \cap J$ and this proves that I is in fact an essential right ideal of R. By Proposition A.2.15, there exists $0 \neq g \in fS$, a regular element of S and $lc(g)$ is a regular element of R. □

The next one is probably the central fact in the study of maximality of orders in PBW extensions. The reader may find rather interesting the fact that the ring, as in the case of Ore Extensions, needs to be both prime and Goldie.

Proposition 1.3.3 ([MZ96], Theorem 4)**.** *If R is a maximal order in a simple Artininan ring $Q(R)$ (which by Proposition A.2.17 makes R a prime Goldie ring), then $S = R\langle x_1, x_2,\dots,x_n\rangle$ is a maximal order in $Q(S)$.*

Proof. Similarly to Proposition A.1.12, $Gr(S) = R[\overline{x_1},\dots,\overline{x_n}]$ is a polynomial ring over R in central variables $\{\overline{x_i}\}$. If C denotes the set of all elements in S whose leading coefficient is a regular element of R, by the previous Proposition, C is a regular Ore set of S and $Q(S) = S_C$. By the way $Gr(S)$ is constructed, c and $Gr(c)$ both have the same leading coefficient and this implies that $Gr(c)$ is a regular element in $Gr(S)$, for any $c \in C$. This fact is used to show, as usual via Proposition 1.1.14, that $O_l(I) = O_r(I) = S$ following the idea of the proof of Proposition 1.1.18. □

According to [MZ96], if $\Delta = \{d_1,\dots,d_n\}$ is the set of derivations of a PBW extension, an ideal I is called **Δ-invariant**, if it is invariant under each $d_i \in \Delta$ (meaning that $d_i(I) \subseteq I$). Furthermore, this leads to the notion of a **Δ-maximal order**, which is what we call a ring R such that $R = O_l(I) = O_r(I)$, for any Δ-invariant integral R-ideal I (see Definition 2.3.3 for the precise definition). As a remark in [MZ96], we find the fact that if $S = R\langle x_1,\dots,x_n\rangle$ is a maximal order, then R is a Δ-maximal order. It is not clear whether the converse statement holds or not.

1.4 Ore-Rees rings

In this section we introduce the notion of a *Ore-Rees ring* which is a ring built in a similar way to an Ore extension. We also state some results of order theory for these objects. In this section R

denotes a Noetherian prime ring.

Definition 1.4.1 ([HMU16], Section 2, Lemma 2.2)**.** Let X be an invertible ideal (which means that there exists an ideal Y of R such that $X \otimes Y \cong R$) of R. Let σ be an automorphism of R and δ a left σ-derivation and let us consider $S = R[Xt;\sigma,\delta] = R \bigoplus Xt \bigoplus \cdots \bigoplus X^n t^n \bigoplus \cdots$. S is a ring if and only if $\sigma(X) = X$ if and only if $\sigma^{-1}(X) = X$. When S is a ring, it is called a **Ore-Rees ring associated to** X. In this case, S is Noetherian and it has the same quotient ring as $R[t;\sigma,\delta]$, which is the simple Artinian ring $Q(S) = Q(R[t;\sigma,\delta])$.

Proposition 1.4.2 ([HMU16], Theorem 2.4.)**.** *If R is a maximal order, then so is the Ore-Rees ring* $S = R[Xt;\sigma,\delta]$.

Proof. For any ideal A of S, consider $C_n(A) = \{a \in R \mid \exists h(t) = at^n + \cdots + a_0 \in A\} \cup \{0\}$, which is an ideal of R (note that $C_n(A) \subseteq X^n$). For $a \in C_n(A)$ there is some $h(t) = at^n + \cdots + a_0 \in A$, then $(Xt)h(t) = X\sigma(a)t^{n+1}+$(the lower degree parts)$\subseteq A$ and so we have $X\sigma(a) \subseteq C_{n+1}(A)$. This means that $X\sigma(C_n(A)) \subseteq C_{n+1}(A)$, or equivalently, $C_n(A) \subseteq X^{-1}\sigma^{-1}(C_{n+1}(A))$ for every n (recall that X is an invertible ideal and σ is an automorphism). Thus we have the following chain of right ideals of R: $C_0(A) \subseteq X^{-1}\sigma^{-1}(C_1(A)) \subseteq X^{-2}\sigma^{-2}(C_2(A)) \subseteq \cdots \subseteq X^{-n}\sigma^{-n}(C_n(A)) \subseteq \cdots$. Given that R is Noetherian, $X^{-m}\sigma^{-m}(C_m(A)) = X^{-(m+k)}\sigma^{-(m+k)}(C_{m+k}(A))$, for some m and for all $k \geq 1$ and equivalently, we get $X^k\sigma^k(C_m(A)) = C_{m+k}(A)$, for all $k \geq 1$. Now let us consider $f \in Q(S)$ such that $fA \subseteq A$, where $Q(S)$ is the quotient ring of S. Let T denote $Q(R)[t;\sigma,\delta]$, the Ore extension with coefficients in $Q(R)$. According to [HMU16], Lemma 2.3, if I is an ideal of S, then IT is an ideal of T. With this in mind, we get that AT is an ideal of T and also $fAT \subseteq AT$. Since T is a maximal order, $f \in O_l(AT) = T$ and so $f = f_k t^k + \cdots + f_0$, where $f_i \in Q$. Take $a \in C_m(A)$ and $h = at^m + a_{m-1}t^{m-1} \cdots + a_0 \in A$. Then $fh = f_k\sigma^k(a)t^{m+k}+$(the lower degree parts)$\in A$ and so $f_k\sigma^k(a) \in C_{m+k}(A)$. Hence $f_k\sigma^k(C_m(A)) \subseteq C_{m+k}(A)$ and since $f_k\sigma^k(C_m(A)) = f_kR\sigma^k(C_m(A)) = f_k\sigma^k X^{-k}X^k(C_m(A)) = f_kX^{-k}C_{m+k}(A)$, we obtain $f_kX^{-k}C_{m+k}(A) \subseteq C_{m+k}(A)$. Thus $f_kX^{-k} \subseteq O_l(C_{m+k}(A)) = R$, since R is a maximal order and we get $f_k \in X^k$. Hence $f_k t^k \in S \subseteq T$. Let us notice that this implies that $f - f_k t^k = f_{k-1}t^{k-1}+$(the lower degree parts)$\in T$ and $(f - f_k t^k)A \subseteq fA - f_k t^k A \subseteq A$ and we obtain $f_{k-1} \in X^{k-1}$ in a similar way. Continuing this process, we have $f \in S$ and so $O_l(A) = S$. Similarly, we obtain that $O_r(A) = S$ and thus S is a maximal order.

□

As the previous proposition states, it is enough to have R being a maximal order for the Ore-Rees ring to be one as well. The reader may see [HMU16] for the proof, which uses Proposition 1.1.14.

Definition 1.4.3 ([HMU16], Section 2)**.** We recall the following notions:

(a) An (R,R)-bimodule I is called $(\sigma,\delta;X)$**-stable**, if $X\sigma(I) = IX$ and $X\delta(I) \subseteq I$.

(b) R is called a $(\sigma,\delta;X)$**-maximal order** in the quotient ring Q of R if $O_l(I) = R = O_r(I)$ for any $(\sigma,\delta;X)$-stable ideal I.

Proposition 1.4.4 ([HMU16], Theorem 3.5., Theorem 4.4.)**.** *We recall the following results:*

(a) Let δ be a derivation of R and X be an invertible ideal. Then R is a $(\delta;X)$-maximal order if and only if the Rees ring $S = R[Xt;\delta]$ (called differential Rees ring, $\sigma := 1$) is a maximal order.

(b) *If we consider $S = R[Xt;\sigma]$ (called skew Rees ring, $\delta := 0$), with $\sigma(X) = X$, then R is a $(\sigma; X)$-maximal order if and only if S is a maximal order in $Q(R)$.*

Proof. (a) Let R be a $(\delta; X)$-maximal order and A be an ideal of S. Since $S \subseteq O_l(A) \subseteq O_l(A_v)$, it suffices to prove that $O_l(A_v) \subseteq S$ so as to prove $O_l(A) = S$. By [HMU16], Lemmas 3.3 and 3.4, it follows that A_v is a v-invertible ideal, which means that $_v((S : A_v)_l A_v) = S = (A_v(S : A_v)_r)_v$ and according to [HMU16], page 411, this implies that $O_l(A_v) = S$, as desired. Similarly, it can be shown that $O_r(A_v) = S$ and thus, S is a maximal order. Conversely, let S be a maximal order and $\mathfrak{a}$ a (δ, X)-stable ideal such that $\mathfrak{a}_v = \mathfrak{a}$. Let us take $A = \mathfrak{a}[XT;\delta]$. By Lemma 2.15 of [HMU16], $A = A_v$ and thus A is a v-ideal. Since $(S : A)_l = (R : \mathfrak{a})_l[Xt;\delta]$, we obtain $S =_v ((S : A)_l A) =_v ((R : \mathfrak{a})_l[Xt;\delta]\mathfrak{a}[Xt;\delta]) =_v ((R : \mathfrak{a})_l\mathfrak{a})[Xt;\delta]$. It follows that $R =_v ((R : \mathfrak{a})_l\mathfrak{a})$ and in a similar way, $R = (\mathfrak{a}(R : \mathfrak{a})_l)_v$. Thus $\mathfrak{a}$ is v-invertible and then $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$. Hence R is a (δ, X)-maximal order.

(b) This part follows in a similar way to part (a).

□

It is important to mention that if a ring R is not a maximal order, this does not imply that the Ore-Rees extension $S = R[Xt;\delta]$ can not be a maximal order. For instance, in [HMU16], Example 5.2. we find a ring D with char$(D) = 0$ which is an HNP ring and satisfies: (a) there is a cycle $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ ($n \geq 2$) so that $\mathfrak{p}_0 = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ is an invertible ideal; and (b) Any maximal ideal different from $\mathfrak{m}_i$ ($1 \leq i \leq n$) is invertible. If we take $R = D[x]$, $X = \mathfrak{p}_0[x]$ and δ to be a derivation on R such that $\delta(x) = 1$ and $\delta(a) = 0$ for all $a \in D$, then $S = R[Xt;\delta]$ is a maximal order but R is not a maximal order. There can also be found examples of a ring R being a $(\sigma; X)$-maximal order which is not a maximal order but $R[Xt;\sigma]$ is; or a ring which is a $(\sigma; X)$-maximal order but not a (σ)-maximal order (Example 5.3. and Example 5.5.).

CHAPTER 2

Maximality on skew PBW extensions

In this chapter we study the notion of maximality of orders for skew PBW extensions. In Section 2.1, the definition, some examples and basic properties of skew PBW extensions are presented. On the other hand, in Section 2.2, we state our results about maximality of orders on skew PBW extensions.

2.1 Skew PBW extensions

In this section we present our objects of interest, *skew PBW extensions*, which generalize PBW extensions and include a great variety of algebras, some of which we illustrate as an example. Ever since their introduction back in 2011, several ring theoretical and homological properties of these objects have been studied by different authors. See for example [LR14], [AL15], [Art15], [LAR15], [RS16b], [RS16c], [GL16], [SLR17], [LV17], [NR17], [AHK17], [RR19], [Lez19], [RS19b], [SR19], [AHK19], [GHK19], [HHR19], among others.

Definition 2.1.1 ([GL11], Definition 1)**.** Let R and A be rings. We say that A *is a skew PBW extension of* R (also called a σ-PBW extension of R), which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ $(x^0 := 1)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n$.

Remark 2.1.2 ([GL11], Remark 2)**.** (i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1.1 are unique. (ii) In Definition 2.1.1, $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.

Proposition 2.1.3 ([GL11], Proposition 3)**.** *Let A be a skew PBW extension of R. For every $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and an σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Definition 2.1.4 ([GL11], Definition 4, and [LAR15], Definition 2.3)**.** Let A be a skew PBW extension of a ring R.

(a) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1.1 are replaced by (iii'): for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; (iv'): for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) A is called *bijective* if σ_i is bijective, for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(c) A is called a skew PBW extension of *endomorphism type*, if $\delta_i = 0$, for every i. In addition, if σ_i is bijective, for each i, A is called a skew PBW extension of *automorphism type*.

Definition 2.1.5 ([GL11], Definition 6)**.** Let A be a skew PBW extension of R with injective endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.1.3.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ denotes a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}^n$). For an element $x^\alpha \in \mathrm{Mon}(A)$, $\exp(x^\alpha) := \alpha \in \mathbb{N}^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R \setminus \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_m X_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$, and a_0 as the constant term of f. Note that $\deg(f) := \text{máx}\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [GL11], Section 3; [Faj19], [GL16], [GL17], [JL16].

Proposition 2.1.6 ([GL11], Theorem 7)**.** *Let A be a polynomial ring over R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(i) *for each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$, if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.*

(ii) *For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$, if $p_{\alpha,\beta} \neq 0$.*

The next example shows that under certain conditions, an Ore extension can be seen as a skew PBW extension.

Example 2.1.7. If $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$ is an iterated Ore extension where

(i) σ_i is injective, for $1 \le i \le n$;

(ii) $\sigma_i(r), \delta_i(r) \in R$, for every $r \in R$ and $1 \le i \le n$;

(iii) $\sigma_j(x_i) = cx_i + d$, for $i < j$, and $c, d \in R$, where c has a left inverse;

(iv) $\delta_j(x_i) \in R + Rx_1 + \cdots + Rx_n$, for $i < j$,

then $R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n] \cong \sigma(R)\langle x_1,\dots,x_n\rangle$ (see [LR14], p. 1212). In particular, note that skew PBW extensions of endomorphism type are more general than iterated Ore extensions $R[x_1;\sigma_1]\cdots[x_n;\sigma_n]$. On the other hand, skew PBW extensions are more general than Ore extensions of injective type (diffusion algebras, univesal enveloping algebras of finite Lie algebras, and others, are examples of skew PBW extensions which can not be expressed as iterated Ore extensions, see [LR14] for more details). Skew PBW extensions contains various well-known groups of algebras such as some types of Auslander-Gorenstein rings, some skew Calabi-Yau algebras, quantum polynomials, some quantum universal enveloping algebras, etc. A detailed list of examples of skew PBW extensions is presented in [LR14], [Rey14], and [LRS15].

It is worth mentioning that all of the following examples have been checked to be skew PBW extensions by diverse authors using Theorem 3.2.6 in [Rey13], which provides an algorithm to guarantee that a factor algebra determined by an ideal of relations in fact fulfills condition (ii) in Definition 2.1.1, which asks A to be a left free R-module with basis Mon(A).

Next we present some examples of skew PBW extensions known as quantum algebras, which are of interest in theoretical physics.

Example 2.1.8 ([JR18], Section 4)**.** (a) The first Weyl algebra $A_1(\mathbb{K})$, and in general, the **n-th Weyl algebra** $A_n(\mathbb{K})$ from Definition 1.2.2. Thus, $A_n(\mathbb{K}) \cong \sigma(\mathbb{K}[x_1,\dots,x_n])\langle y_1,\dots,y_n\rangle$.

(b) **Additive analogue of the Weyl algebra**, the $\mathbb{K}$-algebra $A_n(q_1,\dots,q_n)$ is generated over $\mathbb{K}$ by the indeterminates $x_1,\dots,x_n,y_1,\dots,y_n$ subject to the relations

$$\begin{array}{lll} x_ix_j = x_jx_i,\ y_iy_j = y_jy_i. & 1 \le i < j \le n & \\ y_ix_i = q_ix_iy_i + 1, & 1 \le i \le n & \text{where } q_i \in \mathbb{K}\setminus\{0\} \text{ for every } i. \\ x_jy_i = y_ix_j, & i \ne j & \end{array}$$

Thus, $A_n(q_1,\dots,q_n) \cong \sigma(\mathbb{K}[x_1,\dots,x_n])\langle y_1,\dots,y_n\rangle$. As a matter of fact, in this case we also have $A_n(q_1,\dots,q_n) \cong \sigma(\mathbb{K})\langle x_1,\dots,x_n,y_1,\dots,y_n\rangle$, which means that $A_n(q_1,\dots,q_n)$ can also be seen as a skew PBW extension of $\mathbb{K}$.

(c) **Multiplicative analogue of the Weyl algebra**. This $\mathbb{K}$-algebra, $\mathcal{O}_n(\lambda_{ij})$, is generated by the indeterminates $x_1,\dots,x_n$ subject to the relations $x_jx_i = \lambda_{ij}x_ix_j$, $1 \le i < j \le n$. If $n = 2$ then $\mathcal{O}_2(\lambda_{ij})$ is the **quantum plane**. If $\lambda_{ij} = q^{-2} \ne 0$, for some element $q \in \mathbb{K}\setminus\{0\}$ and every $1 \le i < j \le n$, then $\mathcal{O}_n(\lambda_{ij})$ is the coordinate ring of the so called **quantum affine n-space**. Thus, $\mathcal{O}_n(\lambda_{ij})$ can be seen as a skew PBW extension of $\mathbb{K}$ or as a skew PBW extension of $\mathbb{K}[x_1]$, which means $\mathcal{O}_n(\lambda_{ij}) \cong \sigma(\mathbb{K})\langle x_1,\dots,x_n\rangle$ and $\mathcal{O}_n(\lambda_{ij}) \cong \sigma(\mathbb{K}[x_1])\langle x_2,\dots,x_n\rangle$.

(d) **Quantum Weyl algebra**. This quantum Weyl algebra was introduced with the purpose of studying the Jordan Hecke symmetry. This noncommutative ring can be viewed as a quantization of the usual second Weyl algebra. By definition, $A_2(J_{a,b})$ is the $\mathbb{K}$-algebra generated by the indeterminates $x_1, x_2, \partial_1, \partial_2$ with relations (depending on parameters

$a, b \in \mathbb{K}$) given by

$$
\begin{array}{ll}
x_1x_2 = x_2x_1 + ax_1^2, & \partial_2\partial_1 = \partial_1\partial_2 + b\partial_2^2 \\
\partial_1x_1 = 1 + x_1\partial_1 + ax_1\partial_2, & \partial_1x_2 = -ax_1\partial_1 - abx_1\partial_2 + x_2\partial_1 + bx_2\partial_2 \\
\partial_2x_1 = x_1\partial_2, & \partial_2x_2 = 1 - bx_1\partial_2 + x_2\partial_2
\end{array}
$$

Over any $\mathbb{K}$, if $a = b = 0$, then $A_2(J_{0,0}) \cong A_2$, the usual second Weyl algebra. By the relations that define this algebra we obtain $A_2(J_{a,b}) \cong \sigma(\mathbb{K}[x_1, \delta_2])\langle x_2, \delta_1\rangle$.

(e) q**-Heisenberg algebra**. By definition, it is the $\mathbb{K}$-algebra $\mathbf{h}_n(q)$ generated over $\mathbb{K}$ by the indeterminates x_i, y_i, z_i for $1 \le i \le n$, subject to the relations:

$$
\begin{array}{rl}
x_ix_j = x_jx_i,\ y_iy_j = y_jy_i,\ z_iz_j = z_jz_i, & 1 \le i < j \le n \\
x_iy_j = y_jx_i,\ x_iz_j = z_jx_i,\ y_iz_j = z_jy_i, & i \ne j \\
x_iz_i - qz_ix_i = z_iy_i - qy_iz_i = x_iy_i - q^{-1}y_ix_i + z_i = 0, & 1 \le i \le n
\end{array}
$$

Thus, $\mathbf{h}_n(q) \cong \sigma(K)\langle x_1, \dots, x_n; y_1, \dots, y_n; z_1, \dots, z_n\rangle$

In the next example, we present some algebras that can be seen as skew PBW extensions but can not be expressed as iterated Ore extensions. These algebras illustrate the importance of the study of PBW extensions.

Example 2.1.9 ([Rey15], Section 4; [RS18b], p. 628)**.** (a) If k is a commutative ring and $\mathcal{G}$ is a finite dimensional Lie algebra over k with basis $\{x_1, \dots, x_n\}$. The **universal enveloping algebra of** $\mathcal{G}$, denoted $\mathcal{U}(\mathcal{G})$, is a skew PBW extension of k (see [LR14]), given that $x_ir - rx_i = 0$, $x_ix_j - x_jx_i = [x_i, x_j] \in \mathcal{G} = k + kx_1 + \cdots + kx_n$, $r \in k$, for $1 \le i, j \le n$. Therefore, $\mathcal{U}(\mathcal{G}) \cong \sigma(k)\langle x_1, \dots, x_n\rangle$.

(b) Let k, $\mathcal{G}$, $\{x_1, \dots, x_n\}$ and $\mathcal{U}(\mathcal{G})$ be as in the previous example; let R be a k-algebra containing k. The **tensor product** $A := R \otimes_k \mathcal{U}(\mathcal{G})$ is a skew PBW extension of R and it is a particular case of the **crossed product** $R * \mathcal{U}(\mathcal{G})$ of R by $\mathcal{U}(\mathcal{G})$, which is a skew PBW of R as well.

(c) The **twisted** or *smash product differential operator ring* $R\#_\sigma\mathcal{U}(\mathcal{G})$, where $\mathcal{G}$ is a finite-dimensional Lie algebra acting on R by derivations, and σ is a Lie 2-cocycle with values in R.

(d) The **universal enveloping ring** $\mathcal{U}(V, R, \mathbb{K})$ introduced in [Pas87], where R is a $\mathbb{K}$-algebra and V is a $\mathbb{K}$-vector space which is also a Lie ring containing R and $\mathbb{K}$ as Lie ideals with suitable relations. The enveloping ring $\mathcal{U}(V, R, \mathbb{K})$ is a finite skew PBW extension of R if $\dim_{\mathbb{K}}(V/R)$ is finite, meaning that $\mathcal{U}(V, R, \mathbb{K}) \cong \sigma(R)\langle x_1, \dots, x_n\rangle$

(e) It is known that diffusion algebras arise in physics as a possible way to understand a large class of 1-dimensional stochastic process. A **diffusion algebra** $\mathcal{A}$ with parameters $a_{ij} \in \mathbb{C} \setminus \{0\}$, $1 \le i, j \le n$ is an algebra over $\mathbb{C}$ generated by variables $x_1, \dots, x_n$ subject to the relations $a_{ij}x_ix_j - b_{ij}x_jx_i = r_jx_i - r_ix_j$, whenever $i < j$ for all $b_{ij}, r_i \in \mathbb{C}$. $\mathcal{A}$ admits a PBW basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$, that is, $\mathcal{A}$ is a diffusion algebra if these standard monomials are a $\mathbb{C}$-vector space basis for $\mathcal{A}$. If coefficients $q_{ij} = \frac{b_{ij}}{a_{ij}}$ are all nonzero, then the corresponding diffusion algebra has a basis of standard monomials $x_1^{i_1} \cdots x_n^{i_n}$ and therefore, these algebras are skew PBW extensions, meaning $\mathcal{A} \cong \sigma(\mathbb{C})\langle x_1, \dots, x_n\rangle$. In particular, it is relevant to state that these diffusion algebras can not be seen as PBW extensions as defined in Section 1.3 due to the nature of the relations that define them as algebras.

As it was expressed in [RS17b], skew PBW extensions share relation with other types of algebras. For example, skew PBW extensions contain some type of G-algebras in the sense of Levandovsky, Auslander Gorenstein rings, some Calabi-Yau and skew Calabi-Yau algebras, some Koszul algebras and some Artin-Schelter regular algebras. The reader may see [LR14], [LRS15], [SLR17], [SR17] and [LG19] for a detailed list of examples. The next example shows some PBW extensions that can be seen as other type of object, as the ones mentioned previously.

Example 2.1.10 ([LRS15], Section 3.1)**.** (a) The algebra $A = \mathbb{K}\langle x, y, z\rangle / \langle yz - zy, zx - xz, xy - yx + z^2\rangle$ is a PBW extension of $\mathbb{K}[z]$, meaning $A \cong \sigma(\mathbb{K}(z))\langle x, y\rangle$. This algebra is also Artin-Shelter regular, N-Koszul and Calabi-Yau (see [LRS15], Section 3.1.1 to see the details)

(b) The **Jordan plane** $A = \mathbb{K}\{x, y\}/\langle yx - xy - x^2\rangle$ is a PBW extension of $\mathbb{K}[x]$, meaning $A \cong \sigma(K[x])\langle y\rangle$. It is also a skew Calabi-Yau algebra that can not be seen as a Calabi-Yau algebra (see [LRS15], Section 3.1.3).

(c) The **quantum plane** defined in Example 2.1.8 is an Artin-Shelter regular algebra and a 2-Koszul algebra besides being a skew PBW extension (see [LRS15], Section 3.1.2).

The next example we present is the family of *3-dimensional skew polynomial algebras*, which were also checked to be skew PBW extensions of the field $\mathbb{K}$ in [RS17c], Theorem 4.3., following the ideas and algorithms in [Rey13].

Definition 2.1.11 ([RS17c], Definition 2.1)**.** A 3**-dimensional skew polynomial algebra** $\mathscr{A}$ is a $\mathbb{K}$-algebra generated by the variables x, y, z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$ and $xy - \gamma yx = \nu$, such that $\lambda, \mu, \nu \in \mathbb{K} + \mathbb{K}x + \mathbb{K}y + \mathbb{K}z$, $\alpha, \beta, \gamma \in \mathbb{K}^*$ and the standard monomials $\{x^i y^j z^l \mid i, j, l \geq 0\}$ are a $\mathbb{K}$-basis of the algebra.

There is an interesting property of this family of algebras and it is the fact that they are all classified according to the values of their parameters, as it is stated in the next proposition.

Proposition 2.1.12 ([RS17c], Proposition 2.3)**.** *If $\mathscr{A}$ is a 3-dimensional skew polynomial algebra then $\mathscr{A}$ is one of the following algebras:*

(a) If $\alpha \neq \beta \neq \gamma$ and $\alpha \neq \gamma$, then A is defined by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, $xy - \gamma yx = 0$

(b) If $\beta \neq \alpha = \gamma = 1$, then $\mathscr{A}$ is one of the following algebras:

(i) $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$

(ii) $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$

(iii) $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$

(iv) $yz - zy = 0$, $zx - \beta xz = b$, $xy - yx = 0$

(v) $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$

(vi) $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$

where $a, b \in \mathbb{K}$ are arbitrary and all nonzero values of b give isomorphic algebras.

(c) *If $\beta \neq \alpha = \gamma \neq 1$, then $\mathscr{A}$ is one of the following algebras:*

(i) $yz - \alpha zy = 0$, $zx - \beta xz = y + b$, $xy - \alpha yx = 0$

(ii) $yz - \alpha zy = 0$, $zx - \beta xz = b$, $xy - \alpha yx = 0$

where $b \in \mathbb{K}$ is arbitrary and all nonzero values of b give isomorphic algebras as well.

(d) *If $\alpha = \beta = \gamma \neq 1$, then $\mathscr{A}$ is defined by the relations $yz - \alpha zy = a_1 x + b_1$, $zx - \beta xz = a_2 y + b_2$, $xy - \alpha yx = a_3 z + b_3$. If $a_i = 0$ for $i = 1, 2, 3$, then all nonzero values of b_i gve isomorphic algebras.*

(e) *If $\alpha = \beta = \gamma = 1$, then $\mathscr{A}$ is one of the following algebras:*

(i) $yz - zy = x$, $zx - xz = y$, $xy - yx = z$

(ii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = z$

(iii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = b$

(iv) $yz - zy = -y$, $zx - xz = x + y$, $xy - yx = 0$

(v) $yz - zy = az$, $zx - xz = z$, $xy - yx = 0$

where $a, b \in \mathbb{K}$ are arbitrary and all nonzero values of b give isomorphic algebras.

Example 2.1.13. The next algebras are specific cases of 3-dimensional skew polynomial algebras, and therefore, skew PBW extensions:

(a) **Woronowicz algebra** $\mathscr{W}_\nu(\mathfrak{sl}(2,\mathbb{K}))$: This $\mathbb{K}$-algebra is generated by the indeterminates x, y, z subject to the relations $xz - \nu^4 zx = (1+\nu^2)x$, $xy - \nu^2 yx = \nu z$, $zy - \nu^4 yz = (1+\nu^2)y$, where $\nu \in \mathbb{K} \setminus \{0\}$ is not a root of unity.

(b) **Dispin algebra** $\mathscr{U}(osp(1,2))$: This $\mathbb{K}$-algebra is generated by the indeterminates x, y, z subject to the relations $yz - zy = z$, $zx + xz = y$ and $xy - yx = x$.

The following remark is useful when making explicit calculations with elements of a skew PBW extension.

Remark 2.1.14. ([Rey15], Remark 2.10) If $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$, $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}} \in \mathrm{Mon}(A)$, then, for $a_i, b_j \in R$:

$$\begin{aligned}
a_i X_i b_j Y_j = {} & a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} + a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
& + a_i x_1^{\alpha_{i1}} p_{\alpha_{i2}, \sigma_3^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
& + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} p_{\alpha_{i3}, \sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_4^{\alpha_{i4}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
& + \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}(b_j)} x_n^{\alpha_{in}} x^{\beta_j} \\
& + a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1)}} p_{\alpha_{in}, b} x^{\beta_j}.
\end{aligned}$$

In this way, when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i.

Proposition 2.1.15 ([LR14], Proposition 4.1.)**.** *Let A be a skew PBW extension of a ring R. If R is a domain, then A is a domain.*

Proof. We follow the ideas in [LR14]. Let $f = cx^\alpha + p$, $g = dx^\beta + q$ be two nonzero polynomials in A with $lc(f) = cx^\alpha$ and $lc(g) = dx^\beta$, meaning that $c, d \neq 0$, $x^\alpha \succ lm(p)$ and $x^\beta \succ lm(q)$. Multiplying we obtain:

$$\begin{aligned} fg &= (cx^\alpha + p)(dx^\beta + q) = cx^\alpha dx^\beta + cx^\alpha q + pdx^\beta + pq \\ &= c(\sigma^\alpha(d)x^\alpha + p_{\alpha,d})x^\beta + cx^\alpha q + pdx^\beta + pq \end{aligned}$$

where $p_{\alpha,d} \in A$ is either 0 or $\deg(p_{\alpha,d}) < |\alpha|$, as given by Remark 2.1.14. Then, we have:

$$\begin{aligned} fg &= c\sigma^\alpha(d)x^\alpha x^\beta + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \\ &= c\sigma^\alpha(d)(c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}) + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \\ &= c\sigma^\alpha(d)c_{\alpha,\beta}x^{\alpha+\beta} + c\sigma^\alpha(d)p_{\alpha,\beta} + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \end{aligned}$$

with $0 \neq c_{\alpha,\beta} \in R$, $p_{\alpha,\beta} \in A$ being either 0 or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$.
Then, we see that $\mathrm{lc}(fg) = c\sigma^\alpha(d)c_{\alpha,\beta} \neq 0$ and if we take $h = c\sigma^\alpha(d)p_{\alpha,\beta} + cp_{\alpha,d}x^\beta + cx^\alpha q + pdx^\beta + pq \in A$, then either $h = 0$ or $x^{\alpha+\beta} \succ \mathrm{lm}(h)$, which shows that $fg \neq 0$. Thus, A is indeed a domain. □

Proposition 2.1.16 ([LR14], Proposition 2.3.)**.** *Let A be a quasi-commutative skew PBW extension of a ring R. Then,*

(i) A is isomorphic to an iterated skew polynomial ring of endomorphism type, i.e. $A \cong R[z_1;\theta_1]\cdots[z_n;\theta_n]$.

(ii) If A is bijective, then each endomorphism θ_i is bijective, $1 \leq i \leq n$.

Proposition 2.1.17 ([LR14], Theorem 2.2.)**.** *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by*

$$F_m := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A \mid deg(f) \leq m\}, & \text{if } m \geq 1 \end{cases}$$

and the corresponding graded ring $Gr(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $Gr(A)$ is a quasi-commutative bijective skew PBW extension of R.

Proposition 2.1.18 ([LR14], Corollary 2.4.)**.** *(Hilbert Basis Theorem) Let A be a bijective skew PBW extension of R. If R is a left (right) Noetherian ring then A is also a left (right) Noetherian ring.*

As a way to illustrate this result, we have the following:

Example 2.1.19 ([LR14], Section 3.2)**.** (i) **The algebra of q-differential operators $D_{q,h}[x,y]$**: Take $q, h \in \mathbb{K}$, $q \neq 0$ and consider $\mathbb{K}[y][x;\sigma,\delta]$, $\sigma(y) := qy$ and $\delta(y) := h$. By the definition of skew polynomial ring, we have $xy = \sigma(y)x + \delta(y) = qyx + h$, which gives us $xy - qyx = h$. Thus, $D_{q,h}[x,y] \cong \sigma(\mathbb{K}[y])\langle x\rangle$.

(ii) **The algebra of shift operators S_h**: Consider $h \in \mathbb{K}$. This algebra is defined by $S_h := \mathbb{K}[t][x_h;\sigma_h,\delta_h]$, where $\sigma_h(p(t)) := p(t-h)$ and $\delta_h := 0$. Thus, $S_h \cong \sigma(\mathbb{K}[t])\langle x_h\rangle$.

For the algebras presented in the previous example, given that the field $\mathbb{K}$ is Noetherian, by the classical version of Hilbert's basis theorem, so are $\mathbb{K}[y]$ and $\mathbb{K}[t]$. Therefore, by Proposition **??**, both $D_{q,h}[x,y]$ and S_h are Noetherian as well.

2.2 Results

Our purpose in this section is to take some results obtained in [MZ96] about maximality of orders over PBW extensions and generalize them for the case of skew PBW extensions. Given the difference in the algebraic structure of these extensions, some results require more conditions to be imposed so as to make them work. Thankfully, several properties of skew PBW extensions have been introduced and studied and we can make use of some of them.

We start by recalling the notion of (Σ,Δ)-*Armendariz ring* defined in [NR17], since we make use of it in the next proposition. For more information regarding this and other notions of Armendariz rings and their relation with skew PBW extensions, see [RS16a], [NR17], [RS17b], [RS17d] and [RS18a]. This property is particularly useful when showing that an element in a skew PBW extension is regular, as will be seen in the proof of Theorem 2.2.2.

Definition 2.2.1 ([NR17], Definition 3.4.)**.** Let A be a skew PBW extension of a ring R. We say that R is a (Σ,Δ)-*Armendariz ring*, if for polynomials $f = a_0 + a_1X_1 + \cdots + a_mX_m$ and $g = b_0 + b_1Y_1 + \cdots + b_tY_t$ in A, where $X_i = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$ and $Y_i = x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}}$, the equality $fg = 0$ implies $a_iX_ib_jY_j = 0$, for every i, j.

The following proposition is a generalization of Proposition 1.3.2, originally stated for PBW extensions.

Theorem 2.2.2. *Let R be a semiprime right Goldie ring and A a quasi-commutative skew PBW extension of R. If f is a regular element in A, then there exists a regular element $g \in A$ such that $g \in fA$ and $lc(g)$ is a regular element in R.*

Proof. Let us consider the set $S = \{a \in R \mid a = lc(h) \quad \text{for some} \quad h \in fA\} \cup \{0\}$ and $I = \{S\rangle_R$. Let us suppose that A has been ordered with a total order $\succeq$ and check that I is an ideal of R.

Let J be a right ideal of R and $\sigma(J)\langle x_1,\dots,x_n\rangle = \{h \in A \mid \text{all the coefficients of } h \text{ belong to } J\}$. $\sigma(J)\langle x_1,\dots,x_n\rangle$ is a right ideal of A since: given $h, h' \in \sigma(J)\langle x_1,\dots,x_n\rangle$, then the coefficients of $h + h'$ are in J; if $h = a_0 + \sum_{i=1}^{m} a_iX_i$, then $hr = a_0r + \sum_{i=1}^{m} a_iX_ir = a_0r + \sum_{i=1}^{m} a_ir_iX_i$ where r_i is obtained from r by applying $x_jr = c_{j,r}x_j$ as it may correspond according to Remark 2.1.14. Since $a_ir_i, a_or \in J$, we obtain that $\sigma(J)\langle x_1,\dots,x_n\rangle$ is a right ideal of A. Given that A is a quasi-commutative extension of the semiprime right Goldie ring R, A has these properties as well and since f is a regular element of A, then fA is an essential ideal of A. Therefore, $\sigma(J)\langle x_1,\dots,x_n\rangle \cap fA \neq 0$. This means that there exists $0 \neq q \in \sigma(J)\langle x_1,\dots,x_n\rangle \cap fA$, say $q = ax_1^{\alpha_1}\cdots x_n^{\alpha_n}$ + (lower terms with respect to $\succeq$).
Then $a = lc(h) \in I \cap J$, and this means that I is an essential right ideal of R. Given that R is semiprime right Goldie, by Proposition A.2.15 there exists a regular element $e \in I$. This implies that there exists $0 \neq g \in fA$ such that $lc(g) = e$. Also, g is regular because if we suppose $gp = 0$ for some $p \in A$, then $p = 0$.

□

Theorem 2.2.3. *If R is a maximal order in $Q(R)$ and A is a skew PBW extension of R such that σ_i is bijective for all $1 \leq i \leq n$, then A is a maximal order in $Q(A)$.*

Proof. We know that $\mathrm{Gr}(A)$ is a quasi-commutative and bijective skew PBW extension of R, and according to [LR14], Theorem 2.3., $\mathrm{Gr}(A) \cong R[z_1;\theta_1]\cdots[z_n;\theta_n]$ with $\theta_1 := \sigma_1$,
$\theta_j : R[z_1;\theta_1]\cdots R[z_{j-1};\theta_{j-1}] \to R[z_1;\theta_1]\cdots R[z_{j-1};\theta_{j-1}]$, $\theta_j(z_i) := c_{i,j}z_i$, $1 \le i < j \le n$, $\theta_j(r) := \sigma_j(r)$, for $r \in R$ and each θ_j is bijective. By Hilbert's Basis Theorem, if R is a Noetherian ring, then so is $\mathrm{Gr}(A)$. Let us consider the set C of all elements whose leading coefficient is a regular element of R and check that C is a regular Ore set, which means that every element of C is regular and A satisfies the Ore condition with respect to C: Let f be an element of A and h an element of C. By Theorem 2.2.2, there exists $g \in fA \cap C$, meaning that $g = fp$ for some $p \in A$. Then, $Q(A) = A_C$.
Let us notice that if $q = \sum_{i=1}^{n} q_i X_i + q_0$, then $\mathrm{Gr}(q) = \sum_{i=1}^{n} q_i Z_i + q_0$, with $X_i = x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$ and $Z_i = z_1^{\alpha_{i1}}\cdots z_n^{\alpha_{in}}$, and since q_n is regular in R, $\mathrm{Gr}(q)$ is regular in A. Let I be a nonzero ideal of A and $q \in O_r(I)$. Then $q = fg^{-1}$ for some $f \in A$ and $g \in C$, and also $If \subseteq Ig$. It can be checked by induction on the degree of f that this implies $f \in Ag$. Therefore, $fg^{-1} \in A$ and $O_r(I) = A$. Similarly, we obtain $O_l(I) = A$ and thus A is a maximal order in $Q(A)$. □

2.3 Localization and globalization theorems

Definition 2.3.1. As in [MZ96], but taking Δ as the one defined in Proposition 2.1.3, we say that an ideal of R is Δ-*invariant*, if it is invariant under each derivation in Δ, and a Δ-invariant ideal I of R is called Δ-*prime*, if whenever a product of two Δ-invariant ideals is contained in I, one of the ideals is contained in I (see also [RS19a]).

We will now introduce an important definition that will be used in our following results.

Definition 2.3.2 ([LAR15], Definition 2.1)**.** Let R be a ring, $\Sigma := \{\sigma_1,\dots,\sigma_n\}$ be a finite set of endomorphisms of R and $\Delta := \{\delta_1,\dots,\delta_n\}$ be a finite set such that δ_i is a σ_i-derivation of R, $1 \le i \le n$. If I is an ideal of R, I is called Σ-invariant if $\sigma_i(I) \subseteq I$ and (Σ,Δ)-invariant if it is both Σ-invariant and Δ-invariant.

Definition 2.3.3 ([MZ96], pages 1379-1380)**.** We have the following concepts:

(i) Given $X, Y \subseteq Q(R)$ for a ring R, we define the sets $(X:Y)_l = \{q \in Q(R) \mid qY \subseteq X\}$ and $(X:Y)_r = \{q \in Q(R) \mid Yq \subseteq X\}$.

(ii) If I is a right R-ideal, $I_v = (R:(R:I)_l)_r$ and if $I = I_v$, then I is called a *right v-R-ideal* or *right v-ideal* if no confusion arises. Similarly, ${}_vJ = (R:(R:J)_r)_l$, for any left R-ideal J and J is called *left v-ideal*, if $J = {}_vJ$. An R ideal A is called a v-ideal, if ${}_vA = A = A_v$. An integral v-ideal is called a v-ideal of R. If a v-R-ideal is Δ-invariant, then it is called a Δ-v-ideal.

(iii) R is a Δ-*maximal order* in $Q(R)$ if and only if $R = O_l(I) = O_r(I)$, for any Δ-invariant integral R-ideal I.

Remark 2.3.4 ([MZ96], page 1394)**.** If $S = R\langle x_1,\dots,x_n\rangle$ is a PBW extension of R which is a maximal order, then R is a Δ-maximal order.

Proposition 2.3.5 ([MZ96], Lemma 1)**.** *The following assertions hold for $S = R\langle x_1,\dots,x_n\rangle$ a PBW extension of a ring R.*

(i) *S is a prime ring.*

(ii) *If $Gr(S)$ is a Noetherian ring, then so is S.*

(iii) *Let I be a Δ-invariant ideal of R. Then I is a Δ-prime ideal of R if and only if IS is a prime ideal of S.*

The following propositions are taken from [MZ96] and they use the notion of v-ideals for the study of maximality of orders in PBW extensions. It is worth mentioning that according to [KMU85], page 108, when we work with an Ore extension $R[x;d]$, d can be extended to a derivation of Q, the quotient ring of R by taking $d(ac^{-1}) = d(a)c^{-1} - ac^{-1}d(c)c^{-1}$. We use this fact in the proof we present for the next proposition.

Proposition 2.3.6 ([MZ96], Lemma 5)**.** *Let us suppose that $S = R\langle x_1, \dots, x_n\rangle$ is a PBW extension of R. Then we have the following:*

(i) *Let I be a one-sided Δ-R-ideal, i.e., a Δ-invariant ideal and a one-sided R-ideal. Then $(R:I)_l$, $(R:I)_r$, I_v and ${}_vI$ are Δ-invariant.*

(ii) *Let I be a Δ-R-ideal. Then IS is an S-ideal and $O_l(I)$ and $O_r(I)$ are also Δ-invariant.*

(iii) *Let I be a Δ-R-ideal. Then $(IS)_v = I_vS$ and ${}_v(IS) = {}_vIS$.*

Proof. (i) Let us check that $(R:I)_l$ is Δ-invariant, which means $\delta_i((R:I)_l) \subseteq (R:I)_l$ for all $1 \le i \le n$. Let t be an element in $\delta_i((R:I)_l)$, meaning that $t = \delta_i(q)$ for some $q \in (R:I)_l$. We must check that $t \in Q(R)$ and that $tI \subseteq R$. Let us notice that since δ_i is a derivation of R, we need to extend it to Q by taking $\tilde{\delta}_i(xy^{-1}) = -\frac{x}{y}\frac{\delta_i(y)}{y} + \frac{\delta_i(x)}{y}$, for $xy^{-1} \in Q$ and $x, y \in R$. Let us consider now $k \in I$ and let us see that $\tilde{\delta}_i(q)k \in R$. Let us notice that $qk = r \in R$ given that $qI \subseteq R$. Then, we have $\tilde{\delta}_i(qk) = \tilde{\delta}_i(r) = r' \in R$ and $\tilde{\delta}_i(qk) = q\tilde{\delta}_i(k) + \tilde{\delta}_i(q)k$ by $\tilde{\delta}_i$ being a derivation. Provided that I is Δ-invariant, $\tilde{\delta}_i(k) \in I$, which implies $q\tilde{\delta}_i(k) \in R$ by using the fact that $qI \subseteq R$. Thus $\tilde{\delta}_i(q)k = \tilde{\delta}_i(qk) - q\tilde{\delta}_i(k) \in R$, as we wanted to prove. The proofs for $(R:I)_r$, I_v and ${}_vI$ being Δ-invariant follow similarly.

(ii) We must check that there exist units $a, b \in Q(S)$ such that $aS \subseteq IS \subseteq bS$. Let us recall that $IS = \left\{\sum_{i=1}^{m} k_i s_i \mid m \ge 1, k_i \in I, s_i \in S, 1 \le i \le m\right\}$. Given that I is an R-ideal, there exist units $c, d \in Q(R)$ such that $cR \subseteq I \subseteq dR$. Consider $p \in S$, then cp is a polynomial whose coefficients have the form cr_j for $r_j \in R$. Given that $cR \subseteq I$, such terms are in I and thus cp can be seen as a finite sum of terms of the form $k_i x^i$, where $x^i = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Since $k_\in I$ and $x^i \in S$, we have $cS \subseteq IS$. On the other hand, let us check that $IS \subseteq dS$. Let us consider $x = \sum_{i=1}^{m} k_i s_i \in IS$. then we know that $k_i = dr_i$ for some $r_i \in R$ and $x = \sum_{i=1}^{m} dr_i s_i = d\sum_{i=1}^{m} r_i s_i$. We see that $\sum_{i=1}^{m} r_i s_i \in S$ and therefore, $x \in dS$. We conclude that $cS \subseteq IS \subseteq dS$ and IS is indeed an S-ideal.
Now let us check that $O_l(I)$ is Δ-invariant. Let q be an element in $O_l(I)$, meaning that $qI \subseteq I$; and let us see that $\tilde{\delta}_i(q) \in O_l(I)$, which means $\tilde{\delta}_i(q)I \subseteq I$. If k is an element of I, then $\tilde{\delta}_i(qk) = q\tilde{\delta}_i(k) + \tilde{\delta}_i(q)k$. Since $qk \in I$ by q being in $O_l(I)$, we obtain $\tilde{\delta}_i(qk) \in I$. Thus, $\tilde{\delta}_i(q)k = \tilde{\delta}_i(qk) - q\tilde{\delta}_i(k) \in I$, as required. Similarly, we prove that $O_r(I)$ is Δ-invariant.

□

We now present a generalization to the previous result, in terms of skew PBW extensions. It is worth mentioning that according to [LAC+13], Lemma 2.6, we can extend an endomorphism σ_i and a σ_i-derivation δ_i to R_S (with S being a multiplicative subset of R) by taking $\tilde{\sigma_i}(as^{-1}) = \frac{\sigma_i(a)}{\sigma_i(s)}$ and $\tilde{\delta_i}(as^{-1}) = -\frac{\sigma_i(a)}{\sigma_i(s)}\frac{\delta_i(s)}{s} + \frac{\delta_i(a)}{s}$ as long as $\sigma_i(S) = S$.

Theorem 2.3.7. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of a ring R such that $\sigma_i(S) = S$ for $S = \{$units of $Q(R)\} \cap R$. Then we have the following:*

(i) Let I be a one-sided Δ-R-ideal. If $\sigma((R : I)_l) \subseteq (R : I)_l$, then $(R : I)_l$, $(R : I)_r$, I_v and ${}_vI$ are Δ-invariant.

(ii) Let I be a Δ-R-ideal. Then IA is an A-ideal and $O_l(I)$ and $O_r(I)$ are also Δ-invariant.

(iii) Let I be a Δ-R-ideal. Then $(IA)_v = I_vA$ and ${}_v(IA) = {}_vIA$.

Proof. The proof of this result follows in the same way as the proof of Proposition 2.3.6, but as we mentioned before, we take $\tilde{\sigma_i}(as^{-1}) = \frac{\sigma_i(a)}{\sigma_i(s)}$ and $\tilde{\delta_i}(as^{-1}) = -\frac{\sigma_i(a)}{\sigma_i(s)}\frac{\delta_i(s)}{s} + \frac{\delta_i(a)}{s}$. Let us notice that indeed for item (i), if k is an element in an ideal I of $Q(R)$ and $q \in (R : I)_l$, then $\tilde{\delta_i}(qk) = \tilde{\sigma_i}(q)\tilde{\delta_i}(k) + \tilde{\delta_i}(q)k$. Given that $qI \subseteq R$, we get $\tilde{\delta_i}(q)I \subseteq R$, since $\sigma((R : I)_l) \subseteq (R : I)_l$. Then we have $\tilde{\sigma_i}(q)\tilde{\delta_i}(k) \in R$. On the other hand we have $\tilde{\delta_i}(qk) = \tilde{\delta_i}(r) = r' \in R$ and thus, we obtain $\tilde{\delta_i}(q)k = \tilde{\delta_i}(qk) - \tilde{\sigma_i}(q)\tilde{\delta_i}(k) \in R$. □

Remark 2.3.8. In order to make the previous theorem work, we imposed the condition that $\sigma((R : I)_l) \subseteq (R : I)_l$. Let us notice that in the case of PBW extensions, this condition is trivial, since σ_i is the identity map. In this sense, our theorem is an adequate generalization of Proposition 2.3.6. Imposing this type of conditions is a commonly used technique. For example in [Mar84] for the study of an ideal being σ-invariant, Marubayashi makes use of a condition he calls "condition (C)", which consists in satisfying the maximum condition on right $\underline{C}$-closed ideals and left $\underline{C'}$-closed ideals (see [Mar84], page 1569).

Proposition 2.3.9 ([MZ96], Lemma 6)**.** *The following statements hold:*

(i) Suppose that R is a Noetherian Δ-maximal order in Q. Then a Δ-v-ideal of R is a maximal Δ-v-ideal if and only if it is a Δ-prime v-ideal.

(ii) Suppose that R is a Noetherian maximal order in Q. If $S = R\langle x_1, \dots, x_n\rangle$ is a PBW extension of R, then a v-ideal of S is a maximal v-ideal if and only if it is a prime v-ideal.

Proof. (i) Let A be a maximal Δ-v-ideal of R and I, J be Δ-v-ideals of R such that $IJ \subseteq A$ and $A \subsetneq J$. We have to show that $I \subseteq A$. By Proposition 2.3.6, part i), we have that J_v is also Δ-invariant and we have $A \subsetneq J = J_v$. Since A is maximal, we must have $J_v = R$. Then, it follows that $I = IR = IJ_v \subseteq (IJ_v)_v = (IJ)_v \subseteq A$. Therefore, A is Δ-prime, as we wanted to prove.

Conversely, let us suppose that A is a Δ-prime v-ideal. According to [MZ96], page 1384, since R is a Noetherian Δ-maximal order we have that $\mathscr{D}_\Delta(R)$, the set of all Δ-v-ideals is a free abelian group with the product given by $B \cdot C = (BC)_v$ and the inverse being $B^{-1} = (R : B)_l = (R : B)r = \{q \in Q(R) \mid BqB \subseteq B\}$ and it is generated by maximal Δ-v-ideals of R. With this in mind, A contains a finite product of maximal Δ-v-ideals. Hence, A must be a maximal Δ-v-ideal, as wanted.

(ii) This assertion follows in a similar way to the previous one.

□

We now generalize the previous result to skew PBW extensions.

Theorem 2.3.10. *The following statements hold:*

(i) Suppose that R is a Noetherian Δ-maximal order in Q. Then a Δ-v-ideal of R is a maximal Δ-v-ideal if and only if it is a Δ-prime v-ideal.

(ii) Suppose that R is a Noetherian maximal order in Q. If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension of R, then a v-ideal of A is a maximal v-ideal if and only if it is a prime v-ideal.

Proof. The result follows in a similar way to the proof of Proposition 2.3.9. The key fact is that $\mathscr{D}_\Delta(R)$, the set of all Δ-v-ideals is a free abelian group with the product given by $B \cdot C = (BC)_v$ and the inverse being $B^{-1} = (R : B)_l = (R : B)r = \{q \in Q(R) \mid BqB \subseteq B\}$ and it is generated by maximal Δ-v-ideals of R. □

Proposition 2.3.11 ([MZ96], Corollary 7)**.** *Let A be any maximal Δ-v-ideal of a Noetherian maximal order R and let us suppose that $S = R\langle x_1, \dots, x_n\rangle$ is a PBW extension of R. Then AS is a maximal v-ideal of S.*

Proof. By Propositions 2.3.5 and 1.3.3, S turns out to be a Noetherian maximal order in $Q(S)$. By Proposition 2.3.9, since A is a maximal Δ-v-ideal, then it is a Δ-prime v-ideal. Now, by Proposition 2.3.5, provided that A is a Δ-invariant ideal of R, then A is a Δ-prime ideal of R if and only if AS is a prime ideal of S, which gives us that AS is a prime ideal in S. We know that A is a v-ideal, which means $A = A_v$ and thus $AS = A_vS$. by Proposition 2.3.6, $A_vS = (AS)_v$ and we get $AS = (AS)_v$, which makes AS a v-ideal. Since AS is a v-ideal and it is prime, by Proposition 2.3.9, AS is a maximal v-ideal of S, as wanted. □

Theorem 2.3.12. *Let I be any maximal Δ-v-ideal of a Noetherian maximal order R and let us suppose that $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension of R. Then IA is a maximal v-ideal of A.*

Proof. By Theorem 2.2.3 and Proposition 2.1.18, we obtain that A is a Noetherian maximal order in $Q(A)$ and thus the proof of Proposition 2.3.11, works in this case. □

Proposition 2.3.13 ([MZ96], Lemma 8)**.** *Let R be a Noetherian maximal order in $Q(R)$ and B' be a maximal v-ideal of $T = Q(R)\langle x_1, x_2, \dots, x_n\rangle$. Then $B = B' \cap S$ is a maximal v-ideal of S.*

Proof. By Lemma 2.3 in [Mar83], we get that B is a right v-ideal. If I is an ideal of S, then IT is an ideal of T following Theorem 1.3.1 in [CH80], given that T is Noetherian and $T = S_C$, where C is the set of regular elements of S. Hence B is a prime v-ideal of S since B' is a prime ideal of T and thus B is a maximal v-ideal of S by Proposition 2.3.9. □

Theorem 2.3.14. *Let R be a Noetherian maximal order in $Q(R)$ and B' be a maximal v-ideal of $T = \sigma(Q(R))\langle x_1, x_2, \dots, x_n\rangle$. Then $B = B' \cap S$ is a maximal v-ideal of $A = \sigma(R)\langle x_1, x_2, \dots, x_n\rangle$.*

Proof. This results follows in a similar way as the proof of Proposition 2.3.13, using Theorem 2.3.10. □

Proposition 2.3.15 ([MZ96], Theorem 9)**.** *Let R be a Noetherian maximal order in $Q(R)$ and $S = R\langle x_1,\dots,x_n\rangle$ be a PBW extension of R. Then $\{AS, B'\cap S = B \mid A$ is a maximal Δ-v-ideal of R and B' is a maximal v-ideal of $T\}$ is the full set of all maximal v-ideals of S.*

Proof. By Propositions 2.3.11 and 2.3.13, we get that AS and $B'\cap S$ are maximal v-ideals of S, where A is any maximal Δ-v-ideal of R and B' is any maximal v-ideal of T. Now let M be a maximal v-ideal of S. If $M\cap R = 0$ then there exists a maximal v-ideal B' of T such that $MT \subseteq B' \subseteq T$. So $M \subseteq MT\cap S \subseteq B'\cap S \subseteq S$. According to Proposition 2.3.13, we have that $B'\cap S$ is a maximal v-ideal of S. Because of this, we get that $(M\cap R)S$ is a prime ideal of S and by Proposition 2.3.6, $((M\cap R)S)_v = (M\cap R)_v S = (M\cap R)S$, which makes $(M\cap R)S$ a prime v-ideal. By Proposition 2.3.9, $(M\cap R)S$ is then a maximal v-ideal. Given that $(M\cap R)S \subset M$ and $(M\cap R)S$ is maximal, we have $M = (M\cap R)S$ and thus $M\cap R$ is a maximal Δ-v-ideal by Proposition 2.3.9. □

Theorem 2.3.16. *Let R be a Noetherian maximal order in $Q(R)$ and $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a bijective skew PBW extension of R. Then $\{IA, B'\cap A = B \mid I$ is a maximal Δ-v-ideal of R and B' is a maximal v-ideal of $T\}$, with $T = \sigma(Q(R))\langle x_1,x_2,\dots,x_n\rangle$ is the full set of all maximal v-ideals of A.*

Proof. The proof of this fact follows from Theorems 2.3.12 and 2.3.14 in the same way as the proof of Proposition 2.3.15 with help of Theorems 2.3.7 and 2.3.10. □

Recall that a domain B is a **Dedekind ring**, if it is integrally closed, Noetherian, and all its nonzero prime ideals are maximal ideals. For example, every principal ideal domain (PID) is a Dedekind ring and $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind ring that is not a unique factorization domain, and hence it is not a PID. Now, noncommutative analogs of Dedekind rings are called **left hereditary**. A ring B is *left hereditary*, if every left ideal is a projective B-module. Aside from Dedekind rings, some examples of left hereditary rings are semisimple rings, noncommutative principal ideal rings and free ideal rings (all left ideals are free B-modules). It is well-known that Cartan and Eilenberg proved that the following conditions are equivalent for a domain B: B is a Dedekind ring; every submodule of a projective B-module is projective; every quotient of an injective B-module is injective.

Cohen [Coh50] proved that if B is a Noetherian commutative domain, then the following conditions are equivalent:

(i) B is a Dedekind domain.

(ii) For every maximal ideal $\mathfrak{p}$ in B, the localization $R_{\mathfrak{p}}$ is a discrete valuation ring.

(iii) If $\mathfrak{p}$ is a maximal ideal in B, there is no primary ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ (and hence no ideal at all, since any ideal between $\mathfrak{p}$ and $\mathfrak{p}^2$ is necessarily primary, since $\mathfrak{p}$ is maximal).

(iv) A primary ideal belonging to a maximal ideal is a product of prime ideals.

(v) The primary ideals belonging to a maximal ideal is a product of prime ideals.

(vi) The primary ideals belonging to a maximal ideal are totally ordered (with respect to set inclusion).

With respect to the noncommuative context, Michler [Mic69] proved a noncommutative generalisation of this theorem: Let B be a Noetherian ring. Then B is bounded Asano order in a simple Artinian ring if and only if B satisfies the following conditions:

(i) $B_{\mathfrak{p}}$ exists for each maximal ideal $\mathfrak{p}$ of B and is hereditary;

(ii) $B = \bigcap B_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all maximal ideals of B;

(iii) essential tertiary one-sided ideals of B are primary and their prime ideals are non zero.

Next, Hajarnavis and Lenagan [HL72] showed that such a theorem obtained by Michler holds without B being bounded and that in this general situation conditions (ii) and (iii) are unnecesary.

On the other hand, Kuzmanovich [Kuz72] obtained localization and globalization theorems for a Dedekind prime ring B with classical quotient ring $Q(B)$ ($Q(B)$ is simple Artinian). The localizations of B which are obtained consist of one subring $B_{\mathfrak{m}}$, for every maximal two-sided ideal $\mathfrak{m}$ of B, and one additional simple overring S which coincides with $Q(B)$ when B is bounded. The globalization results establish that $(\cap B_{\mathfrak{m}}) \cap S = B$, and that a homomorphism of B-modules is one-one (onto) if and only if all of the "localized" homomorphisms $f_{\mathfrak{m}}$ and f_S are one-one (onto). About Kuzmanovich's result which establishes that a Dedekind prime ring B is the intersection of the localisations at maximal ideals and a simple overring ($(\cap B_{\mathfrak{m}}) \cap S = B$), Hajarnavis and Lenagan [HL72] proved that this resuls holds in the Asano case.

More recently, Marubayashi [Mar80] has defined Krull orders in simple Artinian rings as a Krull type generalization of noncommutative Dedekind rings. Three years later, in [Mar83], he studied a class of orders which is a Krull type generalization of hereditary Noetherian prime rings (HNP rings) with enough invertible ideals. Let us see the key ideas.

Briefly, if B is an order in a simple Artinian ring $Q(B)$, as it is well known, an B-ideal I is projective as left and right B-modules if and only if $I(B:I)_l := O_l(I) = \{x \in Q(B) \mid xI \subset I\}$ and $(B:I)_r I = O_r(I) = \{y \in Q(B) \mid Iy \subset I\}$, where $(B:I)_l = \{x \in Q(B) \mid xI \subset B\}$ and $(B:I)_r = \{y \in Q(B) \mid Iy \subset B\}$. With this notation in hands, Marubayashi consider the following condition which is a Krull type generalization of projectivities: (K): ${}_V(I(B:I)_l) = O_l(I)$, for any B-ideal I such that $I = {}_V I = (B:(B:I)_r)_l$, that is, I is reflexive as a left B-module, and $((B:I)_r I)_V = O_r(I)$, for any B-ideal I such that $I = I_V$. An order B is said to be *v-hereditary* (simply a VH-order), if B satisfies the condition (K). If ${}_V I = I = I_V$, then it is called a *$v - B$-ideal* (simply v-ideal). A v-ideal J is called *v-invertible*, if $(J(B:J)_r)_V = B = {}_V((B:J)_l J)$. Marubayashi proved that the set $D(B)$ consisting of all v-invertibles ideals is a free abelian group generated by maximal v-invertible ideals of B and that any maximal v-invertible ideal of B is an intersection of a cycle and is prime or semi-prime ideal as in HNP rings if B is a VH-order with enough v-invertible ideals and satisfies the maximum condition on integral v-ideals. Now, Chamarie [Cha81] considered the following condition to get the classical localization B_I of B at a v-ideal I in case B is a maximal order. (C): B satisfies the maximum condition on right C-closed ideals of B and left C'-closed ideals of B. About that, Marubayashi said in his paper that B is a VHC-*order*, if it is a VH-order and satisfies the condition (C). If B is a VHC-order and J is any integral v-invertible ideal, Marubayashi showed that the classical localization B_J of B at J exists by using the properties of v-invertible ideals above. He get the following characterization in terms of localizations ([Mar83], Theorem 2.23): an order is a VHC-order with enough v-invertible ideals if and only if it satisfies the following: $B = \cap B_{\mathfrak{p}} \cap S(B)$, where $\mathfrak{p}$ ranges over all maximal v-invertible ideals of R and $B_{\mathfrak{p}}$ is a semi-local HNP ring with unique maximal invertible ideal $J(B_{\mathfrak{p}})$, the Jacobson radical of $B_{\mathfrak{p}}$, and $S(B)$ is a Krull order in the sense of [Cha81] and is v-simple. Remarkable examples of VHC-orders with enough v-invertible ideals are the Krull orders in the sense of [Cha81] (this notion of Krull order consider a special Gabriel topology on the ring) and HNP rings with enough invertible ideals [ER70]. Precisely, Noetherian maximal orders are Krull orders. This fact is very important in the following proposition, which gives a decomposition of a PBW extension in terms of maximal v-ideals.

Proposition 2.3.17 ([MZ96], Theorem 10)**.** *Let $S = R\langle x_1,\dots,x_n\rangle$ be a PBW extension of R. Then $S = \bigcap_{A\in\mathscr{A}} S_{AS} \cap \bigcap_{B\in\mathscr{B}} S_{B'\cap S} \bigcap \mathscr{S}(S)$ where $\mathscr{A}$ is the set of maximal v-ideals of R and $\mathscr{B}$ is the set of maximal v-ideals of T, and $\mathscr{S}(S) = \cup X^{-1}$ where X runs over all v-invertible ideals of S. Moreover, each S_{AS} and each $S_{B\cap S}$ is a local Dedekind prime ring and $\mathscr{S}(S)$ is v-simple, meaning that it has no proper v-ideals.*

Proof. By the argument about globalization given previously, S has the desired form. Also, by [Cha81], Theorem 4.2.7, we have that $\mathscr{S}(S)$ is v-simple. By the remark of Proposition 2.7 in [Mar83], S_{AS} and $S_{B'\cap S}$ are local Dedekind prime rings. □

Regarding our previous observation, we can establish a similar statement for skew PBW extensions, as follows.

Theorem 2.3.18. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a bijective skew PBW extension of R. Then $A = \bigcap_{I\in\mathscr{I}} A_{IA} \cap \bigcap_{B\in\mathscr{B}} A_{B'\cap A} \bigcap \mathscr{A}(A)$ where $\mathscr{I}$ is the set of maximal v-ideals of R and $\mathscr{B}$ is the set of maximal v-ideals of T, and $\mathscr{A}(A) = \cup X^{-1}$ where X runs over all v-invertible ideals of A. Moreover, each A_{IA} and each $A_{B\cap A}$ is a local Dedekind prime ring and $\mathscr{A}(A)$ is v-simple, meaning that it has no proper v-ideals.*

Proof. By Theorem 2.3.16, we have the classification of all v-ideals and therefore this result follows in the same way as 2.3.15. □

APPENDIX A

Appendix

A.1 Graded and filtered rings

In this section we establish the definition of both graded and filtered rings, their relation and some properties that hold for them. This section follows the ideas presented in [Lez16].

Definition A.1.1 ([Lez16], Definition 2.1.1.)**.** A ring A is said to be $\mathbb{Z}$**-graded**, if there exists a family of subgroups $\{A_p\}_{p\in\mathbb{Z}}$ of its additive subgroup A^+ such that:

(i) $A_pA_q \subseteq A_{p+q}$ for any $p, q \in \mathbb{Z}$

(ii) $A = \sum_{p\in\mathbb{Z}} \bigoplus A_p$

For $p \in \mathbb{Z}$, A_p is called the **homogenous component of degree** p and the elements of A_p are called **homogenous of degree** p. $\{A_p\}_{p\in\mathbb{Z}}$ is called a **graduation** of A. If $A_p = 0$ for $p < 0$, meaning $A = \sum_{p\in\mathbb{N}} A_p$, A is called a **positively graded ring**. If K is an algebra and A is a K-algebra, it is said that A is $\mathbb{Z}$**-graded** if it further satisfies that A_p is a K-subspace of A. If A and B are graded rings (algebras), with graduations $\{A_p\}_{p\in\mathbb{Z}}$ and $\{B_p\}_{p\in\mathbb{Z}}$, respectively, then a ring (algebra) homomorphism $f : A \to B$ is **graded**, if $f(A_p) \subseteq B_p$, for all $p \in \mathbb{Z}$.

Remark A.1.2 ([Lez16], Definition 2.1.2., Definition 2.1.3)**.** (i) From the previous definition, it follows that $1 \in A_0$ and A_0 is a subring (subalgebra) of A.

(ii) Given a $\mathbb{Z}$-graded ring A, we can talk of a **graded module** for a right A-module M when there exists a family of subgroups of its additive group M^+ such that $M_pA_q \subseteq M_{p+q}$, for any $p, q \in \mathbb{Z}$ and $M = \sum_{p\in\mathbb{Z}} \bigoplus M_p$. The names remain the same as in the definition, and an A-homomorphism $f : M \to N$ is **graded**, if $f(M_p) \subseteq N_p$, for every $p \in \mathbb{Z}$.

(iii) If A is a graded ring and M is a graded A-module, a A-submodule N of M is a **graded submodule** of M, if $N = \sum_{p\in\mathbb{Z}} \bigoplus (M_p \cap N)$.

Proposition A.1.3 ([Lez16], Proposition 2.1.4.)**.** *Let A be a graded ring, M a graded A-module and N an A-submodule of M, then the following statements are equivalent:*

(i) N is a graded submodule

(ii) If $u \in N$, then every homogenous component of u belong to N.

(iii) N is generated by homogenous element

(iv) The quotient module M/N is graded: $M/N = \sum_{p\in\mathbb{Z}} \bigoplus [(M_p + N)/N]$

Remark A.1.4 ([Lez16], Proposition 2.1.6.)**.** (i) If we replace submodule by ideal (right, left or bilateral) we acquire the notion of **graded ideal**. If S is a subring of A if $S = \sum_{p\in\mathbb{Z}} \bigoplus (A_p \cap S)$

(ii) If $f : A \to B$ is a graded rng homomorphism, then $\mathrm{Im}(f)$ is a graded subring of B and $\ker(f)$ is a graded bilateral ideal of A.

(iii) Composition of graded ring homomorphisms is again a graded homomorphism.

(iv) If I is a proper bilateral ideal of A then A/I is a graded ring.

(v) If $f : M \to N$ is a graded A-homomorphism, then $\mathrm{Im}(f)$ is a graded submodule of N and $\ker(f)$ is a graded submodule of M.

(vi) Composition of graded A-homomorphisms is again a graded homomorphism.

Definition A.1.5 ([Lez16], Definition 2.2.1.)**.** A ring A is said to be $\mathbb{Z}$**-filtered**, if there exists a family of subgroups $\{F_p(A)\}_{p\in\mathbb{Z}}$ of its additive subgroup A^+ which satisfies:

(i) $F_p(A)F_q(A) \subseteq F_{p+q}(A)$, for every $p, q \in A$

(ii) $\bigcup_{p\in\mathbb{Z}} F_p(A) = A$

(iii) For $p < q$, $F_p(A) \subseteq F_q(A)$

(iv) $1 \in F_0(A)$

The family $\{F_p(A)\}_{p\in\mathbb{Z}}$ is called a **filtration** of A. The filtration is called **separated**, if $\bigcap_{p\in\mathbb{Z}} F_p(A) = 0$. If $F_{-1}(A) = 0$, A is a **positively filtered ring**. If K is a field and A is a K-algebra, then A is $\mathbb{Z}$**-filtered**, if it further satisfies $F_p(A)$ being a K-subspace of A. If A and B are filtered rings (algebras) with filtrations $\{F_p(A)\}_{p\in\mathbb{Z}}$ and $\{F_p(B)\}_{p\in\mathbb{Z}}$, respectively, a ring (algebra) homomorphism $f : A \to B$ is **filtered**, if $f(F_p(A)) \subseteq F_p(B)$, for every $p \in \mathbb{Z}$

Remark A.1.6. 1. $F_0(A)$ is a subring (subalgebra) of A.

2. Every positive filtration is separated.

3. Composition of filtered homomorphisms is again a filtered homomorphism.

Proposition A.1.7 ([Lez16], Proposition 2.2.2.)**.** *Every graded ring A is filtered with graduation is filtered, by taking*

Proof. Let A be a graded ring with graduation $\{A_p\}_{p\in\mathbb{Z}}$. Then A is filtered, with filtration given by $\{F_p(A)\}_{p\in\mathbb{Z}}$, $F_p(A) = \sum_{n\leq p} \bigoplus A_n$. □

Proposition A.1.8 ([Lez16], Proposition 2.2.3.)**.** *If A is a filtered ring, then there exists a graded ring $Gr(A)$ associated to A.*

Proof. Let A be a filtered ring with filtration $\{F_p(A)\}_{p\in\mathbb{Z}}$; we define $Gr(A)_p = F_p(A)/F_{p-1}(A)$, for $p \in \mathbb{Z}$, which is an abelian group. Then we define $Gr(A) = \bigoplus_{p\in\mathbb{Z}} Gr(A)_p$, which is also an abelian group. The product in $Gr(A)$ is defined as follows:

$$F_p(A)/F_{p-1}(A) \times F_q(A)/F_{q-1}(A) \to F_{p+q}(A)/F_{p+q-1}(A)$$
$$(a + F_{p-1}(A), b + F_{q-1}(A)) \mapsto ab + F_{p+q-1}(A)$$

The reader may see [Lez16], Proposition 2.2.3. to check that this product is well defined. By this definition, it follows that $Gr(A)_p Gr(A)_q \subseteq Gr(A)_{p+q}$, for $p, q \in \mathbb{Z}$. If we take $\overline{x} \in Gr(A)$, then $\overline{x} = \sum_{p\in\mathbb{Z}} x_p + F_{p-1} \in Gr(A)$ and this is a direct sum. Therefore, $Gr(A)$ is in fact a graded ring.

□

Definition A.1.9 ([Lez16], Definition 2.2.6.)**.** If A is a filtered ring with filtration $\{F_p(A)\}_{p\in\mathbb{Z}}$ and M is an A-module, M is said to be $\mathbb{Z}$**-filtered**, if there exists a family of subgroups $\{F_p(M)\}_{p\in\mathbb{Z}}$ of M which satisfies:

(i) $F_p(M)F_q(A) \subseteq F_{p+q}(M)$, for every $p, q \in A$

(ii) $\bigcup_{p\in\mathbb{Z}} F_p(A) = A$

(iii) For $p < q$, $F_p(A) \subseteq F_q(A)$

(iv) $1 \in F_0(A)$

The family $\{F_p(M)\}_{p\in\mathbb{Z}}$ is called a **filtration** of M. The filtration is called **separated** if $\bigcap_{p\in\mathbb{Z}} F_p(M) = 0$. If $F_{-1}(M) = 0$, A is a **positively filtered module**. If K is a field and A is a K-algebra, then A is $\mathbb{Z}$**-filtered** if it further satisfies $F_p(M)$ being a K-subspace of M. If M and N are filtered rings (algebras) with filtrations $\{F_p(M)\}_{p\in\mathbb{Z}}$ and $\{F_p(N)\}_{p\in\mathbb{Z}}$, respectively, an A-homomorphism $f : M \to N$ is **filtered** if $f(F_p(M)) \subseteq F_p(N)$ for every $p \in \mathbb{Z}$.

As the following proposition shows, the filtration-graduation technique has been a powerful tool in algebra when trying to prove certain properties of rings, modules and algebras. If the desired property holds for the graded ring, it is of interest to check whether it holds for the filtered ring as well.

Proposition A.1.10 ([Lez16], Proposition 2.4.3., Theorem 2.4.4.)**.** *Let A be a filtered ring and M an A-module, then:*

(i) If M is filtered and $Gr(M)$ is finitely generated as a graded module over $Gr(A)$, then M is finitely generated as an A-module.

(ii) If M_A is finitely generated, then M with its standard filtration satisfies that $Gr(M)$ is finitely generated as a graded module over $Gr(A)$.

(iii) If $Gr(A)$ is a domain, then A is a domain.

(iv) If $Gr(A)$ is a prime ring, then A is a prime ring.

(v) If $Gr(A)$ is right Noetherian, then A is right Noetherian.

Next we mention the definition of a classic object in algebra, which is a *Lie algebra*. Also, we mention two other algebras associated to a Lie algebra, whose graded rings are known.

Definition A.1.11 ([MR01], 1.7.1, 1.7.2, 1.7.12)**.** (i) A **Lie algebra** $\mathscr{G}$ over $\mathbb{K}$ is a $\mathbb{K}$-vector space equipped with a **Lie product**, i.e. a $\mathbb{K}$-bilinear map $\mathscr{G}\times\mathscr{G}\to\mathscr{G}$, $(x,y)\mapsto[x,y]$ such that $[x,y]=-[y-x]$, $[x,x]=0$ and satisfying the **Jacobi identity** $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$. If A is an associative $\mathbb{K}$-algebra, A acquires the structure of Lie algebra by setting $[a,b]=ab-ba$, which is called the **bracket product**. A **representation** of a Lie algebra $\mathscr{G}$, is a Lie algebra homomorphism from $\mathscr{G}$ to such A.

(ii) If $\mathscr{G}$ is a Lie algebra, then the **universal enveloping algebra** of $\mathscr{G}$ is an (associative) $\mathbb{K}$-algebra $U=U(\mathscr{G})$ together with a representation $\theta:\mathscr{G}\to U$ which is universal (meaning that if A is an (associative) algebra and $\varphi:\mathscr{G}\to A$, there exists a unique algebra homomorphism $\psi:U\to A$ such that $\psi\theta=\varphi$). $U(\mathscr{G})$ can be seen as the factor of the free $\mathbb{K}$-algebra F on the set $\{x_i \mid i\in I\}$ by the ideal generated by the elements $x_ix_j-x_jx_i-[x_i,x_j]$.

(iii) If R is a $\mathbb{K}$-algebra with $\mathbb{K}\subseteq R$ and $\mathscr{G}$ is a $\mathbb{K}$-Lie algebra with basis $\{x_i \mid i\in I\}$ then a $\mathbb{K}$-algebra S with $R\subseteq S$ is called a **crossed product** of R by $U(\mathscr{G})$, written $R*U(\mathscr{G})$, if there is a $\mathbb{K}$-module embedding of $\mathscr{G}$ into S, $x\mapsto\overline{x}$ such that:

(a) $\overline{x}r-r\overline{x}\in R$ and $r\mapsto\overline{x}r-r\overline{x}$ is a $\mathbb{K}$-derivation of R.

(b) $\overline{xy}-\overline{yx}\in\overline{[x,y]}+R$ for all $x,y\in\mathscr{G}$

(c) S is a free right (and left) R-module with the standard monomials in $\{\overline{x_i}\}$ as a basis.

Proposition A.1.12 ([MR01], 1.7.4, 1.7.14)**.** *If $\{x_i \mid i\in I\}$ is a $\mathbb{K}$-basis for a Lie algebra $\mathscr{G}$, then:*

(i) $Gr(U(\mathscr{G}))$ is a commutative $\mathbb{K}$-algebra, generated over $\mathbb{K}$ by the set $\{\overline{x_i} \mid i\in I\}$, where $\overline{x_i}$ is the class of x_i in $gr(U(\mathscr{G}))$.

(ii) $Gr(R*U(g))$ is a polynomial ring over R in central variables $\{\overline{x_i} \mid i\in I\}$, where $\overline{x_i}$ is given by the embedding in the definition of crossed product.

A.2 Goldie theory

In this section we state some definitions and properties regarding Goldie Theory. We follow the ideas presented in [Lez16].

Definition A.2.1 ([Lez16], Definition 3.1.1.)**.** Let $M\neq 0$ be an A-module and $0\neq N\leq M$ a submodule such that $X\cap N\neq 0$ for every submodule $X\leq M$. Then N is an **essential submodule** of M, denoted by $N\leq_e M$.

Remark A.2.2 ([Lez16], Proposition 3.1.2., Proposition 3.1.3., Proposition 3.1.4.)**.** (i) Let $M\neq 0$ be an A-module and $0\neq N\leq_e M$. If $N\leq N'\leq M$, then $N'\leq_e M$.

(ii) Let $M\neq 0$ be an A-module. If $N\leq M$, then there exists $N'\leq M$ such that $N\cap N'=0$ and $N\bigoplus N'\leq_e M$.

(iii) For $1\leq i\leq t$, if $N_i\leq_e M_i$, then $N_1\bigoplus\cdots\bigoplus N_t\leq_e M_1\bigoplus\cdots\bigoplus M_t$

Proposition A.2.3 ([Lez16], Proposition 3.1.6.)**.** *An A-module $M \neq 0$ is semisimple if and only if M possesses no proper essential submodules.*

Definition A.2.4 ([Lez16], Definition 3.2.1., Definition 3.2.2.)**.** (i) A module U is **uniform** if $U \neq 0$ and every submodule of U is essential.

(ii) A module has **finite uniform dimension** if it does not possess infinite direct sums of nonzero submodules.

Proposition A.2.5 ([Lez16], Definition 3.2.4.)**.** *Let $0 \neq M$ be an A-module of finite uniform dimension. Then M has a uniform submodule.*

Proposition A.2.6 ([MR01], 2.2.9)**.** *Let M be a module of finite uniform dimension and let $\bigoplus_{i=1}^n U_i$ be a finite direct sum of uniform submodules of M which is essential in M. Then:*

(i) Any direct sum of nonzero submodules of M has at most n summands.

(ii) A direct sum of uniform submodules of M is essential in M if and only if it has precisely n summands.

The number n is called the ***uniform dimension*** *or* ***Goldie dimension*** *of M, noted $udim(M)$.*

Proof. We follow the ideas in [[Lez16], Theorem 3.2.5].

(i) Let $V = \bigoplus_{i=1}^k V_i \leq_e M$ be a direct sum of uniform submodules of M. If $W = \bigoplus_{i=2}^k V_i$, then $W \not\leq_e M$ since $W \cap V_1 = 0$. Let us suppose $W \cap U_i \neq 0$ for all i. Then, $W \cap U_i \leq_e U_i$ since U_i is uniform. Therefore, $\bigoplus_{i=1}^n (W \cap U_i) \leq_e \bigoplus_{i=1}^n U_i$, but given that $\bigoplus_{i=1}^n (W \cap U_i) \subseteq W \cap \bigoplus_{i=1}^n U_i$, we would obtain $W \cap \bigoplus_{i=1}^n U_i \leq_e \bigoplus_{i=1}^n U_i$. From $\bigoplus_{i=1}^n U_i \leq_e M$, we get $V_1 \cap \bigoplus_{i=1}^n U_i \neq 0$, then $(V_1 \cap \bigoplus_{i=1}^n U_i) \cap (W \cap \bigoplus_{i=1}^n U_i) \neq 0$ and thus, $V_1 \cap W \neq 0$, which is a contradiction. Then, we must have $W \cap U_i = 0$ for some i, say i=1. Then, $U_1 \cap \bigoplus_{i=2}^k V_i$ is a direct sum of nonzero submodules. If we repeat this process for $W_2 = U_1 \cap \bigoplus_{i=3}^k V_i$, and suppose $W_2 \cap U_i \neq 0$ for all $i \geq 2$, we obtain $W_2 \cap V_2 \neq 0$, which is a contradiction. Repeating this process k times, we obtain $k \leq n$.

(ii) ($\Rightarrow$) If we have a direct sum of m summands, by part (i), we have $m \leq n$. If we apply (i) but taking m instead of n, we obtain $n \leq m$. Thus, $n = m$.
($\Leftarrow$) Let us consider $B = \bigoplus_{i=1}^n B_i$ with $B_i \leq M$, uniform for each i. If we suppose $B \not\leq_e M$, then there exists $B_{n+1} \leq M$ such that $B \bigoplus B_{n+1}$ is a direct sum, but by part (i), any direct sum must have at most n summands.

□

Proposition A.2.7 ([Lez16], Definition 3.2.8.)**.** *Let M be a nonzero A-module.*

(i) $udim(M) = 1$

(ii) If $udim(M) = n$ and $N \leq M$, then $udim(N) \leq n$ and the equality holds if and only if $N \leq_e M$

(iii) $udim(M_1 \bigoplus M_2) = udim(M_1) + udim(M_2)$

It is pertinent to state some results about uniform dimension for skew PBW extensions studied in [Rey14].

Proposition A.2.8 ([Rey14], Proposition 3.2)**.** *If A is a bijective skew PBW extension of a right Noetherian domain R, then rudim$(A) = 1$.*

Proof. We have rudim$(R) = 1$ for being a right Noetherian domain. By Proposition **??**, A is also right Noetherian and by Proposition 2.1.15, A is also a domain. Thus, rudim$(A) = 1$. □

Proposition A.2.9 ([Rey14], Theorem 3.5)**.** *Let R be a prime right Goldie ring. If A is a bijective skew PBW extension of R, then rudim$(A) \leq$ rudim(R).*

Proof. Lemma 3.4 in [Rey14] states that rudim$(A) \leq$ rudim$(Gr(A))$. By Proposition 2.1.17, we know that $Gr(A) \cong R[z_1;\theta_1]\cdots[z_n;\theta_n]$ (with θ_i bijective, for all $1 \leq i \leq n$) and Proposition 3.3 in [Rey14] says that the uniform dimension is preserved by iterated polynomial rings of automorphism type. Therefore, rudim$Gr(A) =$ rudim(R) and the result follows. □

Definition A.2.10 ([BG88], Section 2)**.** Let R be a right Noetherian ring and let U be a uniform right R-module. Then there is a unique prime ideal P of R which is the largest annihilator of any nonzero submodule of U. This prime ideal is called the **assassinator** of U and we say that U is **tame** if it contains a copy of a nonzero right ideal of R/P. An arbitrary right R-module M

By making use of the previous definition, in [Rey14] it was possible to establish some conditions in order to preserve the dimension from a ring R to a skew PBW extension A of R. Let us recall that according to [Rey14], if $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ is a PBW extension of R, then an ideal I of R is called (Σ,Δ)-invariant if $\sigma_i(I) \subseteq I$ and $\delta_i(I) \subseteq I$, for all $1 \leq i \leq n$.

Proposition A.2.11. *Let A be a bijective skew PBW extension of a right Noetherian ring. Suppose that R is tame as a right R-module over itself and that any prime annihilator ideal in R is (Σ,Δ)-invariant. Then rudim$_R(R) =$ rudim$_A(A)$*

Next, we present the definition of *semiprime* ideal, which is a notion of great usage in algebra.

Definition A.2.12 ([Lez16], Section 3.3.)**.** A bilateral proper ideal I of a ring A is **semiprime** if it is intersection of prime ideals. The ring A is semiprime if 0 is semiprime $(rad(A) = 0)$

The following proposition gives us several ways to check whether a given ideal is semiprime.

Proposition A.2.13 ([Lez16], Proposition 3.3.1., Definition 3.3.2.)**.** *Let I be a bilateral ideal of a ring A. The following statements are equivalent:*

(i) *I is semiprime*

(ii) *If $x \in A$ is such that $xAx \subseteq I$, then $x \in I$*

(iii) *If J is a bilateral ideal of A such that $J^2 \subseteq I$, then $J \subseteq I$*

(iv) *If J is a right ideal of A such that $J^2 \subseteq I$, then $J \subseteq I$*

(v) *If J is a left ideal of A such that $J^2 \subseteq I$, then $J \subseteq I$*

(vi) If J is a bilateral ideal of A such that $J^2 \subseteq I$ and $I \subseteq J$, then $I = J$

Definition A.2.14 ([MR01], 2.3.1)**.** A ring R is a **right Goldie ring**, if it has finite right uniform dimension and satisfies the ascendent chain condition on right annihilators.

Proposition A.2.15 ([Lez16], Lemma 3.3.10.)**.** *(Goldie's Regular Element Lemma) Let A be a semiprime right Goldie ring. Then, a nonzero right ideal I of A is essential if and only if it contains a regular element.*

Proposition A.2.16 ([Lez16], Proposition 3.3.12., Proposition 3.3.13.)**.** *Let A be a ring and S be a set of elements which are not zero divisors such that S satisfies the right Ore condition. then:*

(i) If AS^{-1} is right Goldie, then A is right Goldie.

(ii) If $I \leq_e A_A$, then $IS^{-1} \leq_e (AS^{-1})_{AS^{-1}}$

(iii) If $J \leq_e (AS^{-1})_{AS^{-1}}$, then $I \leq_e A_A$ with $I = \{a \in A \mid \frac{a}{1} \in J\}$

Proposition A.2.17 ([Lez16], Definition 3.3.14.)**.** *(Goldie's Theorem) Let A be a ring. Then:*

(i) $Q_r(A)$ exists and it is semisimple if and only if A is semiprime right Goldie.

(ii) Let A be semiprime right Goldie. $Q_r(A)$ is simple if and only if A is prime.

(iii) $Q_r(A)$ exists and it is simple right artinian if and only if A is prime right Goldie.

Future work

Currently in the literature there are some ideas about a possible theory of noncommutative rings from the point of view of multiplicative ideal theory. Maximal orders, Krull orders (rings), unique factorization rings, generalized Dedekind prime rings and hereditary Noetherian prime rings are the main classes of noncommutative rings that illustrate these ideas. We think that the first step in the research about these topics for skew PBW extensions is the notion of Krull order. Precisely, this notion has been studied for PBW extensions by Marubayashi, Yang and Zhang in [MYZ98] where the authors investigated sufficient conditions to guarantee that a PBW extension of a Krull order is a Krull order and gave some examples of Krull orders, so the natural task for us is to extend the results of [MYZ98] to the more general context of skew PBW extensions. After doing this (we have faith that it is possible), we could consider the ideal theory in HNP (Hereditary Noetherian prime) rings following [ER70] and the proposal of a polynomial-type generalization of these rings established by Akalan and Marubayashi in [AM03] (the G-HNP rings). The reason of our interest lies in the fact that at the end of [AM03] the authors formulate several questions about necessary and sufficient conditions for an Ore extension or an Ore-Rees ring to be a G-HNP, so these questions can also be asked for PBW and skew PBW extensions. As we can see, fortunately the research in this topic does not stop; it goes on.

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