Some combinatorial topics in representation theory of algebras and its applications

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Bogotá, D.C.
February, 2020
Some combinatorial topics in representation theory of algebras and its applications

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DISSERTATION PRESENTED FOR THE DEGREE OF
Doctor in Mathematics

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Departamento de Matemáticas
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February, 2020
Title in English
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Título en español
Algunos temas combinatoriales en la teoría de representación de álgebras y sus aplicaciones.

Abstract: This thesis deals with a range of questions in combinatorial aspects of the representation theory of algebras and its applications. In particular, we provide a solution to the following problems:

1. How is it possible to give a geometric model, in the spirit of [18], for the categories of socle-projective modules over incidence algebras of some particular kind of posets?

2. How can the finitely generated Zavadskij modules over a finite-dimensional \( k \)-algebra be combinatorially characterized?

3. What kind of integer sequences arise from enumerating partitions induced by the \( \tau \)-orbits in the Auslander-Reiten quiver of a Dynkin algebra?

4. What are the algebraic properties of the base matrices in the lattice-based visual secret sharing scheme for color and gray-scale images given in [54]?

Regarding the first question, we give a geometric realization to the category of finitely generated socle-projective modules over the incidence algebra of a poset of type \( A \) in the spirit of [18]. In particular, the Auslander-Reiten quiver of such a category can be calculated using a geometric model based on a triangulation of a regular polygon, a special set of diagonals called sp-diagonals, and a kind of moves between sp-diagonals. Furthermore, for each poset of type \( A \), the category of sp-diagonals defines a categorification for certain subalgebra of a cluster algebra, then we study such subalgebras. To tackle the second topic, we use the mast of an indecomposable uniserial module introduced by Zimmermann (see [13, 93]) to give a combinatorial characterization to the finitely generated Zavadskij modules over finite-dimensional \( k \)-algebras. Additionally, we give explicit formulas for the number of indecomposable Zavadskij modules over Dynkin algebras. The third question allows us to define the \( \tau \)-orbit partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of integer numbers, where the parts \( \lambda_i \) are defined by the length of the \( \tau \)-orbits in the Auslander-Reiten quiver of a Dynkin algebra; in such a situation, we describe and enumerate those partitions obtaining the integer sequences A016116 [83] and A000034 [84] in The On-Line Encyclopedia of Integer Sequences (in short, OEIS). Finally, in order to answer the last question, we associate certain matrix problems to the scheme in [54], which gives an algebraic interpretation to the base matrices as matrix representations of a color-lattice.

Resumen: Esta tesis trata con un rango de preguntas en aspectos combinatórios de representaciones de álgebras y sus aplicaciones. En particular, damos una solución a los siguientes problemas:

1. ¿Cómo dar un modelo geométrico, en el espíritu de [18], para las categorías de módulos zócalo proyectivos bajo álgebras de incidencia de alguna clase de conjuntos parcialmente ordenados?
2. ¿Cómo describir combinatorialmente los módulos de Zavadskij finitamente generados bajo cualquier k-álgebra de dimensión finita?

3. ¿Qué tipo de sucesiones enteras resultan de enumerar particiones inducidas por las \( \tau \)-órbitas en el carcaj de Auslander-Reiten de una álgebra Dynkin?

4. ¿Cuáles son las propiedades algebraicas de las matrices base en el esquema de criptografía visual para imágenes a escala de grises y a color dado en \([54]\)?

En relación a la primera pregunta, damos una realización geométrica a la categoría de módulos zócalo-proyectivos finitamente generados bajo el álgebra de incidencia de conjuntos ordenados de tipo \( A \) en el espíritu de \([18]\). En particular, el carcaj de Auslander-Reiten de dicha categoría puede construirse usando un modelo geométrico basado en: una triangulación de un polígono regular, un conjunto especial de diagonales llamadas sp-diagonales y un tipo de movimientos entre sp-diagonales. Además, para cada conjunto ordenado de tipo \( A \), la categoría de sp-diagonales define una categorificación para cierta subálgebra de una álgebra de conglomerado, entonces nosotros estudiamos dichas subálgebras. En el segundo tema, usamos el mástil de un módulo uniserial indecomponible introducido por Zimmermann (ver \([13,93]\)) para dar una caracterización combinatorial a los módulos Zavadskij finitamente generados bajo una álgebra finitodimensional. Adicionalmente, damos fórmulas explícitas para el número de módulos Zavadskij indecomponibles bajo álgebras Dynkin. En el tercer punto definimos las particiones de \( \tau \)-orbita \( \lambda = (\lambda_1, \ldots, \lambda_n) \) de números enteros, donde las partes \( \lambda_i \) son definidas por las longitudes de las \( \tau \)-órbitas en el carcaj de Auslander-Reiten de una álgebra Dynkin; en tal situación, describimos y enumeramos esas particiones obteniendo las sucesiones enteras A016116 \([83]\) y A000034 \([84]\) en la On-Line Encyclopedia of Integer Sequences (En corto, OEIS). Finalmente, para responder la última pregunta, asociamos ciertos problemas matriciales al esquema dado en \([54]\), lo cual da una interpretación algebraica a las matrices base como representaciones matriciales de un retículo de colores.

**Keywords:** Representation theory of quivers, Representation theory of partially ordered sets, Auslander-Reiten theory, Matrix problem, Cluster algebra, Cluster category, Zavadskij modules, Partitions of integer numbers, Integer sequences, Visual cryptography.

**Palabras clave:** Teoría de representación de carcajes, Teoría de representación de conjuntos parcialmente ordenados, Teoría de Auslander-Reiten, Problema matricial, Álgebras de conglomerado, Categorías de conglomerado, Módulos Zavadskij, Particiones de números enteros, Sucesiones enteras, Criptografía visual.
Acceptation Note

Thesis Work

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Bogotá, D.C., February, 2020
Dedicated to

Paola, Pablo, Victoria, and Rachelle.
Acknowledgments

I owe my deepest gratitude to God for his infinite love and being with me at every step I take.

I am deeply grateful to my advisors Dr. Hernán Giraldo and Dr. Agustín Moreno Cañadas for giving me the honor to work with them on this thesis and letting me be a part of their research on the representation theory of algebras and applications. You both have imparted to me your wisdom, passion, skill, and dedication not only in mathematics but also in personal life.

I extend my sincere gratitude to my examiners, Dr. Ryan Kinser, Dr. Edson Ribeiro, and Dr. Omar Saldarriaga for their constructive comments and suggestions. Moreover, I would like to express my gratitude to Dr. Ralf Schiffler for the great opportunity to work with him not only during my visit to the University of Connecticut but also by email and video calls.

A special thanks to Dr. Daniel Simson, Dr. Wolfgang Rump, Dr. Axel Boldt, and Dr. Justyna Kosakowska for replying to my emails with questions about some of their works. Moreover, a special thanks to my academic brothers Veronica, Alejandra, Isaías, Pedro, and Gabriel, for your friendship and help along the way. Together we have shared fun moments at academy.

I owe my eternal gratitude to my wife Paola and my children Pablo, Victoria, and Rachelle, together we have achieved it. You have given me invaluable help from the beginning to the end of this work and, although we have had to sacrifice family time, all of you have remained brave and loving.

I want to thank my parents David and Ana for the prayers and for teaching me with their example of hard work. I also want to say thanks to my parents in law Lazaro and Gloria for their help and love.

Last but not least, I would like to offer my special thanks to the Pedagogical and Technological University of Colombia and to Colciencias for the financial support during my studies.
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Introduction

The present work is a contribution to the development of combinatorial representation theory of algebras. Specifically, we consider some topics around three of its techniques: quiver representations, poset representations, and Auslander-Reiten theory. Moreover, we describe some links with cluster algebras and visual cryptography.

The development of the theory of quiver representations was started by Gabriel in the early 1970s with the famous Gabriel’s theorem [35]. But, intuitively, quiver representations were used before to treat problems of linear algebra as the classification of tuples of subspaces of a prescribed vector space [39]. The study of finite-dimensional algebras is reduced to the study of bound quiver algebras. Thus, quiver representations play an important role in representation theory of finite-dimensional algebras; however, they also occur in Kac-Moody Lie algebras, quantum groups, Coxeter groups, geometric invariant theory, and most recently in cluster algebras (see [34, 65]).

The notion of a matrix representation of a poset $\mathcal{P}$ over an algebraically closed field $\mathbb{k}$ was also introduced in the 1970s by Nazarova and Roiter [58]. The aim of their investigations was to give a solution to the second Brauer-Thrall conjecture. Aside from matrix representations of a poset $\mathcal{P}$, the concept of representation of a poset $\mathcal{P}$ over $\mathbb{k}$ was suggested by Gabriel [36] in connection with the investigation of representations of quivers. The classification theorems and others important developments have been given using matrix problems and algorithms of differentiation or reduction (see [6, 8, 38, 52, 60, 61, 81, 90, 91]). In general, poset representations play an important role in the study of lattices over orders, in the classification of indecomposable lattices over some simple curve singularity and in the classification of abelian groups of finite rank. (see [1, 81]).

On the other hand, Auslander-Reiten theory was introduced by Auslander and Reiten [5] in 1975; their work deals with problems in the representation theory considered directly with module theoretic techniques. In addition to the classical module theory available, including homological methods, they introduced the notion of almost split sequences. Although they initially developed their ideas in the case of the category mod $A$ of finitely generated modules over an Artin algebra $A$, this theory has been extended to a lot of other categories including categories of representations of ordinary posets [6, 81, 90] and of posets with additional structures [9, 81].

V
Regarding representation theory of both posets and of quivers, this thesis deals with a range of objectives, among them: to give a geometric model for the category of socle-projective modules over the incidence algebra of some kind of posets using the category of diagonals given in [18], to describe combinatorially the Zavadskij modules over any finite-dimensional $k$-algebra, to determine what kind of integer sequences appear by counting $\tau$–orbit partitions of type $A$ and $D$, and finally, to give an algebraic interpretation of the base matrices, in the lattice-based visual secret sharing scheme for color and gray-scale images [54], using matrix representations of posets.

The main results of this research are:

- In Definition 2.1 we introduce a class of $r$–peak posets (posets with $r$ maximal points) which we call posets of type $A$. Those posets and their categories of socle-projective modules over the incidence algebra are described in Proposition 2.2 and Lemma 2.3.

- Theorem 2.9 gives a geometric realization to the category of socle-projective modules over the incidence algebra of a poset of type $A$ using diagonals of polygons.

- In section 2.5 we define an algebra associated to a poset $P$ of type $A$ which is a subalgebra $A(P)$ of a cluster algebra $A$. Thus, Theorem 2.14 states that if $P$ is the poset whose Hasse quiver is a Dynkin quiver $Q$ of type $A_n$, then the subalgebra $A(P)$ is the cluster algebra $A$.

- Theorems 3.5 and 3.7 characterize the Zavadsij modules over a finite-dimensional algebra $A$ using the mast associated to a uniserial module.

- Theorem 3.9 and corollaries 3.11 and 3.12 state formulas for the number of indecomposable Zavadskij $A$–modules over a Dynkin algebra.

- In Definition 4.1 we introduce $\tau$–orbit partitions of integer numbers. They are characterized in Lemma 4.2 and theorems 4.4 and 4.6.

- Via Theorem 4.3 and Corollary 4.5 we obtain the sequences A016116 [83] and A000034 [84] in the Online Encyclopedia of Integer Sequences counting $\tau$–orbit partitions of type $A_n$ and $D_n$ respectively.

- Algorithm 4.8 states a procedure to compute the length of $\tau$–orbit partitions of type $A_n$ using tiled orders.

- In section 5.3 theorems 5.1-5.3 interpret the visual cryptography schemes for color and gray-scale images given in [54] as certain matrix representations of color-lattices.

The results in Chapters 4 and 5 are joint work with my advisors Agustín Moreno Cañadas and Hernán Giraldo which were published in [21, 22]. Moreover, results described in Chapter 3 is a submitted joint work with Ralf Schiffler [73], whereas Chapter 2 is joint work (in progress) with Agustín Moreno Cañadas and Cesar Espinosa [23]. The author has socialized some of the results of those works in the following conferences:

- Second international colloquium on representations of algebras and its applications; Alexander Zavadskij, Universidad Nacional de Colombia, Bogotá Colombia, 2017.
Throughout this thesis, $k$ denotes an algebraically closed field; $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of the natural, integer, and rational numbers respectively and we always suppose that the modules are finitely generated.

Outline

This thesis consists of five chapters. In Chapter 1, we recall some definitions and well known facts in representation theory of Artin algebras which will be used in the work; particularly, it deals with quiver representations (Section 1.1), poset representations (Section 1.2 and 1.3), and Auslander-Reiten theory (Section 1.4).

In Chapter 2, we introduce the posets of type $A$ (Section 2.1). Moreover, we consider the category of diagonals introduced in [18] and its full subcategory given by the notion of sp-diagonals (Section 2.2). Then, we prove a categorical equivalence between the category of socle-projective representations of a poset of type $A$ and the category of sp-diagonals associated to the poset (Section 2.3). Finally, we recall the notion of cluster algebra (Section 2.4) and we give properties of the subalgebra of a cluster algebra defined by the sp-diagonals (Section 2.4).

In Chapter 3, we recall some facts about Zavadskij modules (Section 3.1) giving a combinatorial characterization of Zavadskij modules over finite-dimensional $k$-algebras (Section 3.2). Finally, we consider the number of indecomposable Zavadskij modules over Dynkin algebras (Section 3.3).

In Chapter 4, we recall a combinatorial construction of the Auslander-Reiten quiver of a Dynkin algebra (Section 4.1). We also describe and enumerate the $\tau$-orbit partitions (Section 4.2). Finally, we give an algorithm to calculate those partitions using tiled orders (Section 4.3).

In the last chapter, we give a short motivation to visual cryptography (Section 5.1). After that, we recall some definitions and concepts about visual cryptography schemes (Section 5.2). Finally, we define an algebraic approach of a visual cryptography scheme for color and gray-scale images using matrix representations of certain lattices (Section 5.3).
In this chapter we give a short introduction to three of the methods in the representation theory of Artin algebras: quiver representations, representations of partially ordered sets (in short, posets), and almost split sequences. We give some of the central theorems along the way following the exposition of [2,65,72,81,82].

An **Artin algebra** is a ring which is finitely generated as a module over a commutative Artin ring.

**Example 1.1.**
(a) Factor rings of polynomial rings like $k[x]/(x^n)$ and $k[x,y]/(x,y)^2$, where $x$ and $y$ are indeterminates.
(b) Group algebras $kG$ for a finite group $G$.
(c) Matrix algebras like the $k$-algebra $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ of matrices $[\lambda_{ij}] \in M_2 (k)$, with $\lambda_{12} = 0$ and the $k$-algebra $\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$ of matrices $[\lambda_{ij}] \in M_3 (k)$, with $\lambda_{32} = 0$ and $\lambda_{pq} = 0$ for $p < q$; where addition and multiplication are the ordinary operations on matrices.

Representation theory of Artin algebras $A$ deals with the study of the $A$-modules. Some work has been done on modules of infinite $k$-dimension, but most attention has been given to the category of finitely generated (right) modules, which we denote by $\text{mod} A$. For example, when $A$ is equal to $k$, $\text{mod} A$ is the category of finite-dimensional vector spaces over $k$.

A nonzero module $M$ is said to be **indecomposable** if it cannot be written as a direct sum of two nonzero submodules. For example, $k$ is the only (up to isomorphism) indecomposable $k$-module. A basic starting point is the theorem of Krull-Schmidt, which states the following.

**Theorem 1.1.** [2, Theorem 4.10] Let $A$ be an Artin algebra. Each module $M$ in $\text{mod} A$ can be written uniquely as a direct sum of indecomposable modules, up to isomorphism and order of summands.

A lot of investigations in the area have centered around **algebras of finite representation type**, that is, having only a finite number of isoclasses of indecomposable finitely generated modules; otherwise, they are called **algebras of infinite representation type**.
Example 1.2. The matrix $k$-algebra
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & k & 0 \\
0 & 0 & k
\end{pmatrix}
\]
of matrices $[\lambda_{ij}] \in M_3(k)$, with $\lambda_{32} = \lambda_{42} = \lambda_{43} = 0$ and $\lambda_{pq} = 0$ for $p < q$ is of finite representation type, whereas the $k$-algebra
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
k & 0 & 0 & 0 \\
0 & k & 0 & 0 \\
0 & 0 & k & 0 \\
0 & 0 & 0 & k
\end{pmatrix}
\]
of matrices $[\lambda_{ij}] \in M_5(k)$, with $\lambda_{32} = \lambda_{42} = \lambda_{43} = \lambda_{52} = \lambda_{53} = \lambda_{54} = 0$ and $\lambda_{pq} = 0$ for $p < q$ is of infinite representation type.

The Brauer-Thrall conjectures, which were first formulated for group algebras, have been important and influential problems in the area. Many developments were started in an effort to attack these conjectures, which are the following.

Brauer-Thrall I. If $A$ is a $k$-algebra of infinite type, there is no bound on the $k$-dimension of indecomposable finitely generated $A$-modules.

Brauer-Thrall II. If $A$ is a $k$-algebra of infinite type (over an infinite field $k$), then for an infinite number of dimensions there is an infinite number of indecomposable modules of this dimension.

The first conjecture was proved by Roiter in \cite{68}, and a proof for the second one was published by Bautista in \cite{7}.

Another type of problem has been to find classes of modules which are possible to classify or understand better for algebras of infinite representation type. Special attention has been given to the so-called tame algebras, where it is possible to classify all indecomposable modules. Formally, an algebra $A$ is tame or of tame representation type if $A$ is not of finite representation type, and if for each $d$ there exist finitely many $A - k[x]$-bimodules $M_1, \ldots, M_t$, which are free of finite rank as right $k[x]$-modules, such that (up to isomorphism) all but finitely many indecomposable $d$-dimensional $A$-modules are isomorphic to a module of the form $M_i \otimes_{k[x]} S$ with $S$ a simple $k[x]$-module. Recall that the simple $k[x]$-modules are of the form $S_\lambda = k[x]/(x - \lambda)$ with $\lambda \in k$, and $S_\lambda \cong S_{\lambda'}$ if and only if $\lambda = \lambda'$. In contrast, an algebra $A$ is called wild or of wild representation type if there exists a faithful exact $k$-linear functor $\text{mod } k\langle x, y \rangle \to \text{mod } A$ which respects isomorphism classes and indecomposables. In other words, the classification of the indecomposables in $\text{mod } A$ essentially contains the problem of classifying the modules of finite length over the free algebras $k\langle x, y \rangle$ with two generators. According to the above, we have the fundamental representation type result of Drozd \cite{32}.

Theorem 1.2. \cite{81} Theorem 14.14] Every finite-dimensional $k$-algebra $A$ over an algebraically closed field $k$ is representation-finite, representation-tame or representation-wild and these types are mutually exclusive.
1.1 Quiver representations

The use of quivers and their representations gives us the possibility to visualize very concretely the finite-dimensional modules of a given finite-dimensional \(k\)-algebra. In this section we recall main definitions and results regarding quiver representations, we refer the reader to [2,4,35,07,72] for a more detailed study of the development of the research.

**Definition 1.1.** A **quiver** is a directed graph where loops and multiple arrows between two vertices are allowed. Formally, a quiver \(Q = (Q_0, Q_1, s, t)\) is a quadruple consisting of a set of vertices \(Q_0\), a set of arrows \(Q_1\) between the vertices and two maps \(s, t : Q_1 \to Q_0\) which associates to each arrow \(\alpha \in Q_1\) its source \(s(\alpha) \in Q_0\) and its target \(t(\alpha) \in Q_0\), respectively.

Assume for simplicity that \(Q_0\) and \(Q_1\) are finite. For example, we have the following quiver

\[
\begin{array}{c}
\bullet_1 \\
\begin{array}{c}
\alpha \\
\downarrow \\
\beta \\
\end{array} \\
\begin{array}{c}
\bullet_2 & \leftrightarrow \bullet_3
\end{array}
\end{array}
\]

A vertex \(x \in Q_0\) is said to be a **sink vertex** (resp. **source vertex**) if there is no arrow \(\alpha\) in \(Q_1\) such that \(s(\alpha) = x\) (resp. \(t(\alpha) = x\)).

A **finite-dimensional representation** \(M = (M_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}\) of a quiver \(Q\) over \(k\) is a collection of finite-dimensional \(k\)-vector spaces \(M_x\) for each \(x \in Q_0\), and a collection of \(k\)-linear maps \(\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}\) one for each arrow \(\alpha \in Q_1\). The **dimension vector** \(\dim M\) of \(M\) is the vector \((\dim_k M_x)_{x \in Q_0}\) of the dimensions of the \(k\)-vector spaces and the **support** \(\supp M\) of \(M\) is the set of vertices \(x \in Q_0\) such that \(M_x \neq 0\). A **morphism** \(f : M \to M'\) between two representations \(M = (M_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}\) and \(M' = (M'_x, \varphi'_\alpha)_{x \in Q_0, \alpha \in Q_1}\) of \(Q\) is a collection of linear maps \(f_x : M_x \to M'_x\), one for each vertex \(x \in Q_0\) satisfying the identity \(\varphi'_\alpha f_x = f_y \varphi_\alpha\) for any arrow \(\alpha\) from \(x\) to \(y\).

We denote by \(\text{rep}(Q, k) = \text{rep} Q\) the category of finite-dimensional representations of \(Q\) over \(k\). This is an abelian category (see [2]).

There is a \(k\)-algebra defined for a quiver \(Q\) in the following way. A **path of length** \(l\) in \(Q\) is a sequence of arrows \((\alpha_1, \ldots, \alpha_l)\) of \(Q\), of length \(l\), such that \(s(\alpha_{i+1}) = t(\alpha_i)\). A path of length \(0\), from a vertex \(x\) to \(x\) is denoted by \(e_x\) and it is called **stationary path**. An arrow \(x \xrightarrow{\alpha} y\) is a path of length one and if \(x = y\) then \(\alpha\) is a **loop**. A path \((\alpha_1, \ldots, \alpha_l)\) of length \(l \geq 1\) such that \(s(\alpha_1) = t(\alpha_l)\) is called an **oriented cycle**. Thus a loop is an oriented cycle of length one. The **path algebra** \(kQ\) of \(Q\) is defined to be the \(k\)-algebra whose underlying \(k\)-vector space has as a basis, the set of all paths \((x|\alpha_1, \ldots, \alpha_l|y)\) of length \(\geq 0\). The product of two basis elements \((x|\alpha_1, \ldots, \alpha_l|y)\) and \((z|\beta_1, \ldots, \beta_m|w)\) of \(kQ\) is defined as,

\[
(x|\alpha_1, \alpha_2, \ldots, \alpha_l|y) \cdot (z|\beta_1, \beta_2, \ldots, \beta_m|w) = \delta_{yz}(x|\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m|w),
\]

where

\[
\delta_{yz} = \begin{cases} 
0, & \text{if } y \neq z, \\
1, & \text{if } y = z.
\end{cases}
\]
The path algebras \( kQ \) are **hereditary** \( k \)-algebras, that is, \( k \)-algebras such that the submodules of projective modules are projective. Now, the importance of quiver representations here is that they are closely connected with modules in the following sense.

**Theorem 1.3.** [2, Lemma 1.4 and Theorem 1.6] If \( Q \) has no oriented cycles, then

(a) \( \text{rep} \, Q \) is equivalent to \( \text{mod} \, kQ \).

(b) \( kQ \) is a finite-dimensional \( k \)-algebra.

**Example 1.3.** For an obvious generalization of the algebras in examples 1.1 (c) and 1.2, we assume that \( Q_n \) is the quiver

\[
\begin{array}{cccccc}
1 & 2 & \cdots & n-1 & n \\
& \downarrow & & \downarrow & \\
& 0 & & & \\
\end{array}
\]

then \( kQ_n \) is isomorphic to the \((n + 1) \times (n + 1)\) matrix algebra

\[
\begin{pmatrix}
k & 0 & 0 & \cdots & 0 \\
k & k & 0 & \cdots & 0 \\
k & 0 & k & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
k & 0 & 0 & \cdots & k \\
k & 0 & 0 & \cdots & 0 \\
\end{pmatrix}.
\]

Moreover, if \( Q \) is the loop \( \bullet \) then \( \text{rep} \, Q \) is equivalent to the category of \( k[x] \)-modules of finite length. Additionally, when \( Q \) is the quiver \( \bullet \quad \bullet \) then \( \text{rep} \, Q \) is equivalent to the category of \( k \langle x, y \rangle \)-modules of finite length [2].

When studying representation theory of hereditary algebras, it turns out that the quadratic form plays a prominent role. Given a quiver \( Q \) with \( n \) vertices, the **Tits quadratic form** associated to \( Q \) is defined to be the form

\[
q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{s(a)}x_{t(a)},
\]

where \( x = [x_1, \ldots, x_n]^t \in \mathbb{Z}^n \). Given \( x \in \mathbb{Z}^n \) such that \( q_Q(x) = 1 \) then \( x \) is called a **root** of \( q_Q \). As an example, all the vectors of the canonical basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{Z}^n \) are clearly roots of \( q_Q \).

**Classification theorems in the hereditary case.** Recall that the (Jacobson) radical \( \text{rad} \, A \) of a ring \( A \) is the intersection of all maximal right ideals in \( A \). According to [4], if the square of the radical of a \( k \)-algebra \( A \) is zero, there is an associated quiver \( Q \) such that \( \text{mod} \, A \) and \( \text{rep} \, Q \) are closely related, in particular \( \text{mod} \, A \) has a finite number of indecomposables if and only is \( \text{rep} \, Q \) does. The following important finite type criterion due to Gabriel [36] applies to the \( k \)-algebras which are hereditary.

**Theorem 1.4.** [2, Theorem 5.10] Let \( Q \) be a connected quiver. Then

(a) The \( k \)-algebra \( kQ \) is of finite representation type if and only if the underlying graph \( \overline{Q} \) of \( Q \) is one of the Dynkin diagrams
(b) If $kQ$ is of finite representation type, the map $M \mapsto \dim M$ induces a bijection from the isomorphism classes of indecomposable representations of $Q$ to the positive roots of the quadratic form $q_Q$.

(c) If $Q$ is the Dynkin diagram $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$ then the number of the isomorphism classes of indecomposable $kQ$-modules equals $n(n+1)/2$, $n^2 - n$, 36, 63, or 120, respectively.

When the underlying graph $Q$ of $Q$ is a Dynkin diagram of type $A$, $D$, or $E$, we say that $Q$ is a simply laced Dynkin quiver and that $kQ$ is a Dynkin algebra.

The above theorem gives a connection with other areas of mathematics via the appearance of the Dynkin diagrams. In that way, Bernstein-Gelfand-Ponomarev gave a proof of Gabriel’s Theorem, using the Weyl group associated with $Q$ (see [2]). This method has also been useful for the study of quivers of tame representation type, where a classification is in terms of extended Dynkin diagrams as first proved by Nazarova [59] and Donovan and Freislich [30].

**Theorem 1.5.** [81, Theorem 14.15] Let $Q$ be a connected quiver. The $k$-algebra $kQ$ is of tame representation type if and only if the underlying graph $Q$ of $Q$ is one of the extended Dynkin diagrams.
Bound quiver algebras. So far, we know that the study of quivers with no oriented cycles and their representations describe the study of modules over hereditary algebras and algebras $A$ with square radical zero. Now, it is possible to study more general classes of algebras in that way.

A bound quiver algebra is the quotient $kQ/I$ of a path algebra $kQ$ by an admissible ideal $I$, that is, a two-sided ideal $I$ of $kQ$ that satisfies $R_Q^m \subset I \subset R_Q^{2m}$ for some integer $m \geq 2$, where $R_Q$ is two-sided ideal generated by all arrows in $Q$ and for all $l \geq 1$, $R_l^Q$ denotes the ideal of $kQ$ generated, as a $k$-vector space, by the set of all paths of length $\geq l$. Also, the pair $(Q, I)$ is called a bound quiver.

Let $Q$ be a finite quiver and $M = (M_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$ be a representation of $Q$. For any nontrivial path $w = \alpha_1\alpha_2 \ldots \alpha_l$ from $x$ to $y$ in $Q$, the evaluation of $M$ on the path $w$ is the $k$-linear map from $M_x$ to $M_y$ defined by $\varphi_w = \varphi_\alpha_l \varphi_\alpha_{l-1} \ldots \varphi_\alpha_1$. The definition of evaluation extends to $k$-linear combinations of paths with a common source and a common target which are called relations. Let $\rho = \sum_{i=1}^m \lambda_i w_i$ be a relation, where $\lambda_i \in k$ and $w_i$ is a path in $Q$, for each $i$, then $\varphi_\rho = \sum_{i=1}^m \lambda_i \varphi_{w_i}$. The aforementioned relation $\rho$ is called a zero relation if $m = 1$ and it is a commutativity relation when it is of the form $w_1 - w_2$, where $w_1, w_2$ are two paths in $Q$.

Let $Q$ be a finite quiver and $I$ be an admissible ideal of $kQ$. A representation $M = (M_x, \varphi_\alpha)$ of $Q$ is said to be bound by $I$, if we have that $\varphi_\rho = 0$, for all relations $\rho \in I$. We denote by $\text{rep}(Q, I)$ the full subcategory of $\text{rep}Q$, consisting of all representations of $Q$ bounded by $I$.

**Theorem 1.6.** [2, Theorem 1.6] If $A$ is a finite-dimensional $k$-algebra then $\text{mod}A$ is equivalent to the category of representations $\text{rep}(Q_A, I_A)$ of an associated bound quiver $(Q_A, I_A)$, where $A$ and $kQ_A/I_A$ are isomorphic $k$-algebras.

Assuming that the relations involve no path of length one or zero, the associated quiver $Q_A$ is unique, and it is called the ordinary quiver of $A$. Moreover, since any admissible ideal of a path algebra is always generated by a finite set of relations, it is not always easy to decide when different sets of relations give isomorphic algebras.

**Example 1.4.** The category $\text{mod}A$ of the matrix $k$-algebra $A = \left( \begin{array}{ccc} k & 0 & 0 \\ k & k & 0 \\ k & 0 & k \\ k & k & k \end{array} \right)$ is equivalent to the full subcategory $\text{rep}(Q, I)$ of $\text{rep}Q$, consisting of the representations $(M_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$ of $Q$ equal to

![Diagram](https://via.placeholder.com/150)

satisfying $\varphi_\beta \varphi_\alpha = \varphi_\delta \varphi_\gamma$, where the admissible ideal $I$ is generated by the relation $\alpha \beta - \gamma \delta$.

Particularly, in the following two cases of bound quiver algebras there is a classification theorem for finite representation type:
(1) Algebras given by fully commutative quivers. A bound quiver \((Q, I)\) is fully commutative when there are no oriented cycles in \(Q\) and the admissible ideal \(I\) is generated by all the commutativity relations in \(Q\). This case was solved by Loupias [55] and Shkabara and Zavadskij [75].

(2) Algebras given by quivers which are trees, where the only possible relations are zero relations. Such a classification is due to Bongartz and Ringel [14].

1.2 Representations of ordinary posets

In this section we recall main definitions and results regarding the theory of representation of ordinary partially ordered sets, we refer the reader to [1, 36, 38, 58, 67, 81] for a more detailed study of the development of the research.

A partially ordered set (in short, poset) is a couple \((\mathcal{P}, \preceq)\) of a set \(\mathcal{P}\) and a binary relation \(\preceq\) contained in \(\mathcal{P} \times \mathcal{P}\), called the order on \(\mathcal{P}\), such that \(\preceq\) is reflexive, antisymmetric and transitive [74]. The elements of \(\mathcal{P}\) are called the points of \(\mathcal{P}\). We will write \(x < y\) for \(x \preceq y\) and \(x \neq y\), in this case we will say that \(x\) is strictly less than \(y\). An ordered set will be called finite (resp. infinite) if and only if the underlying set is finite (resp. infinite). Henceforth, we will assume that any poset \((\mathcal{P}, \preceq)\) or simply \(\mathcal{P}\) is finite.

Given an element \(a\) in a poset \(\mathcal{P}\), the sets \(a^\triangledown = \{x \in \mathcal{P} \mid a \preceq x\}\) and \(a_\Delta = \{x \in \mathcal{P} \mid x \preceq a\}\) are the ordinary up-cone and the ordinary down-cone associated to \(a\) respectively, whereas the sets \(a^\uparrow = a^\triangledown \setminus \{a\}\) and \(a_\uparrow = a_\Delta \setminus \{a\}\) are called truncated cones (up and down respectively) associated to \(a\). Moreover, if \(A \subset \mathcal{P}\), we define the subsets, \(A^\triangledown\) and \(A_\Delta\) such that \(A^\triangledown = \bigcup_{a \in A} a^\triangledown\) and \(A_\Delta = \bigcup_{a \in A} a_\Delta\).

A poset \(C\) is called a chain (or a totally ordered set or a linearly ordered set) if and only if for all \(x, y \in C\) we have \(x \preceq y\) or \(y \preceq x\), i.e. \(x\) and \(y\) are comparable points. On the other hand, an ordered set \(\mathcal{P}\) is called an antichain if and only if for all \(x \preceq y\) in \(\mathcal{P}\) we have \(x = y\) [27]. An antichain of \(n\) points is denoted by \(S_n\). The maximal cardinality of antichains in a poset \(\mathcal{P}\) is called the width \(w(\mathcal{P})\) of \(\mathcal{P}\). If some subsets \(X_1, \ldots, X_n\) of \(\mathcal{P}\) do not intersect mutually (but may have comparable points) then their union \(X_1 \uplus \cdots \uplus X_n\) is called a sum and is denoted by \(X_1 + \cdots + X_n\). Now, we recall the Dilworth’s theorem.

Theorem 1.7. [74] Theorem 2.5.7 Let \(\mathcal{P}\) be a poset. If the width \(w(\mathcal{P}) = n\) then \(\mathcal{P}\) is a sum of \(n\) chains.

An element \(x\) of a poset \(\mathcal{P}\) is said to be a maximal point (resp minimal point) of \(\mathcal{P}\) when there is no point \(y\) of \(\mathcal{P}\) with \(x < y\) (resp. \(y < x\)) in \(\mathcal{P}\). We denote by \(\max \mathcal{P}\) (resp. \(\min \mathcal{P}\)) the set of all maximal (resp. minimal) points of \(\mathcal{P}\). We say that \(y\) covers \(x\) if \(x < y\) but there is no element \(z\) such that \(x < z < y\). The Hasse diagram of \(\mathcal{P}\) is the graphical representation that represents each element of \(\mathcal{P}\) as a vertex in the plane and draws a line segment or curve that goes upward from \(x\) to \(y\) whenever \(y\) covers \(x\). These curves may cross each other but must not touch any vertices other than their endpoints. Such a diagram, with labeled vertices, uniquely determines its partial order.
Example 1.5. The Hasse diagram of the poset \( \mathcal{P} = \{1 \prec 3 \prec 4; 2 \prec 3 \prec 5\} \) is the following.

\[\begin{array}{c}
4 \\
3 \\
1 \\
\end{array}\begin{array}{c}
5 \\
2 \\
\end{array}\]

Sometimes we use the symbol ⋆ instead of o for the maximal points in \( \mathcal{P} \) (see Section 3).

Matrix representations. This notion was introduced by Roiter and Nazarova in 1972 \cite{58}. Let \( \mathcal{P} = \{x_1, \ldots, x_n\} \) be a finite poset with \( n \) elements. A matrix representation of \( \mathcal{P} \) over a field \( k \) is an arbitrary block matrix

\[
M = \begin{bmatrix}
    M_{x_1} & \ldots & M_{x_n}
\end{bmatrix}
\]

partitioned into \( n \) vertical blocks called strips, where for all \( x \in \mathcal{P} \), \( M_x \) is a \( (d_0 \times d_x) \)-matrix with entries in \( k \). Moreover, some of these matrices can be empty. In that case we are in fact dealing with a representation of a subposet.

The vector \( \text{cdim} \ M = (d_0, d_x \mid x \in \mathcal{P}) \), where \( d_0 \) is the number of rows of the matrix \( M \) and \( d_x \) is the number of columns of the matrix \( M_x \) for all \( x \in \mathcal{P} \), is called the coordinate vector of \( M \).

Recall that an elementary row transformation over a field \( k \) for an arbitrary matrix is one of the followings:

(i) Interchanging two rows.
(ii) Multiplying a row (on the left) by a nonzero scalar, i.e. an element of \( k \).
(iii) Replacing a row by itself plus a scalar multiple of another row.

Analogously there can be introduced the elementary column transformations. Here the scalar multiplications of columns are on the right. Let \( M \) be a matrix representation of a poset \( \mathcal{P} \). The admissible transformations for \( M \) are the following.

(1) Elementary row transformations on the whole matrix \( M \).
(2) Elementary column transformations within each vertical strip \( M_x \), \( x \in \mathcal{P} \).
(3) Additions of scalar multiples of columns of a strip \( M_x \) to columns of a strip \( M_y \) if \( x \prec y \) holds in \( \mathcal{P} \).

\footnote{Nazarova and Roiter wrote \cite{58}: The definition (of a matrix representation) presented appears to be totally absurd because the order relation on \( \mathcal{P} \) is totally unaccounted for. However the order relation will be taken into account when defining the similarity (or isomorphism) of representations. Thus, the classes of similar representations, which are the main objects of study in any theory, are already defined in an intelligent manner.}
Two matrix representations $M$ and $M'$ of $\mathcal{P}$ are isomorphic if and only if one of them can be obtained from the other by a sequence of admissible transformations. The direct sum of two matrix representations $M$ and $M'$ is a matrix representation $M \oplus M'$ which is equal to the block matrix

$$M \oplus M' = \begin{bmatrix} M_{x_1} & 0 & \cdots & 0 \\ 0 & M_{x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{x_n} \end{bmatrix}$$

Then we have the identity $\text{cdim}(M \oplus M') = \text{cdim} M + \text{cdim} M'$. A matrix $M$ is indecomposable if it is not the direct sum of two nonzero matrices. According to the above, the problem of classification (up to isomorphism) of the indecomposable matrix representations of a poset is a matrix problem.

Another definition of representations of a finite poset over a field $k$ was suggested by Gabriel [36] in the language of vector spaces. A representation (or filtered $k$-linear representation or $\mathcal{P}$-space) $U$ of a poset $\mathcal{P}$ is a system of finite-dimensional $k$-vector spaces of the form

$$U = (U_0, U_x \mid x \in \mathcal{P}) \quad (1.1)$$

where $U_0$ is a finite-dimensional $k$-space and $U_x$ is a subspace in $U_0$ for each $x \in \mathcal{P}$, such that $U_x \subseteq U_y$ provided that $x \preceq y$. A morphism $\varphi : U \rightarrow V$ between two representations $U$ and $V$ is a $k$-linear transformation $\varphi : U_0 \rightarrow V_0$ such that $\varphi(U_x) \subseteq V_x$, for each $x \in \mathcal{P}$. The morphism $\varphi$ is an isomorphism if $\varphi(U_x) = V_x$ for each $x \in \mathcal{P}$. The radical of a representation $U$ is the representation $\text{rad} U = (U_0, U_x \mid x \in \mathcal{P})$ where $U_x = \sum_{y < x} U_y$ is the radical subspace of $U_x$. The vector $\text{cdim} U = (d_0, d_x \mid x \in \mathcal{P})$, where $d_0 = \text{dim}_k U_0$ and $d_x = \text{dim}_k U_x/U_0$ is called the coordinate vector of the representation $U$.

A matrix representation $M$ of $\mathcal{P}$ corresponds to a representation $U$ of $\mathcal{P}$, if the columns of each strip $M_x$ are the coordinates (with respect a some basis fixed $B_0$ of $U_0$) of some system $B_x$ of generators of $U_x$ modulo the radical subspace $U_x$. Note that, the admissible transformations of $M$ arise from the change of basis $B_0$ and of the system $B_x$ when the number of generators in each $B_x$ remains invariant.

We have that $\text{cdim} M = \text{cdim} U$ if and only if the systems of generators $B_x$ for all $x \in \mathcal{P}$ are minimal (in other words, the matrix $M$ is reduced). Otherwise, we have $\text{cdim} M > \text{cdim} U$. Therefore, if the representations $U$ and $V$ correspond to the reduced matrix representations $M$ and $M'$ respectively then $M$ is isomorphic to $M'$ if and only if $U$ is isomorphic to $V$.

Let $\text{rep}\mathcal{P}$ denote the category of representations of $\mathcal{P}$.

The direct sum between two objects $U, V \in \text{rep}\mathcal{P}$ is an object $U \oplus V = W = (W_0; W_x \mid x \in \mathcal{P})$ such that $W_0 = U_0 \oplus V_0$ and $W_x = U_x \oplus V_x$, for any $x \in \mathcal{P}$. An object $U \in \text{rep}\mathcal{P}$ is said to be indecomposable provided that in a decomposition of the form $U = U_1 \oplus U_2$ either $U_1 = 0$ or $U_2 = 0$, otherwise $U$ is a decomposable representation. The category $\text{rep}\mathcal{P}$ is a non-abelian Krull-Schmidt category.

Let $\text{Ind}\mathcal{P}$ denote a complete set of isoclasses of indecomposable objects in $\text{rep}\mathcal{P}$. One of the main problems regarding poset representation is the classification of $\text{Ind}\mathcal{P}$ for any
poset $\mathcal{P}$. Clearly, describing the indecomposable representations of a poset is equivalent to describing its indecomposable matrix representations, excluding those formal matrices that do not have rows but have columns. If $|\text{Ind}\mathcal{P}| < \infty$ then the poset $\mathcal{P}$ is said to be of **finite representation type**, otherwise $\mathcal{P}$ is of **infinite representation type**.

Given a representation $U$ of a poset $\mathcal{P}$ over a field $k$ such that $\dim_k U_0 = 1$ then $U$ is a **trivial representation**. For instance, if $A \subset \mathcal{P}$ then $k(A)$ is the indecomposable representation of $\mathcal{P}$, where $U_0 = k$ and

$$
U_x = \begin{cases} 
k, & \text{if } x \in A^\vee, \\
0, & \text{otherwise}. 
\end{cases}
$$

In particular, the representation $k(\varnothing)$ has the field $k$ as the ground vector space $U_0$ and $U_x = 0$ for any point $x \in \mathcal{P}$. We write $k(a_1, \ldots, a_s)$ instead of $k(A)$ when $A = \{a_1, \ldots, a_s\}$.

**Example 1.6.** A complete list of the isoclasses of indecomposable representations of the poset given by three incomparable points $S_3 = \{1, 2, 3\}$ is as follows.

$$
\text{Ind } S_3 = \{k(\varnothing), k(1), k(2), k(3), k(1, 2), k(1, 3), k(2, 3), k(1, 2, 3), U\},
$$

where $U$ is isomorphic to $(k \oplus k \oplus 0 \oplus k, (1 + 1)k)$, with $(1 + 1)k = \{(\lambda, \lambda) : \lambda \in k\}$. On the other hand, a complete list of isoclasses of indecomposable matrix representations of $S_3$ is $M_{(1,0,0,0)}, M_{(0,1,0,0)}, M_{(0,0,1,0)}, M_{(0,0,0,1)}$, where $M_d$, $d \in \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is the unique formal matrix whose coordinate vector is $d = (d_0, d_1, d_2, d_3)$, that is, the matrix $M_{(1,0,0,0)}$ has one row and zero columns in each vertical strip, whereas the matrices $M_{(0,1,0,0)}, M_{(0,0,1,0)}, M_{(0,0,0,1)}$ do not have rows and for some $x \in S_3$ the vertical strip $M_x$ has just one column if $d_x = 1$. In the notation above, the number $x$ is on top of the vertical strip $M_x$ of each representation $M$; also, if $M_x$ is an empty matrix then the block $M_x$ does not appear in the matrix $M$.

**The main classification theorems.** Kleiner [52] found out the following finite type criterion using an algorithm of differentiation known as **differentiation with respect to a maximal point** (see [81]).

**Theorem 1.8.** [81, Theorem 10.1] A finite poset $\mathcal{P}$ is of finite representation type if and only if the poset $\mathcal{P}$ does not contain as a full subposet any of the following Kleiner’s critical posets

<table>
<thead>
<tr>
<th>$\mathcal{K}_1 = S_4$</th>
<th>$\mathcal{K}_2 = (2,2,2)$</th>
<th>$\mathcal{K}_3 = (1,3,3)$</th>
<th>$\mathcal{K}_4 = (N,4)$</th>
<th>$\mathcal{K}_5 = (1,2,5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="critical_posets.png" alt="Critical Posets" /></td>
<td><img src="critical_posets.png" alt="Critical Posets" /></td>
<td><img src="critical_posets.png" alt="Critical Posets" /></td>
<td><img src="critical_posets.png" alt="Critical Posets" /></td>
<td><img src="critical_posets.png" alt="Critical Posets" /></td>
</tr>
</tbody>
</table>

A $\mathcal{P}$-space $U$ is said to be a **sincere representation** if $d_x \neq 0$ for any $x \in \mathcal{P}$ and $\mathcal{P}$ is said to be a **sincere poset** if it has at least one indecomposable sincere representation.
Moreover, the set $\text{csupp } U = Q = \{x \in P : d_x \neq 0\}$ is said to be the **coordinate support** of the representation $U$ of $P$ and it is a sincere subposet of $P$, clearly any indecomposable representation of $P$ can be viewed as a sincere representation of a sincere subposet of $P$.

**Remark 1.1.** It is well known that the classification of isomorphism classes of indecomposable representations of posets of a given shape is reduced to the problem of determining which of them are sincere and its corresponding indecomposable sincere representations.

Kleiner [53] established the following result which gives the complete list of the sincere posets of finite representation type.

**Theorem 1.9.** [81, Theorem 10.2] A poset of finite representation type is sincere if and only if it has one of the forms

For a poset $P$ in Table 1.1, it holds that $P$ has a unique indecomposable sincere representation if and only if $P \in \{F_0, F_1, F_7, F_8, \ldots, F_{13}\}$. In this case, the values of the coordinate vector appear next to each point of the poset. In the remaining cases, there are at least two indecomposable sincere representations for each sincere poset (see [81, Table 10.20]).

**Example 1.7.** Assume that $P$ is as in Example 1.5. An object in the category $\text{rep } P$ consists of collections $U = (U_0; U_1, U_2, U_3, U_4, U_5)$ of vector spaces for which it is possible to associate the diagram

![Diagram](chart.png)

of embeddings. Since the width of $P$ is two, Theorem 1.8 implies that the category $\text{rep } P$ is of finite representation type. In this case, we have 8 isoclasses of indecomposable representations of $P$ which correspond to 8 sincere subposets of $P$. They are: 5 subposets of the form $F_1$, 2 subposets of the form $F_7$ and the poset $F_0$. Thus, the isoclasses of
indecomposable representations of $\mathcal{P}$ are: $k(1)$, $k(2)$, $k(3)$, $k(4)$, $k(5)$, $k(1, 2)$, $k(4, 5)$ and $k(\emptyset)$.

Recall that a series associated to a poset $\mathcal{P}$ of infinite representation type is a representation $U$ of $\mathcal{P}$ over the ring $k[t]$. For a fixed coordinate vector $d = (d_0; d_x \mid x \in \mathcal{P})$, $\mu(d)$ denotes the minimum number of series containing almost all indecomposable representations $U \in \text{rep}_k \mathcal{P}$ with $\text{cdim} U = d$. Then, a poset $\mathcal{P}$ is said to be tame or of tame representation type if and only if $\mu(d) < \infty$. Intuitively, a poset is wild or of wild representation type if and only if its matrix problem contains as a subproblem the known problem of simultaneous similarity of pairs of matrices over a field. According to the Drozd’s theorem (see [81, Theorem 15.2]) any poset is of finite, tame or wild representation type and these types are mutually exclusive.

Nazarova [60] used differentiation with respect to a maximal point to prove the following tameness criterion.

**Theorem 1.10.** [81, Theorem 15.3] A finite poset $\mathcal{P}$ of infinite representation type is tame if and only if $\mathcal{P}$ does not contain as a full subposet the following critical posets of Nazarova $N_1 = S_5$, $N_2 = (1, 1, 1, 2)$, $N_3 = (2, 2, 3)$, $N_4 = (1, 3, 4)$, $N_5 = (N, 5)$ and $N_6 = (1, 2, 6)$.

Here $(l_1, ..., l_m)$ is the cardinal sum of $m$ chains of cardinalities $l_1, ..., l_m$ and $(N, l)$ is the cardinal sum of a four-point set $N = \{a < b > c \prec d\}$ with a chain of cardinality $l$.

**Example 1.8.** The poset $S_3$ (the triad) is of finite representation type, $S_4$ (the tetrad) is a tame poset, whereas the poset $S_5$ is wild.

### 1.3 Socle-projective representations of posets

As a generalization to the representations of ordinary posets, the category of socle-projective representations of an $r$-peak poset was introduced by Simson in 1991 [80, 82]. He gave the finiteness criterion for those categories and his student J. Kosakowska classified the sincere posets and its sincere representations in the case of finite representation type [42–44]. Moreover, the tameness criterion was given by Kasjan and Simson in [49].

Let $\mathcal{P}$ be a finite poset, and let $\text{max} \mathcal{P}$ be the set of all maximal points of $\mathcal{P}$. A poset $\mathcal{P}$ is called an $r$-peak poset if $|\text{max} \mathcal{P}| = r$. We recall that a full subposet $\mathcal{P}'$ of $\mathcal{P}$ is said to be a peak-subposet if $\mathcal{P}' \cap \text{max} \mathcal{P} = \text{max} \mathcal{P}'$. In the sequel, we denote $\mathcal{P}^- = \mathcal{P} \setminus \text{max} \mathcal{P}$.

A peak $\mathcal{P}$-space or a socle-projective representation (in short, sp-representation) $U$ of $\mathcal{P}$ over a field $k$ is a system of finite-dimensional $k$-vector spaces $U = (U_x)_{x \in \mathcal{P}}$, satisfying the following conditions:
(a) For each $x \in \mathcal{P}$, $U_x$ is a $k$-subspace of the space

$$U^* = \bigoplus_{z \in \max \mathcal{P}} U_z.$$ 

(b) The inclusion $U_z \hookrightarrow U^*$ is defined as usual for each $z \in \max \mathcal{P}$.

(c) For each $x < y$ in $\mathcal{P}$ it holds that $\pi_y(U_x) \subseteq U_y$, where $\pi_y \in \text{End} \ U^*$ is the composition of the direct summand projection of $U^*$ on

$$U^*_y = \bigoplus_{y \leq z \in \max \mathcal{P}} U_z$$

with the natural embedding homomorphism $U^*_y \hookrightarrow U^*$.

(d) If $z \in \max \mathcal{P}$ and $x \notin z_\lambda$ then $\pi_z(U_x) = 0$.

A morphism $f : U \to V$ between two sp-representations $U$ and $V$ is a collection of $k$-linear maps $f = (f_z : U_z \to V_z)_{z \in \max \mathcal{P}}$ such that for all $x \in \mathcal{P}$

$$\left( \bigoplus_{z \in \max \mathcal{P}} f_z \right)(U_x) \subseteq V_x.$$

The category of sp-representations of $\mathcal{P}$ over $k$ is denoted by $\hat{\text{rep}} \mathcal{P}$. The coordinate vector of a sp-representation $U$ is the vector $d = \text{cdim} U = (d_x)_{x \in \mathcal{P}} \in \mathbb{Z}^\mathcal{P}$, where $d_x = \dim_k U_x$ for all $x \in \max \mathcal{P}$ and $d_x = \dim_k(U_x/U_x)$ with $U_x = \sum_{y \leq x} \pi_x(U_y)$ for all $x \in \mathcal{P}^-$.

**Example 1.9.** Assume that $\mathcal{P}$ is the poset in Example 1.5 where 4 and 5 are its maximal points. An object in the category $\hat{\text{rep}} \mathcal{P}$ consists of collections $U = (U_1, U_2, U_3, U_4, U_5)$ of vector spaces $U_x$, $x \in \mathcal{P}$, that satisfies the embedding system

$$U_1 \hookrightarrow U_3 \hookrightarrow U_4 \oplus U_5.$$ 

$$U_2 \hookrightarrow U_3 \hookrightarrow U_4 \oplus U_5.$$ 

Clearly, the maps $\pi_x : U_4 \oplus U_5 \longrightarrow U_4 \oplus U_5$ are such that $\pi_3$ is the identity, $\pi_4(u, v) = (u, 0)$ and $\pi_5(u, v) = (0, v)$. Note that every such a collection $U$ can be viewed as a system of linear maps of the form

$$U_1 \xrightarrow{\varphi} U_4 \xleftarrow{\psi} U_3 \xrightarrow{\psi} U_5,$$ 

$$U_2 \xrightarrow{\varphi} U_4 \xleftarrow{\psi} U_3 \xrightarrow{\psi} U_5,$$

where the left hand arrows represent embeddings, and $\varphi, \psi$ are $k$-linear maps such that $\ker \varphi \cap \ker \psi = 0$. Moreover, maps $\varphi$ and $\psi$ are compositions of the form $U_3 \hookrightarrow U_4 \oplus U_5 \longrightarrow U_4$ and $U_3 \hookrightarrow U_4 \oplus U_5 \longrightarrow U_5$ respectively. In this case, the category $\hat{\text{rep}} \mathcal{P}$ has an infinite number of indecomposable representations. To see this, we fix $m \geq 1$, $\lambda \in k$, and a
\(\kappa\)-linear endomorphism \(\varphi_\lambda : k^m \rightarrow k^m\) such that the matrix of \(\varphi_\lambda\) in the standard basis of \(k^m\) is the simple Jordan cell \(J(m, \lambda) \in M_m(k)\) with the eigenvalue \(\lambda\). Let us define the sp-representation \(U^{(m, \lambda)} = (U^{(m, \lambda)}_1, U^{(m, \lambda)}_2, U^{(m, \lambda)}_3, U^{(m, \lambda)}_4, U^{(m, \lambda)}_5)\) of \(\mathcal{P}\), where \(U^{(m, \lambda)}_4 = U^{(m, \lambda)}_5 = k^m\), \(U^{(m, \lambda)}_3 = k^m \oplus k^m\), \(U^{(m, \lambda)}_2 = \{(v, \varphi_\lambda(v)) \in U^{(m, \lambda)}_2 : v \in k^m\}\) and \(U^{(m, \lambda)}_1 = \{(v, v) \in U^{(m, \lambda)}_1 : v \in k^m\}\). The endomorphism algebra \(\text{End}(U^{(m, \lambda)})\) of \(U^{(m, \lambda)}\) is isomorphic to the local algebra \(k[t]/(t - \lambda)^m\) and therefore for every \(m \geq 1\) and \(\lambda \in \kappa\) the peak \(\mathcal{P}\)-space \(U^{(m, \lambda)}\) is indecomposable and \(\text{cdim} U^{(m, \lambda)} = (m, m, 0, m, m)\) which shows that the category \(\hat{\text{rep}} \mathcal{P}\) is of infinite representation type.

Although the definition of the categories \(\hat{\text{rep}} \mathcal{P}\) and \(\text{rep} \mathcal{P}\) seems to be very similar there are many differences between them; for instance, for the poset \(\mathcal{P}\) in Example 1.5 the category \(\text{rep} \mathcal{P}\) has only a finite number of nonisomorphic indecomposable \(\mathcal{P}\)-spaces (see Example 1.7) while the category \(\hat{\text{rep}} \mathcal{P}\) is of infinite representation type (see Example 1.9). However, representations of a ordinary poset \(\mathcal{P}\) in the sense of \([35, 58]\) coincide with the socle-projective representations of the poset \(\mathcal{P}^*\), obtained from \(\mathcal{P}\) by the addition of a unique maximal point \(*\), that is, \(\mathcal{P}^* = \mathcal{P} \cup \{\star\}\) with \(x \prec \star\) for all \(x \in \mathcal{P}\). This means that the categories \(\text{rep} \mathcal{P}\) are precisely the categories \(\hat{\text{rep}} \mathcal{P}\) where \(I\) is a one-peak poset.

Recall that the peak-subposet \(\text{csupp}(U) = \{x \in \mathcal{P} : (\text{cdim} U)_x \neq 0\}\) of \(\mathcal{P}\) is called coordinate support of \(U\). We say that an object \(U\) in \(\hat{\text{rep}} \mathcal{P}\) is a sincere sp-representation if it is indecomposable and \(\text{csupp}(U) = \mathcal{P}\). Additionally, \(\mathcal{P}\) is a sincere peak poset if there exists a sincere sp-representation of \(\mathcal{P}\).

Given a peak-subposet \(S\) of \(\mathcal{P}\), the subposet induced functor \(T_S : \hat{\text{rep}} S \rightarrow \hat{\text{rep}} \mathcal{P}\) assigns to the sp-representation \((U_x)_{x \in S}\) of \(S\) the sp-representation \((\hat{U}_x)_{x \in \mathcal{P}}\) of \(\mathcal{P}\), where \(\hat{U}_x\) is defined by

\[
\hat{U}_x = \begin{cases} U_x, & \text{if } x \in \max S, \\ 0, & \text{if } x \notin (\max S)_\triangle, \\ \sum_{y \leq x} \pi_x(U_y), & \text{if } x \in (\max S)_\triangle. \end{cases}
\]

Then, we have the following.

**Remark 1.2.** Up to isomorphism, any indecomposable object \(U\) in \(\hat{\text{rep}} \mathcal{P}\) is the image \(T_S(V)\) of a sincere sp-representation \(V\) of \(S\), where \(S\) is a sincere peak-subposet of \(\mathcal{P}\). As a consequence, \(S = \text{csupp}(U)\) (see \([81\text{, Proposition 5.14}]\)).

The classification of all sincere \(r\)-peak posets of finite representation type with \(r \geq 1\) was given by M. Kleiner (see Theorem 1.9), for the case \(r = 1\), and by J. Kosakowska \([42, 44]\), in the remaining cases. Moreover, the matrix sp-representations of peak posets and the associated matrix problem were introduced by Simson ([82 section 2]) and it generalizes the matrix problem induced by matrix representations of ordinary posets.

**Module theoretic approach.** Let \(\mathcal{P}\) be a finite poset. The incidence algebra \(k\mathcal{P}\) is a bound quiver algebra \(k\mathcal{Q}/I\) induced by the quiver \(\mathcal{Q}\) whose vertices are the points of \(\mathcal{P}\) and there is an arrow \(\alpha : x \rightarrow y\) for each pair \(x, y \in \mathcal{P}\) such that \(y\) covers \(x\). The ideal \(I\) is generated by all the commutativity relations \(\gamma - \gamma'\) with \(\gamma\) and \(\gamma'\) parallel paths in \(\mathcal{Q}\). We let \(\text{mod}(k\mathcal{P})\) denote the category of finitely generated right \(k\mathcal{P}\)-modules.
Recall that the **socle** of a module $M$ in $\text{mod}(kP)$ is defined to be the sum of all the simple submodules of $M$. In that way, a module $M$ in $\text{mod}(kP)$ is **socle-projective** if the socle of $M$ is projective. Moreover, $M$ is **prinjective** if $M$ has a projective resolution of the form

$$0 \to P' \to P \to M \to 0$$

in $\text{mod}(kP)$, where $P$ is projective and $P'$ is semisimple projective.

**Remark 1.3.** Each finitely generated right $kP$-module $M = (M_x, yh_x)$ for each $x, y \in P$ is identified with a collection of finite-dimensional $k$-vector spaces $M_x$, one for each point $x \in P$, and a collection of $k$-linear maps $yh_x : M_x \to M_y$, one for each relation $x \preceq y \in P$, satisfying the following condition [82].

(a) $h_x$ is the identity of $M_x$ for all $x \in P$ and $yhx = yhx$ for all $x \preceq y \preceq w$ in $P$.

In particular, $M = (M_x, yh_x)_{x,y \in P}$ is a socle-projective right $kP$-module if and only if additionally $M$ satisfies the following condition [82]:

(b) $\bigcap_{x \in \mathcal{P} \cap \max P} \ker h_x = 0$ for all $x \in \mathcal{P}^-$.

Let $M = (M_x, yh_x)_{x,y \in P}$ and $N = (N_x, yh'_x)_{x,y \in P}$ be two objects in $\text{mod}(kP)$. A **morphism** of $kP$-modules $f : M \to N$ is a collection $f = (f_x)_{x \in P}$ of linear maps $f_x : M_x \to N_x$ such that for each relation $x \preceq y$ in $P$ the diagram

$$
\begin{array}{ccc}
M_x & \xrightarrow{yhx} & M_y \\
| & f_x | & | f_y |\\
N_x & \xrightarrow{yh'_x} & N_y
\end{array}
$$

commutes, that is,

$$f_y \circ yhx = yh'_x \circ f_x.$$

Henceforth, $\text{mod}_{sp}(kP)$ denotes the category of socle-projective $kP$-modules, whereas $\text{prin}(kP)$ denotes the category of prinjective $kP$-modules.

The category $\hat{\text{rep}} \mathcal{P}$ is considered a full subcategory of $\text{mod}(kP)$ via the **embedding functor**

$$\rho : \hat{\text{rep}} \mathcal{P} \to \text{mod}(kP)$$

defined by $\rho(U) = (U_y, y\pi_x)_{x,y \in \mathcal{P}}$, where $y\pi_x : U_x \to U_y$ is the unique $k$-linear map making the diagram

$$
\begin{array}{ccc}
U_x & \xleftarrow{y\pi_x} & U^* \\
\downarrow & \downarrow & \downarrow \\
U_y & \xrightarrow{\pi_y} & U^*
\end{array}
$$
commutative. Moreover, the adjustment functor \[ \theta : \text{mod}(k\mathcal{P}) \rightarrow \hat{\text{rep}} \mathcal{P} \] (1.3)
is given by \( \theta(M_{x,y}h_{x}, y \in \mathcal{P}) = (U_{x})_{x \in \mathcal{P}} \), such that

\[ U_{x} = \begin{cases} M_{x} & \text{if } x \in \text{max } \mathcal{P}, \\ \text{Im}(f : M_{x} \rightarrow \bigoplus_{z \in \text{max } \mathcal{P}} M_{z}) & \text{otherwise.} \end{cases} \]

where \( f = (z_{f_{x}})_{z \in \text{max } \mathcal{P}} \) and

\[ z_{f_{x}} = \begin{cases} z_{h_{x}} & \text{if } x \prec z \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases} \]

Lemma 1.11. \[ 82 \text{ Lemma 2.1] Let } \mathcal{P} \text{ be a finite poset. The following statements holds.} \]

(a) The global dimension of the \( k \)-algebra \( k\mathcal{P} \) is finite.

(b) The embedding functor induces an equivalence of categories \( \rho : \hat{\text{rep}} \mathcal{P} \rightarrow \text{mod}_{\text{sp}}(k\mathcal{P}) \) and the inverse of \( \rho \) is the restriction of the adjustment functor \( \theta \) to the category \( \text{mod}_{\text{sp}}(k\mathcal{P}) \).

(c) The adjustment functor \( \theta \) induces a full dense additive adjustment functor

\[ \theta_{\mathcal{P}} : \text{prin}(k\mathcal{P}) \rightarrow \hat{\text{rep}} \mathcal{P} \]

such that \( \ker \theta_{\mathcal{P}} \) consist of all maps in \( \text{prin}(k\mathcal{P}) \) having a factorisation through a direct sum of copies of the projective \( k\mathcal{P} \)-modules. Furthermore, the functor \( \theta_{\mathcal{P}} \) preserves and reflects finite representation type, and induces an equivalence of categories

\[ \text{prin}(k\mathcal{P})/\ker \theta_{\mathcal{P}} \cong \hat{\text{rep}} \mathcal{P}. \]

The coordinate vector of a sp-representation in the category \( \hat{\text{rep}} \mathcal{P} \) is interpreted in the categories \( \text{mod}_{\text{sp}}(k\mathcal{P}) \) and \( \text{prin}(k\mathcal{P}) \) as the coordinate vector \( \text{cdim } M = (d_{x})_{x \in \mathcal{P}} \in \mathbb{Z}^{P} \) of a module \( M \) in \( \text{mod}(k\mathcal{P}) \) which is defined by the formula

\[ d_{x} = \begin{cases} \dim_{k} M_{x}, & \text{if } x \in \text{max } \mathcal{P}, \\ \dim_{k}(\text{top } M)e_{x}, & \text{otherwise.} \end{cases} \]

where, \( \text{top } M \) denotes the module \( M/\text{rad } M \) and \( \text{rad } M \) is the Jacobson radical of \( M \).

Finiteness criterion. The arguments described above allow us to enunciate the following finiteness criterion.

Theorem 1.12. \[ 82 \text{ Theorem 3.1] The following conditions are equivalent } \]

(a) The category \( \hat{\text{rep}} \mathcal{P} \) is of finite representation type.

(b) The category \( \text{prin}(k\mathcal{P}) \) is of finite representation type.

(c) The poset \( \mathcal{P} \) does not contain as a peak- subposet any of the posets presented in Appendix A.
Actually, there is a bijection between the isomorphism classes of indecomposable matrices of the matrix problem associated to the poset $\mathcal{P}$ and the isomorphism classes of indecomposable prinjective $k\mathcal{P}$-modules. Moreover, in the case of finite representation type $\hat{\text{rep}} \mathcal{P}$ and $\text{prin}(k\mathcal{P})$ have a description of its indecomposable objects by means of the positive roots of the Tits quadratic form $q_{\mathcal{P}}: \mathbb{Z}^{\mathcal{P}} \rightarrow \mathbb{Z}$ given by the formula

$$q_{\mathcal{P}}(x) = \sum_{i \in \mathcal{P}} x_i^2 + \sum_{\langle i,j \rangle \in \mathcal{P}} x_i x_j - \sum_{p \in \max \mathcal{P}} (\sum_{i \prec p} x_i) x_p.$$

**Corollary 1.13.** [82, Corollary 3.2] If the category $\hat{\text{rep}} \mathcal{P}$ is of finite representation type then:

(a) The map $U \mapsto \dim U$ establishes a bijection between the isomorphism classes of indecomposable objects in $\text{prin}(k\mathcal{P})$ and positive roots of the quadratic form $q_{\mathcal{P}}$. Furthermore, there is a bijection between the isomorphism classes of indecomposable objects in $\hat{\text{rep}} \mathcal{P}$ and the positive roots of the quadratic form $q_{\mathcal{P}}$ which are distinct from the standard basis vectors $e_x$, $x \in \mathcal{P}^-$ of $\mathbb{Z}^{\mathcal{P}}$.

(b) If $U$ is an indecomposable object in $\text{prin}(k\mathcal{P})$ then $\text{End}_{k\mathcal{P}}(U, U) = 0$ and $(\dim U)_x \leq 6$ for all $x \in \mathcal{P}$.

Thus, a sincere $k\mathcal{P}$-module $M$ is prinjective if and only if $M$ is socle-projective.

### 1.4 Auslander-Reiten theory

This section is a short introduction to the Auslander-Reiten quiver of three particular categories: the module category $\text{mod} A$, the category $\text{prin}(k\mathcal{P})$ of prinjective $k\mathcal{P}$-modules and the category $\hat{\text{rep}} \mathcal{P}$ of sp-representations of a poset $\mathcal{P}$ which can be identified with the category $\text{mod}_{sp}(k\mathcal{P})$ of socle-projective $k\mathcal{P}$-modules, where $k\mathcal{P}$ is the incidence algebra of $\mathcal{P}$.

The study of representations of one-peak posets (or more generally $l$-hereditary 1-Gorenstein algebras) in terms of irreducible morphisms and almost split sequences is due to Bautista and Martínez [6]. On the other hand, Auslander-Reiten theory for categories of socle-projective representations of peak posets and categories of prinjective modules was described by Simson and De la Peña (see [64, 81, 82]). It is well known that those categories are additive, closed under extensions, have Auslander-Reiten sequences, source maps and sink maps and have enough relative projective and relative injective objects; also, the category $\text{prin}(k\mathcal{P})$ is a hereditary subcategory of $\text{mod}(k\mathcal{P})$ in the sense that $\text{Ext}^2_{k\mathcal{P}}(M, N) = 0$ for all $M, N \in \text{prin}(k\mathcal{P})$ [64].

**Existence of almost split sequences.** In the sequel, $\mathcal{C}$ denotes either the category $\text{mod} A$ or $\text{prin}(k\mathcal{P})$ or $\hat{\text{rep}} \mathcal{P}$. Recall that a **short exact sequence** in $\mathcal{C}$ is a sequence of morphisms $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ such that $f$ is injective, $g$ is surjective and $\text{im} f = \text{kerg}$.

A morphism $X \xrightarrow{f} Y$ is called **section** if there exists a morphism $Y \xrightarrow{h} X$ such that $h \circ f = \text{id}_X$. A morphism $Y \xrightarrow{g} Z$ is called **retraction** if there exists a morphism $Z \xrightarrow{h} Y$. 
such that $g \circ h = \text{id}_Z$. We say that a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ splits if $f$ is section. It is well known that $f$ is section if and only if $g$ is retraction.

A nonsplit exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{C}$ is said to be almost split (or Auslander-Reiten sequence) if $X$ and $Z$ are indecomposable, and given any $W \xrightarrow{h} Z$ which is not an isomorphism, where $W$ is indecomposable, there is some $W \xrightarrow{s} Y$ such that $gs = h$.

Then, we are ready to give the theorem of existence of almost split sequences.

**Theorem 1.14.** [81, Theorem 11.27] and [64, Theorem 3.4] Let $Z$ be indecomposable nonprojective in $\mathcal{C}$ (or $X$ indecomposable noninjective). Then there is an almost split sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$, which is unique up to isomorphism.

**Auslander-Reiten quiver.** We recall a class of morphisms which are closely related to almost split sequences. Let $X$ and $Y$ be indecomposable objects in the category $\mathcal{C}$. A morphism $X \xrightarrow{f} Y$ is irreducible if $f$ is not an isomorphism, and whenever

$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{h} \\
Z & \xrightarrow{h} & Z
\end{array}$

is a commutative diagram, then either $g$ is a section or $h$ is a retraction.

Recall that $\text{rad}_C$ denotes the radical of the category $\mathcal{C}$. If $X$ and $Y$ are indecomposable modules in $\mathcal{C}$, then $\text{rad}_C(X, Y)$ is the $k$–vector space of all noninvertible morphisms from $X$ to $Y$ and the space $\text{rad}_C^2(X, Y)$ consists of all morphisms of the form $hg$, where $g \in \text{rad}_C(X, Z)$ and $h \in \text{rad}_C(Z, Y)$ for some (not necessarily indecomposable) object $Z$ in $\mathcal{C}$. It is clear that $\text{rad}_C^2(X, Y) \subseteq \text{rad}_C(X, Y)$. Now, we recall that the quotient space

$\text{Irr}(X, Y) = \text{rad}_C(X, Y)/\text{rad}_C^2(X, Y)$

measures the number of irreducible morphisms between indecomposable modules $X$ and $Y$.

**Lemma 1.15.** [2, Lemma 1.6] Let $X$, $Y$ be indecomposable objects in $\mathcal{C}$. A morphism $X \xrightarrow{f} Y$ is irreducible if and only if $f \in \text{rad}_C(X, Y) \setminus \text{rad}_C^2(X, Y)$.

Further, the relationship between irreducible morphisms and almost split sequences is as follows.

**Theorem 1.16.** [81, Corollary 11.12] If $X$ is indecomposable noninjective and $Y$ indecomposable in the category $\mathcal{C}$, then $X \xrightarrow{f} Y$ is irreducible if and only if there is an almost split sequence $0 \to X \to Y \oplus Y' \to Z \to 0$.

---

2I. Reiten wrote the following [65]: This theorem expresses a finiteness condition. The crucial part is the special property of the map $g : Y \to Z$, and the surprising fact is that it is possible to find some $Y$, which is a direct sum of a finite number of indecomposable modules, such that any other map to $Z$ factors through $Y$.
The information given by almost split sequences and irreducible maps is recorded and summarized in the **Auslander-Reiten quiver** (in short, AR-quiver) $\Gamma(\mathcal{C})$ of the category $\mathcal{C}$, that is, the quiver whose vertices correspond to the isomorphism classes of indecomposable modules $[X]$ in $\mathcal{C}$, and the arrows from $[X]$ to $[Y]$ are in bijective correspondence with the vectors of a basis of the $k$-vector space $\text{Irr}(X,Y)$. This quiver is equipped with maps $\tau$ and $\tau^-$, where $\tau [Z] = [X]$ if there is an almost split sequence of the form $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$, for the vertices $[Z]$ such that $Z$ is nonprojective in $\mathcal{C}$. In the same way, we can define $\tau^- [X] = [Z]$, for vertices $[X]$ such that $X$ is noninjective in $\mathcal{C}$. The maps $\tau$ and $\tau^-$ are called **Auslander-Reiten translations**. Thus, the AR-quiver is a translation quiver in the sense of [66]. For simplicity, sometimes we shall use $X$ instead $[X]$.

Recall that for each vertex $x$ in a bound quiver $(Q,I)$, the indecomposable projective (resp. injective) $kQ/I$-module $P_x$ (resp. $I_x$) is the $k$-vector space with basis the set of all paths $w$ starting (resp. ending) at $x$ that are not in $I$. A connected component $P$ of $\Gamma(\mathcal{C})$ is called **preprojective** (resp. **preinjective**) if $P$ is acyclic and, for any indecomposable object $M$ in $\mathcal{P}$ there exist $t \geq 0$ and an indecomposable projective (resp. injective) module $P_x$ (resp. $I_x$) in $\mathcal{C}$ such that $M = \tau^{-t} P_x$ (resp. $M = \tau^t I_x$).

Given a poset $\mathcal{P}$, we can build the preprojective component of the category $\text{rep}\mathcal{P}$ using the computational algorithm given in [81, Section 11.10].

**Example 1.10.** In Example [7], we describe the isoclasses of indecomposable representations in $\text{rep}\mathcal{P}$. Since this category is of finite representation type, the preprojective component is the Auslander-Reiten quiver $\Gamma(\text{rep}\mathcal{P})$ given by

In general, the algorithm used to build the preprojective component of the category $\text{prin}k\mathcal{P}$ is given in [50, Section 4]. Moreover, according to [82, Corollary 3.2], if the category $\text{prin}(k\mathcal{P})$ is of finite representation type, the Auslander-Reiten quiver $\Gamma(\hat{\text{rep}}\mathcal{P})$ and $\Gamma(\text{prin}(k\mathcal{P}))$ of the categories $\hat{\text{rep}}\mathcal{P}$ and $\text{prin}(k\mathcal{P})$ respectively coincide with their preprojective components and the translation quiver $\Gamma(\hat{\text{rep}}\mathcal{P})$ is obtained from $\Gamma(\text{prin}(k\mathcal{P}))$ by deleting the points $[e_i k\mathcal{P}^\perp]$ which correspond to the indecomposable projective $k\mathcal{P}^\perp$-modules.

**Example 1.11.** Let $\mathcal{P}$ be the three-peak poset

Note that, $\mathcal{P}$ can be viewed as a Dynkin quiver of type $E_7$. Thus, the Auslander-Reiten quiver $\Gamma(\text{mod}(k\mathcal{P}))$ of the module category $\text{mod}(k\mathcal{P})$ can be built using the knitting algorithm (see [72]) and it has the form
In the diagram, we have drawn with blue numbers the dimensions of indecomposable socle-projective modules and with red numbers the dimensions of the indecomposable projectives $\mathbb{kP}$-modules. Hence, the Auslander-Reiten quiver $\Gamma(\text{mod}_{sp}(\mathbb{kP}))$ of the category of finitely generated socle-projective modules $\text{mod}_{sp}(\mathbb{kP})$ has the form
A geometric realization of socle-projective representations of posets of type $\mathbb{A}$

Geometric realizations of algebraic structures using the combinatorial geometry of the surfaces have been developed by different authors in recent years (see [16, 18, 45, 48, 63, 71]). Also, this kind of work has been used for research combinatorics and theoretical aspects of the categories; essentially, it plays an important role in cluster-tilting theory and in representation theory in general. As an example, the category $\mathcal{C}$ of all diagonals (not including boundary edges) in a regular polygon introduced by Caldero-Chapoton-Schiffler [18] is a geometric realization of cluster categories of type $\mathbb{A}$; which, in greater generality, were simultaneously defined by Buan-Marsh-Reiten-Reineke-Todorov [17]. They defined cluster categories as orbit categories of the bounded derived category of hereditary algebras. As an application in [18], the module category of a cluster tilted algebra of type $\mathbb{A}_n$ can be described by a category of diagonals $\mathcal{C}_T$ in a regular polygon $\Pi_{n+3}$ with $n+3$ vertices, where $T$ is a triangulation of $\Pi_{n+3}$.

The present work links the theory of cluster algebras introduced by Fomin and Zelevinsky [33] and cluster categories with the theory of socle-projective representations of partially ordered sets through a geometric realization inspired by the one in [18].

In this chapter, we introduce a class of $r$-peak posets which we call posets of type $\mathbb{A}$. Roughly speaking, they are posets with $n$ elements, $n \geq 1$, whose category of socle-projective representations is embedded in the category of representations of a Dynkin quiver $Q$ of type $\mathbb{A}_n$. Then, we define a subcategory $\mathcal{C}_{(T,F)}$ of the category of diagonals of a triangulated polygon $\Pi_{n+3}$ with $n + 3$ vertices $\mathcal{C}_T$ to give a geometric realization of the category of sp-representations of posets of type $\mathbb{A}_n$, where $T$ is the triangulation of $\Pi_{n+3}$ associated to $Q$. Moreover, we define a subalgebra $\mathcal{A}(\mathcal{P})$ of the cluster algebra $\mathcal{A}_Q$, generated by the cluster variables associated to diagonals in $\mathcal{C}_{(T,F)}$ and diagonals in $T$; then, we establish that if $\mathcal{P}$ is the poset whose Hasse quiver is a Dynkin quiver $Q$ of type $\mathbb{A}_n$ then $\mathcal{A}_Q = \mathcal{A}(\mathcal{P})$. 

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2.1 Posets of type $A$

We introduce a family of posets which we call posets of type $A$ because of a characterization using a type $A$ Dynkin quiver given in Proposition 2.2.

**Definition 2.1.** A finite connected poset $P$ is said to be **poset of type $A$** if $P$ does not contain as peak-subposet any of the following posets:

\[
\begin{array}{c}
R_1 \quad \begin{tikzpicture}
  \node (1) at (0,0) [circle, draw, fill=black] {$\star$};
  \node (2) at (0,-1) [circle, draw, fill=white] {}; 
  \node (3) at (1,-1) [circle, draw, fill=white] {}; 
  \draw (1) edge (2) edge (3);
\end{tikzpicture} \\
R_2 \quad \begin{tikzpicture}
  \node (1) at (0,0) [circle, draw, fill=black] {$\star$};
  \node (2) at (0,-1) [circle, draw, fill=white] {}; 
  \node (3) at (0.5,-1) [circle, draw, fill=white] {}; 
  \draw (1) edge (2) edge (3);
\end{tikzpicture} \\
R_3 \quad \begin{tikzpicture}
  \node (1) at (0,0) [circle, draw, fill=black] {$\star$};
  \node (2) at (0,-1) [circle, draw, fill=white] {}; 
  \node (3) at (0.5,-1) [circle, draw, fill=white] {}; 
  \node (4) at (1,-1) [circle, draw, fill=white] {}; 
  \draw (1) edge (2) edge (3) edge (4);
\end{tikzpicture} \\
R_{4,n}, n \geq 0 \quad \begin{tikzpicture}
  \node (1) at (0,0) [circle, draw, fill=black] {$\star_1$};
  \node (2) at (0.5,0) [circle, draw, fill=black] {$\star_2$}; 
  \node (3) at (1,0) [circle, draw, fill=black] {$\star_3$}; 
  \node (4) at (1.5,0) [circle, draw, fill=black] {$\cdots$}; 
  \node (5) at (2,0) [circle, draw, fill=black] {$\star_{n+2}$}; 
  \node (6) at (0,-1) [circle, draw, fill=white] {}; 
  \node (7) at (0.5,-1) [circle, draw, fill=white] {}; 
  \node (8) at (1,-1) [circle, draw, fill=white] {}; 
  \node (9) at (1.5,-1) [circle, draw, fill=white] {}; 
  \node (10) at (2,-1) [circle, draw, fill=white] {}; 
  \draw (1) edge (6) edge (7) edge (8) edge (9) edge (10) 
  edge (2) edge (3) edge (4) edge (5);
\end{tikzpicture}
\end{array}
\]

Two maximal points $z$ and $z'$ in a poset $P$ are **neighbors** if $z_\Delta \cap z'_\Delta \neq \emptyset$.

**Lemma 2.1.** Let $P$ be an $r$-peak poset of type $A$ with $r \geq 2$. The following statements hold:

(a) The points $z, z' \in \text{max } P$ are neighbors if and only if $z_\Delta \cap z'_\Delta = \{x\}$, for some $x \in \text{min } P$.

(b) There exists a point $z \in \text{max } P$ such that $z$ has a unique neighbor.

**Proof.** Since $R_2$ is not peak-subposet of $P$ then $x \in z_\Delta \cap z'_\Delta$ implies that $x \in \text{min } P$ because otherwise there exists $y \prec x$ and then the subposet $\{y, x, z, z'\}$ is of the form $R_2$. Now, if $x \neq x' \in z_\Delta \cap z'_\Delta$ then $R_{4,0}$ is peak-subposet of $P$ which is a contradiction. Thus, the set $z_\Delta \cap z'_\Delta$ is a singleton. Clearly the converse implication is true. On other hand, since $P$ is a connected poset then each maximal point $z$ has at least one neighbor, but $z$ does not have three neighbor points. Indeed, if $z_1, z_2, z_3$ are distinct neighbors of $z$ with $x_i \in z_\Delta \cap (z_i)_\Delta$ then we have the subposet

\[
\begin{tikzpicture}
  \node (z1) at (0,0) [circle, draw, fill=black] {$z_1$};
  \node (z) at (0,1) [circle, draw, fill=black] {$z$};
  \node (z2) at (1,1) [circle, draw, fill=black] {$z_2$};
  \node (z3) at (2,1) [circle, draw, fill=black] {$z_3$};
  \draw (z1) -- (z) -- (z2) -- (z3);
\end{tikzpicture}
\]

If $x_1, x_2, x_3$ are three distinct points then $\{z, x_1, x_2, x_3\}$ is a peak-subposet of type $R_1$, a contradiction. Suppose that two of the $x_i$ are equal, for instance $x_1 = x_2$. Then $\{z_1, z, z_2, x_1\}$ is a peak-subposet of type $R_3$, a contradiction. Thus $z$ has at most two neighbors. Finally, if each maximal point has exactly two neighbor points then $R_{4,n}$ is peak-subposet of $P$ for some $n \geq 0$, which is contradictory, and we are done.

Actually, the posets of type $A$ can be viewed as posets associated to certain quivers which are obtained from Dynkin quivers of type $A$ by adding some new arrows. To explain this, we need the following definitions:

Let $Q$ be an acyclic quiver and let $P_Q = Q_0$ be its set of vertices. We define an order on $P_Q$ by $x \preceq y$ if and only if there exists a path from $x$ to $y$ in $Q$. We say that $P_Q$ is the **poset associated to the quiver $Q$**. Note that there is a unique poset associated
to a finite acyclic quiver, but the converse is false in general. As an example, the poset associated to the quiver

\[
\begin{array}{c}
1 \\
\uparrow \\
3 \quad 2
\end{array}
\]

is \(\{1 < 3 < 2\}\). However, the Hasse quiver of this poset is \(1 \rightarrow 3 \rightarrow 2\). Thus, the two quivers have the same associated poset. As another example, corresponding to the poset \(P = \{1, 2\}\) together with the usual ordering \(1 < 2\), we get countably many quivers with \(n\) arrows from 1 to 2 for any natural number \(n \in \mathbb{N}\).

Let \(Q\) be a Dynkin quiver of type \(A\) and let \(z \in Q_0\) be a sink vertex. The maximal full subquiver \(Q^{(z)}\) of \(Q\) with \(z\) as the unique sink vertex is called the \(z\)-subquiver of \(Q\). In other words, \(Q^{(z)}\) is the support of the indecomposable injective representation \(I_z\) at vertex \(z\).

**Example 2.1.** The quiver \(Q = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7\) of type \(A_7\) contains the 2-subquiver \(1 \rightarrow 2 \leftarrow 3\), the 5-subquer \(3 \rightarrow 4 \leftarrow 5 \leftarrow 6\), and the 7-subquiver \(6 \leftarrow 7\).

We will now add new arrows to our quiver \(Q\) as follows.

**Definition 2.2.** A set \(F = \{\alpha_1, \ldots, \alpha_t\}\) of new arrows for \(Q\) is called an alien set for \(Q\) if the following conditions hold.

(a) For each \(\alpha \in F\) there exists a sink vertex \(z\) in \(Q\) such that \(s(\alpha), t(\alpha) \in \text{Supp } I_z\).

(b) \(t(\alpha)\) is not a source vertex in \(Q\) unless it is an extremal vertex in \(Q\).

(c) For all \(\alpha \in F\), the arrow \(\alpha\) is the unique path from \(s(\alpha)\) to \(t(\alpha)\) in \(Q^F\), where \(Q^F\) is the quiver such that \(Q^F_0 = Q_0\) and \(Q^F_1 = Q_1 \cup F\).

(d) The quiver \(Q^F\) is acyclic.

The arrows in an alien set for \(Q\) will be called alien arrows.

**Example 2.2.** Let \(Q\) be the quiver in Example 2.1. Then \(F = \{\alpha, \beta\}\) be an alien set for \(Q\) given by the arrows \(\alpha : 3 \rightarrow 1\) and \(\beta : 6 \rightarrow 4\). Thus,

\[
\begin{array}{c}
1 \quad \alpha \quad 3 \\
\downarrow \quad \downarrow \quad \downarrow \\
2 \quad 4 \quad 6 \\
\beta \quad \downarrow \quad \downarrow \\
5 \quad 7
\end{array}
\]

is the acyclic quiver \(Q^F\). Moreover, the poset \(\mathcal{P}_{Q^F}\) associated to the quiver \(Q^F\) is the poset in Example 1.11

The following proposition characterizes posets of type \(A\).
Proposition 2.2. A poset \( \mathcal{P} \) is of type \( \mathbb{A} \) if and only if there exists a Dynkin quiver \( Q \) of type \( \mathbb{A} \) and an alien set \( F \) for \( Q \) such that \( \mathcal{P} = \mathcal{P}_Q^F \) is the poset associated to the quiver \( Q^F \).

Proof. In order to prove the necessary condition we proceed by induction on the number \( r \) of peaks in \( \mathcal{P} \). First we suppose that \( \mathcal{P} \) is a one-peak poset with a maximal point \( z \). Since \( R_1 \) is not peak-subposet of \( \mathcal{P} \) we conclude that \( w(\mathcal{P}) \leq 2 \). Thus, if \( w(\mathcal{P}) = 1 \) then \( \mathcal{P} \) is a chain and it can be viewed as a linearly oriented quiver \( Q \) of type \( \mathbb{A} \). Clearly, if \( F = \emptyset \) then \( \mathcal{P} \) is the poset associated to the quiver \( Q^F \). On the other hand, if \( w(\mathcal{P}) = 2 \) then by Dilworth's theorem \( \mathcal{P}^- \) is a sum of two chains \( \mathcal{P}_1 = \{x_1 < \cdots < x_s\} \) and \( \mathcal{P}_2 = \{y_1 < \cdots < y_t\} \). Given the quiver

\[
Q = x_1 \rightarrow \cdots \rightarrow x_s \rightarrow z \leftarrow y_t \leftarrow \cdots \leftarrow y_1,
\]

the set \( F = F_1 \cup F_2 \) such that \( F_1 = \{\alpha : x \to y \mid y \in \mathcal{P}_2 \text{ covers } x \in \mathcal{P}_1\} \) and \( F_2 = \{\alpha : y \to x \mid x \in \mathcal{P}_1 \text{ covers } y \in \mathcal{P}_2\} \) is an alien set for \( Q \). Let \( \alpha : x \to y \) be an alien arrow in \( F \). We suppose that there is another path from \( x \) to \( y \) in \( Q^F \), then there exists an alien arrow \( \alpha' : x' \to y' \) in \( Q^F \) such that \( x \preceq x' \), \( y' \preceq y \) and \( x \neq x' \) or \( y \neq y' \). However, in this case, \( y \) does not cover \( x \) which is a contradiction. Thus, \( F \) is an alien set for \( Q \) and \( \mathcal{P} \) is the poset \( \mathcal{P}_Q^F \) associated to the quiver \( Q^F \).

Now, we suppose that the assertion is true for any \( h \)-peak poset of type \( \mathbb{A} \), for all \( 1 \leq h \leq r - 1 \). Let \( \mathcal{P} \) be a \( r \)-peak poset of type \( \mathbb{A} \). By Lemma 2.1 part (b) we can choose a point \( z \in \max \mathcal{P} \) such that \( z \) has a unique neighbor. The peak-subposets \( \mathcal{P} = \{z_1, \ldots, z_{r-1}\} \) and \( \mathcal{P}_z = z \) of \( \mathcal{P} \) are two posets of type \( \mathbb{A} \), where \( \max \mathcal{P} = \{z_1, \ldots, z_{r-1}, z\} \). By induction there are two Dynkin quivers \( Q' \) and \( Q'' \) of type \( \mathbb{A} \) and two alien sets \( F' \) and \( F'' \) for \( Q' \) and \( Q'' \) respectively such that \( \mathcal{P} \) is the poset associated to the quiver \( Q'^{F'} \) and \( \mathcal{P}_z \) is the poset associated to the quiver \( Q''^{F''} \). We suppose that \( z' \in (\max \mathcal{P}) \setminus \{z\} \) is the neighbor of the point \( z \). By Lemma 2.1 part (a) we conclude that \( z_0 \cap z_0' = \{x\} \), where \( x \in \min \mathcal{P} \), in other words, \( x \) is source vertex in \( Q' \) and \( Q'' \). Clearly \( \bar{\mathcal{P}} \cap \bar{\mathcal{P}}_z = \{x\} \), otherwise \( z \) would have two neighbors. Now we are going to prove that the point \( x \) is an extremal vertex of both quivers \( Q' \) and \( Q'' \). Since \( Q'' \) has a unique sink vertex \( z \) and \( x \) is a source vertex in \( Q'' \) then \( x \) is an extremal vertex in \( Q'' \). Moreover, if \( x \) is a source vertex which is not an extremal vertex in \( Q' \) then \( R_3 \) would be a peak-subposet of \( \mathcal{P} \) and in this way we get a contradiction. Then the quiver \( Q = (Q_0, Q_1) \) such that \( Q_0 = Q_0' \cup Q_0'' \) and \( Q_1 = Q_1' \cup Q_1'' \) is a Dynkin quiver of type \( \mathbb{A} \). Note also that \( F = F' \cup F'' \) is an alien set for \( Q \) because there is no alien arrow ending at \( x \), otherwise \( R_2 \) would be a peak-subposet of \( \mathcal{P} \). Furthermore, \( \mathcal{P} \) is the poset associated to the quiver \( Q^F \).

The sufficiency of the assertion is proved as follows; let us suppose that \( \mathcal{P} \) is the poset \( \mathcal{P}_Q^F \) associated to a quiver \( Q^F \), where \( Q \) is a Dynkin quiver of type \( \mathbb{A} \) and \( F \) is an alien set for \( Q \), we shall prove that \( \mathcal{P} \) is of type \( \mathbb{A} \). Locally an alien arrow \( \alpha \in F \) with \( s(\alpha), t(\alpha) \in \text{Supp} \ I_z \), where \( z \) is a sink vertex in \( Q \) is such that \( s(\alpha) \neq z \), otherwise the quiver \( Q^F \) would be cyclic. Then the maximal points in \( \mathcal{P} \) are exactly the sink vertices in the quiver \( Q \). Since the subposet \( z_0 = Q_0^{(z)} \) of \( \mathcal{P} \) is a poset of width at most two then \( R_1 \) is not a peak-subposet of \( \mathcal{P} \). On the other hand, by Lemma 2.1 part (b), if \( z, z' \in \max \mathcal{P} \) are neighbors and \( x \in z_0 \cap z_0' \) then \( x \in \min \mathcal{P} \), thus \( x \in \min \mathcal{P}_Q \). Since \( Q \) is a Dynkin quiver of type
A, then $x$ is a source vertex in $Q$. However, $x$ is a non extremal vertex in $Q$ because an alien arrow always connects two vertices in the same $z$-subquiver. Definition 2.2 part (b) implies that $P$ does not contain $R_2$ as peak-subposet. Now, we suppose that $P$ contains $R_3$ as peak-subposet, that is, there are three maximal points $z, z', z''$ in $P$ and a point $x \in P$ such that $x \in z, z' \cap z''$. Thus, by the same arguments as above, $z, z'$ and $z''$ are sink vertices in $Q$. Moreover, since $R_2$ is not a peak-subposet of $P$, then $x$ is a minimal point in $P$ which implies that $x$ is a source vertex in the quiver $Q$. Moreover, since $Q$ is a Dynkin quiver of type $A$, we can suppose that there is no path in $Q$ from $x$ to $z''$; thus, Definition 2.2 implies that $x \not\in z''$ in $P$, a contradiction. These arguments allow us to conclude that $R_3$ is not peak-subposet of $P$. In the same way, we can see that for all $n \geq 0$, $R_{4,n}$ is not a peak-subposet of $P$. 

A poset $P$ is said to be locally of width $n$ or have local width $n$ if $n$ is the minimum integer such that for each $z \in \text{max} P$ it holds that $w(z) \leq n$. Clearly a poset of type $A$ has local width less than or equal to two. The following lemma describes sincere posets of type $A$ and their socle-projective indecomposable modules.

**Lemma 2.3.** Let $P$ be a poset of type $A$. The following statements hold:

(a) $\text{mod}_{\mathcal{P}} kP$ is of finite representation type.

(b) $P$ is a sincere poset if and only if $P$ is isomorphic to one of the following posets:

\[
\begin{array}{ccc}
\mathcal{S}_1^{(r)} & \mathcal{S}_2^{(r)} & \mathcal{S}_3^{(r)} \\
\star_1 & \star_2 & \star_3 & \cdots & \star_r \\
\odot & \odot & \odot & \cdots & \odot \\
\end{array}
\]

for some $r \geq 1$. Moreover, the socle-projective $kP$-module $M = (M_x, y h_x)_{x,y \in P}$ such that $M_x = k$ for all $x \in P$ and $y h_x = \text{id}_k$ for each $x < y$ in $P$ is the unique sincere indecomposable object in $\text{mod}_{\mathcal{P}}(kP)$.

**Proof.** In order to prove (a) we observe that no minimal infinite prinjective poset listed in Appendix A is a peak-subposet of $P$. Indeed, the posets of the series $P_{2,n+1}, P_{2,n}^\prime, P_{3,n}, n \geq 0$ and the poset $P_{2,0}$ contain $R_3$ as peak-subposet. Moreover, the posets of the series $P'_{2,n+1}, P_{3,n}^\prime, n \geq 0$ contain $R_1$ as peak-subposet and the posets of the series $P'_{3,n}, n \geq 0$ contain $R_2$ as peak-subposet. Note that by definition $P$ does not contain as a peak-subposet a poset of the series $P_{1,n}, n \geq 0$. Moreover, we note that any poset of the form $\{P_1, \ldots, P_{110}\}$ contains as peak-subposet to $R_i$ for some $i = 1, 2, 3$. Thus, no minimal poset of infinite type is a peak-subposet of $P$ then Theorem 1.12 allows to conclude the desired result.

In order to prove (b), first we consider that $P$ is one-peak poset. In this case, according to the list of sincere one peak-posets (see [53]) we have that $P = \mathcal{S}_i^{(r)}$ for some $i = 1, 2, 3$. Moreover, we observe in the known lists of sincere $r$-peak posets of finite prinjective type that the posets $\mathcal{F}_1^{(2)} = \mathcal{S}_1^{(2)}, \mathcal{F}_2^{(2)} = \mathcal{S}_2^{(2)}, \mathcal{F}_3^{(2)} = \mathcal{S}_3^{(2)}$ are the sincere two-peak posets of type $A$ (see [42]), the posets $\mathcal{F}_4^{(3)} = \mathcal{S}_1^{(3)}, \mathcal{F}_5^{(3)} = \mathcal{S}_2^{(3)}, \mathcal{F}_6^{(3)} = \mathcal{S}_3^{(3)}$ are the sincere three-peak posets of type $A$ (see [43]) and the posets $\mathcal{F}_8^{(r)} = \mathcal{S}_1^{(3)}, \mathcal{F}_9^{(r)} = \mathcal{S}_2^{(3)}, \mathcal{F}_{10}^{(r)} = \mathcal{S}_3^{(3)}$ are the sincere $r$-peak posets of type $A$, with $r \geq 4$ (see [44]). Thus, the first part of (b) is true.
On the other hand, we observe in the mentioned lists that for each \(i = 1, 2, 3\) and for each \(r \geq 1\) the sincere \(r\)-peak poset \(S^{(r)}_i\) has only one sincere prinjective indecomposable \(\mathbb{k}S^{(r)}_i\)-module \(M = (M_x, yh_x)_{x,y \in S^{(r)}_i}\) such that \(M_x = \mathbb{k}\) and \(yh_x = \text{id}_\mathbb{k}\) for each \(x \preceq y\). In this way, Lemma 1.11 implies that the second part of (b) is true. \(\square\)

The following lemma will be used to prove the categorical equivalence proposed in Theorem 2.9.

**Lemma 2.4.** Let \(\mathcal{P}\) be a poset of type \(\Delta\) associated to the quiver \(Q^F\) as in Proposition 2.2. Then

(a) Up to isomorphism, any indecomposable \(\mathbb{k}\mathcal{P}\)-module \(M = (M_x, yh_x)_{x,y \in \mathcal{P}}\) in \(\text{mod}_{\mathcal{P}}(\mathbb{k}\mathcal{P})\) is such that \(M_x = \mathbb{k}\) for all \(x \in \mathcal{P}\) and \(yh_x = \text{id}_\mathbb{k}\) for all \(x \preceq y\) in \(\text{supp} M\).

(b) The support \(\text{supp} M\) of an indecomposable object in the category \(\text{mod}_{\mathcal{P}}\mathbb{k}\mathcal{P}\) is connected as a subset of the quiver \(Q\).

**Proof.** Let \(M = (M_x, yh_x)_{x,y \in \mathcal{P}}\) be an indecomposable object in \(\text{mod}_{\mathcal{P}}(\mathbb{k}\mathcal{P})\). Then the image \(\theta(M) = (\theta(M)_x)_{x \in \mathcal{P}}\) of \(M\) by the adjustment functor \(\theta\) defined in Equation (1.3) is an indecomposable object in \(\text{rep} \mathcal{P}\). Let \(S = \text{cupp}(\theta(M))\) be the coordinate support of \(\theta(M)\). Then the poset \(S\) is a peak-subposet of \(\mathcal{P}\). Thus, Definition 2.1 implies that \(S\) is a poset of type \(\Delta\). Indeed, if \(\mathcal{R} \in \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5\}\) is peak-subposet of \(\mathcal{S}\) then \(\mathcal{R}\) is peak-subposet of \(\mathcal{P}\), a contradiction. Also, \(S\) is a sincere poset because the peak \(S\)-space (\(\theta(M)_x)_{x \in S}\) is sincere. Lemma 2.3 part (b) implies that for some \(r \geq 1\) and \(i = 1, 2, 3\), we have \(S = S_i^{(r)}\). Moreover, by Proposition 1.2 we have that \(\theta(M)\) is isomorphic to \(T_S(L)\) where \(L = (L_x)_{x \in S}\) is a sincere peak \(S\)-space in \(S\)-repr and \(T_S\) is the subposet induced functor. Lemma 1.11 implies that \(\rho(L) = N = (N_x, yg_x)_{x,y \in S}\) is a sincere module in \(\text{mod}_{\mathcal{P}}\mathbb{k}S\), and Lemma 2.3 then proves that \(N_x = \mathbb{k}\) for all \(x \in S\) and \(yg_x = \text{id}_\mathbb{k}\) for each \(x \preceq y\) in \(S\). Hence, Lemma 1.11 implies that \(L = \theta(N)\). Now, by definition of \(\theta\) we have that \(L_x = \mathbb{k}\) for all \(z \in \text{max} S\). Let \(\{e_z \mid z \in \text{max} S\}\) be the standard basis of the space \(L^*\), then for all \(x \in S^\prec\) we have that \(L_x\) is the subspace of \(L^*\) generated by the vector \(w_x = \sum_{z \succ x} e_z\). Moreover, let \(\hat{L} = (L_x)_{x \in \mathcal{P}} = T_S(L)\) be the image of \(L\) by the functor \(T_S\) then

\[
\hat{L}_x = \begin{cases} 
\langle w_x \rangle, & \text{if } x \in (\text{max} S)_{\Delta} \cap S^\prec, \\
0, & \text{otherwise}.
\end{cases}
\]

Thus, the image of \(\hat{L}\) by the functor \(\rho\) is the \(\mathbb{k}\mathcal{P}\)-module \(M = M_\hat{L} = (\hat{L}_x, y\pi_x)_{x,y \in \mathcal{P}}\) where \(y\pi_x : \hat{L}_x \rightarrow \hat{L}_y\) is such that \(\lambda w_x \mapsto \lambda w_y\) if \(x \preceq y\) in \((\text{max} S)_{\Delta} \cap S^\prec\) and \(y\pi_x = 0\) if \(x \preceq y\) and either \(x\) or \(y\) is not in \((\text{max} S)_{\Delta} \cap S^\prec\). It is easy to see that there is a natural isomorphism between \(M_\hat{L}\) and the representation described in (a).

To prove (b) it is enough to see that the set \((\text{max} S)_{\Delta} \cap S^\prec\) is connected as a subset of the quiver \(Q\) for any sincere peak-subposet \(S\) of \(\mathcal{P}\). Note that the poset \(S_i^{(r)}\) is a peak-subposet of \(S = S_i^{(r)}\) for all \(i = 1, 2, 3\). We suppose that

\[
S_i^{(r)} = \{z_1 \succ x_2 \prec z_2, z_2 \succ x_3 \prec z_3, \ldots, z_{r-1} \succ x_r \prec z_r\}
\]
then \( \{z_1, \ldots, z_r\} \subseteq \max P \) and since \( R_2 \not\subseteq P \) we have that \( \{x_2, \ldots, x_r\} \subseteq \min P \). Thus, for each \( z_i \), with \( 2 \leq i \leq r - 1 \) the \( z_i\)-subquiver \( Q^{(z_i)} \) of \( Q \) has the form \( x_i \longrightarrow \cdots \longrightarrow z_i \longrightarrow \cdots \longrightarrow x_{i+1} \). Since each vertex in \( Q^{(z_i)} \) belongs to the set \( \{z_i\}_{\triangle} \cap \{x_i, x_{i+1}\} \), then \( Q^{(z_i)} \) \( \subset \) supp \( M \) for each \( 2 \leq i \leq r - 1 \). Let \( w \) (respectively \( w' \)) be the left (respectively right) extremal vertex of the quiver associated to \( S \) and let \( x \) (respectively \( y \)) be minimal element in \( (x_2^T \cap (P \setminus S)) \cup \{w\} \) (respectively \( (x_2^T \cap (P \setminus S)) \cup \{w'\} \)) then it is easy to see that \( \text{supp } M = [x, y]_Q \), where \( [x, y]_Q \) denote a interval of \( Q \) which is a connected subset of \( Q \). \qed

2.2 Category of diagonals \( \mathcal{C}_T \)

We recall some results and notation of \([18]\) (see also Chapter 3 in \([72]\)) which are used in this work. A diagonal in a regular polygon is a straight line segment that joins two of the vertices and goes through the interior of the polygon. A triangulation of the polygon is a maximal set of noncrossing diagonals. Such a triangulation cuts the polygon into triangles.

Let \( T = \{\tau_1, \ldots, \tau_n\} \) be a triangulation of a regular polygon with \( n + 3 \) vertices (or \((n + 3)-gon\)) \( \Pi_{n+3} \) and let \( \gamma \) and \( \gamma' \) be diagonals which are not in \( T \). The diagonal \( \gamma \) is related to the diagonal \( \gamma' \) by a pivoting elementary move if they share a vertex on the boundary (this vertex is called pivot), the other vertices of \( \gamma \) and \( \gamma' \) are the vertices of a boundary edge of the polygon and the rotation around the pivot is positive (for the trigonometric direction) from \( \gamma \) to \( \gamma' \). Let \( P_\gamma : \gamma \longrightarrow \gamma' \) denote the pivoting elementary move from \( \gamma \) to \( \gamma' \) with pivot \( v \). Compositions of pivoting elementary moves are called pivoting paths.

The combinatorial \( k \)-linear additive category \( \mathcal{C}_T \) of diagonals is defined as follows: The objects are positive integral linear combinations of diagonals which are not in \( T \). By additivity, it is enough define morphisms between diagonals. To do that, we recall that the mesh relation is the equivalence relation between pivoting paths induced by identifying every couple of pivoting paths of the form

\[
\gamma P_{v_1} \beta P_{v_2}' \gamma' = \gamma P_{v_2} \beta' P_{v_1}' \gamma'
\]

where \( v_1 \neq v_2' \) and \( v_2 \neq v_1' \) (see Figure 2.1).

![Figure 2.1. Mesh relations \( P_{v_2}' P_{v_1} = P_{v_1}' P_{v_2} \) in \( \mathcal{C}_T \)](image)

In these relations, diagonals in \( T \) or boundary edges are allowed with the following convention: If one of the intermediate edges (\( \beta \) or \( \beta' \)) is either boundary edge or diagonal in
The corresponding term in the mesh relation is replaced by zero. Thus, the space of morphisms from a diagonal \( \gamma \notin T \) to a diagonal \( \gamma' \notin T \) is the quotient of the vector space over \( k \) spanned by pivoting paths from \( \gamma \) to \( \gamma' \) modulo the mesh relation.

The following lemma describes the relative positions of diagonals \( \gamma \) and \( \gamma' \), where there exist a nonzero morphism between them.

**Lemma 2.5.** [18 Lemma 2.1] The vector space \( \text{Hom}_{\mathcal{C}T}(\gamma, \gamma') \) is nonzero if and only if there exists a diagonal \( \tau_i \in T \) such that \( \tau_i \) crosses the diagonals \( \gamma \) and \( \gamma' \) and the relative positions of them are as in Figure 2.2. That is, let \( v_1, v_2 \) be the endpoints of \( \tau_i \) and \( u_1, u_2 \) (resp. \( u'_1, u'_2 \)) be the endpoints of \( \gamma \) (resp. \( \gamma' \)). Then ordering the vertices of the polygon in the positive trigonometric direction starting at \( v_1 \), we have \( v_1 < u_1 \leq u'_1 < v_2 < u_2 \leq u'_2 \). In this case, \( \text{Hom}_{\mathcal{C}T}(\gamma, \gamma') \) is of dimension one.

A triangulation \( T \) of the \((n + 3)\)-gon is said to be **triangulation without internal triangles** if each triangle has at least one side on the boundary of the polygon. It is important to recall that every triangulation gives rise to a cluster-tilted algebra of type \( \mathbb{A}_n \), and every cluster tilted algebra is of this form. In particular, every Dynkin quiver of type \( \mathbb{A}_n \) corresponds to a triangulation without internal triangles. The map associates a quiver \( Q_T \) to the triangulation \( T = \{\tau_1, \ldots, \tau_n\} \) of \( \Pi_{n+3} \) as follows. The vertices of \( Q_T \) are \( (Q_T)_0 = \{1, 2, \ldots, n\} \) and there is an arrow \( x \rightarrow y \) in \( (Q_T)_1 \) precisely if the diagonals \( \tau_x \) and \( \tau_y \) bound a triangle in which \( \tau_y \) lies counter-clockwise from \( \tau_x \) (see Figure 2.3 and Example 2.3).

**Example 2.3.** Given the quiver \( Q \) in Example 2.1, we have that \( Q = Q_T \), where \( T \) is the following triangulation
A vertex \( x \in (Q_T)_0 \) belongs to the support \( \text{supp} \gamma \) of a diagonal \( \gamma \notin T \) if the diagonal \( \tau_x \in T \) crosses \( \gamma \). The following lemma permits to see diagonals which are not in \( T \) as indecomposable objects in the category \( \text{mod} \ kQ_T/I \), where \( I \) is the two-sided ideal generated by all length two subpaths of oriented 3-cycles in \( Q_T \).

**Lemma 2.6.** [18, Lemma 3.2] Let \( \gamma \) be a diagonal which does not belong to \( T \). The set \( \text{supp} \gamma \) is connected as a subset of the quiver \( Q_T \).

In [18], the authors defined a \( k \)-linear additive functor \( \Theta \) from \( C_T \) to the category \( \text{mod} \ kQ_T/I \) of finitely generated \( kQ_T/I \)-modules. The image of the diagonal \( \gamma \notin T \) is the representation \( M^\gamma = (M^\gamma_x, f^\gamma_\alpha) \) defined as follows. For each vertex \( x \) in \( Q_T \),

\[
M^\gamma_x = \begin{cases} 
k & \text{if } x \in \text{supp} \gamma, \\
0 & \text{otherwise}. \end{cases}
\]

For any arrow \( \alpha : x \to y \) in \( Q_T \),

\[
f^\gamma_\alpha = \begin{cases} 
\text{id}_k & \text{if } M^\gamma_x = M^\gamma_y = k, \\
0 & \text{otherwise}. \end{cases}
\]

Moreover, for any pivoting elementary move \( P : \gamma \to \gamma' \) they defined the morphism \( \Theta(P) \) from \( (M^\gamma_x, f^\gamma_\alpha) \) to \( (M'^\gamma_x, f'^\gamma_\alpha) \) to be \( \text{id}_k \) whenever possible and 0 otherwise. The category \( C_T \) gives a geometric realization of the category of finitely generated \( kQ_T/I \)-modules in the following sense.

**Theorem 2.7.** [18, Theorems 4.4 and 5.1]

(a) The functor \( \Theta \) gives an equivalence of categories.

(b) The irreducible morphisms of \( C_T \) are direct sums of the generating morphisms given by pivoting elementary moves.

(c) The mesh relations of \( C_T \) are the mesh relations of the Auslander-Reiten quiver of \( C_T \).

(d) The Auslander-Reiten translation is given on diagonals by \( r^- \). Here, \( r^- \) (resp. \( r^+ \)) denotes the clockwise (resp. counter-clockwise) elementary rotation of the regular polygon \( \Pi_{n+3} \).

(e) The projective indecomposable objects of \( C_T \) are the diagonals in \( r^+(T) \).

(f) The injective indecomposable objects of \( C_T \) are the diagonals in \( r^-(T) \).

### 2.3 Category of sp-diagonals

In this section we define a category \( C_{(T,F)} \) of diagonals associated to the poset \( P \) of type \( A \) as a full subcategory of the category of diagonals \( C_T \). To do that, we define fans, peak-diagonals and sp-diagonals.

Let \( P \) be a poset of type \( A \) associated to the quiver \( Q^F \) as in Proposition 2.2. Thus, \( Q \) is a Dynkin quiver of type \( An \) and \( F \) is an alien set for \( Q \). Let \( T = \{\tau_1, \ldots, \tau_n\} \) be the
triangulation of a \((n+3)\)-gon \(\Pi_{n+3}\) such that \(Q_T = Q\). A fan in \(T\) is a maximal subset \(\Sigma_v \subseteq T\) of at least two diagonals such that all the diagonals in \(\Sigma_v\) share the vertex \(v\) of \(\Pi_{n+3}\). A diagonal \(\tau \in \Sigma_v\) is said to be the peak-diagonal of \(\Sigma_v\) if it is maximal in \(\Sigma_v\) in accordance with the order \(\tau_x < \tau_y\) if and only if there is a path from the vertex \(x\) to the vertex \(y\) in the quiver \(Q\). Geometrically, the peak-diagonal of a fan \(\Sigma_v\) is the diagonal that can be obtained from each other diagonal in \(\Sigma_v\) by a clockwise rotation around the vertex \(v\) (see Figure 2.4).

![Figure 2.4. Fan of a triangulation](image)

**Definition 2.3.** A diagonal \(\gamma \notin T\) is an \(sp\)-diagonal if it satisfies the following conditions:

(a) If \(\gamma\) crosses \(\tau \in T\) then \(\gamma\) crosses the peak-diagonal of a fan \(\Sigma\) in \(T\) such that \(\tau \in \Sigma\). Henceforth, any diagonal \(\gamma \notin T\) satisfying this condition will be called a \(\star\)-diagonal.

(b) For all alien arrows \(\alpha \in F\) with \(s(\alpha), t(\alpha) \in \text{supp} I_z\), if \(\gamma\) crosses \(\tau_{s(\alpha)}\) and \(\tau_z\) then \(\gamma\) also crosses \(\tau_{t(\alpha)}\). Diagonals \(\gamma \notin T\) satisfying this condition will be called non-frozen diagonals.

**Example 2.4.** Let \(Q\) be the quiver in Example 2.1 then \(Q = Q_T\), where \(T\) is the following triangulation

![Triangulation](image)

In this case, the sets \(\{\tau_1, \tau_2\}, \{\tau_2, \tau_3\}, \{\tau_3, \tau_4, \tau_5\}, \{\tau_5, \tau_6\}\) and \(\{\tau_6, \tau_7\}\) are fans of \(T\). We have used bold font for the peak-diagonal of each fan. Note that, the peak-diagonal corresponds to a sink vertex in the quiver \(Q_T\). Moreover, let \(Q^F\) be the quiver in the Example 2.2. Then, the diagonals

![Diagonals](image)

are such that \(\text{supp} \gamma_1 = \{3, 4\}\) and \(\text{supp} \gamma_2 = \{1, 2, 3\}\). Thus, \(\gamma_1\) is not a \(\star\)-diagonal because it crosses \(\tau_4\) but it does not cross the peak-diagonal \(\tau_5\) in the unique fan \(\{\tau_3, \tau_4, \tau_5\}\) of \(\tau_4\), whereas \(\gamma_2\) is a \(\star\)-diagonal because it crosses \(\tau_2\) which is the peak-diagonal in the fans \(\{\tau_1, \tau_2\}\) and \(\{\tau_2, \tau_3\}\) for \(\tau_1, \tau_2\) and \(\tau_3\).
Given the alien arrows $\alpha : 3 \rightarrow 1$ and $\beta : 6 \rightarrow 4$ (see Example 2.2), a diagonal $\gamma$ is frozen by $\alpha$ if $\gamma$ crosses $\tau_3$ and $\tau_2$ but not $\tau_1$; whereas the diagonals frozen by $\beta$ cross $\tau_6$ and $\tau_5$ but not $\tau_4$ (see Figure 2.5).

![Figure 2.5. Diagonals frozen by $\alpha$ (left) and by $\beta$ (right).]

Note that, $\gamma_2$ is an sp-diagonal because it is non-frozen and $\star$-diagonal.

The following lemma describes the relation between $\star$-diagonals and socle-projective modules in $\text{mod} \ kQ_T$.

**Lemma 2.8.** Let $\Theta : C_T \rightarrow \text{mod} \ kQ_T$ be the equivalence of categories of Theorem 2.7, where $Q_T$ is a Dynkin quiver of type $A$. Then $\gamma$ is a $\star$-diagonal if and only if $\Theta(\gamma)$ is socle-projective.

**Proof.** Since $Q_T$ is a Dynkin quiver of type $A$, then $T = \{\tau_x \mid x \in (Q_T)_0\}$ is a triangulation without internal triangles. First, we suppose that $\gamma$ is a $\star$-diagonal. Let $x$ be a vertex in $Q_T$ such that the indecomposable simple $kQ_T$-module $S_x$ at vertex $x$ is a submodule of $\Theta(\gamma) = M^\gamma$. We shall prove that $S_x$ is a projective $kQ_T$-module. Since $\text{Hom}(S_x, M^\gamma) \neq 0$, then $M_x^\gamma = k$, that is, $\tau_x$ crosses $\gamma$. By hypothesis, there exists a fan $\Sigma$ containing $\tau_x$ such that $\gamma$ crosses the peak-diagonal $\tau_z$ of $\Sigma$. If $x \neq z$ then $\tau_x < \tau_z$, that is, there is a path $p$ in $Q_T$ from $x$ to $z$ whose vertices are in $\text{supp} \gamma$. Moreover, a nonzero morphism $f = (f_x)_{x \in (Q_T)_0}$ of representations from $S_x$ to $M^\gamma$ is such that $f_t = 0$ for all $t \neq x$ because $S_x$ is the simple representation at vertex $x$. Let $x \rightarrow y$ be the arrow in $p$ starting in $x$, then the diagram

$$
\begin{array}{c}
(S_x)_x \\
\downarrow f_x \\
M_x^\gamma
\end{array}
\xrightarrow{0}
\begin{array}{c}
(S_x)_y \\
\downarrow 0 \\
M_y^\gamma
\end{array}

$$

commutes because $f$ is a morphism of representations of the quiver $Q_T$. Since $(S_x)_y$ is zero and $M_x^\gamma = M_y^\gamma = k$, then $f_x = 0$. Therefore, the morphism $f$ is zero, a contradiction. Thus, we conclude that $x = z$, that is, $\tau_x$ is a peak-diagonal. In other words, $x$ is a sink vertex in $Q_T$ and then $S_x$ is projective. Since all simple submodules of $M^\gamma$ are projectives, we have that soc $M$ is projective.

In the other direction, we have that $\Theta(\gamma)$ is socle-projective. Let $\tau_x$ be a diagonal in $T$ crossing $\gamma$. If $\tau_x$ is a peak-diagonal then the definition of $\star$-diagonal is trivially satisfied. If $\tau_x$ is not a peak-diagonal, we suppose that for all fans $\Sigma$ containing $\tau_x$, $\gamma$ does not cross the peak-diagonal in $\Sigma$. We have that the number $s$ of fans containing $\tau_x$ is either one or two. In the case $s = 1$, let $\tau_y$ be the maximal diagonal in the fan $\Sigma$ which crosses $\gamma$. Then
Let $C_{(T,F)}$ be the full subcategory of the category of diagonals $C_T$ generated by all sp-diagonals in $C_T$. Moreover we denote by $E(T,F)$ the set whose elements are the sp-diagonals in $C_{(T,F)}$, the diagonals in $T$, and the boundary edges in $\Pi_{n+3}$.

The irreducible morphisms in $C_{(T,F)}$ cannot be factorized through sp-diagonals. Then, we introduce the notion of a pivoting sp-move from $\gamma \in E(T,F)$ to $\gamma' \in E(T,F)$, that is, a composition of pivoting elementary moves of the form

$$P : \gamma = \gamma_0 \xrightarrow{P_1^{(1)}} \gamma_1 \xrightarrow{P_1^{(2)}} \cdots \xrightarrow{P_1^{(s)}} \gamma_s = \gamma'$$

with the same pivot $v$ such that $\gamma_1, \ldots, \gamma_{s-1}$ are not sp-diagonals in $\Pi_{n+3}$.

According to the mesh relations defined in the category $C_T$ we analyse the situation in the full subcategory $C_{(T,F)}$ of $C_T$. We suppose that $\gamma$ and $\gamma'$ are sp-diagonals and that the compositions $\gamma \xrightarrow{P_2} \beta \xrightarrow{P_2} \gamma'$ and $\gamma \xrightarrow{P_3} \beta' \xrightarrow{P_3} \gamma'$ of two pivoting sp-moves are as in Figure 2.6. We have that

$$\gamma \xrightarrow{P_4} \beta \xrightarrow{P_4} \gamma' = \gamma \xrightarrow{P_4} \beta' \xrightarrow{P_4} \gamma',$$

taking into account the following conventions:

(i) If one of the intermediate edges ($\beta$ or $\beta'$) is a boundary edge, the corresponding term in the identity is replaced by zero.

(ii) If one of the intermediate edges ($\beta$ or $\beta'$) is a diagonal in $T$, the corresponding term in the identity is replaced by zero.

The functor $\Omega$. Let $P$ be the poset of type $A$ associated to the quiver $Q^F$, where $Q$ is a quiver of Dynkin type $A$ and $F$ an alien set for $Q$ and denote by $T$ a triangulation associated to $Q$. Let us define a $\mathbb{k}$-linear additive functor

$$\tau_x \leq \tau_y < \tau_z,$$

where $\tau_z$ is the peak-diagonal in $\Sigma$. In other words, there is a path $p$ from $x$ to $z$ in $Q_T$ passing by $y$, such that the vertices $x, \ldots, y$ in $p$ belong to supp $\gamma$, whereas the others vertices in $p$ are not in supp $\gamma$. In particular, $M^\gamma_x = M^\gamma_y = k$ and $M^\gamma_z = 0$. Let $S_y$ be the simple representation of $Q_T$ at vertex $y$. Because the diagram

$$\begin{array}{ccc}
(S_y)_x & \xrightarrow{0} & (S_y)_y \\
\downarrow 0 & & \downarrow \lambda \\
M^\gamma_x & \xrightarrow{1} & M^\gamma_y \\
\downarrow 0 & & \downarrow 0 \\
M^\gamma_z & \xrightarrow{0} & M^\gamma_z
\end{array}$$

commutes, we conclude that there is a nonzero injective morphism from $S_y$ to $M^\gamma$. Therefore, $S_y$ is a non-projective module which is a submodule of $M^\gamma$, a contradiction to the hypothesis. In the case $s = 2$, if $\tau_y$ (respectively $\tau'_y$) is the maximal diagonal in $\Sigma$ (respectively $\Sigma'$) crossing $\gamma$. By the above arguments, we conclude that $S_y$ and $S_y'$ are non-projective summands of soc $M^\gamma$, a contradiction to the hypothesis. Therefore, $\gamma$ is a $*$-diagonal. □
CHAPTER 2. A GEOMETRIC REALIZATION OF SP-REPRESENTATIONS OF POSETS OF TYPE A

Figure 2.6. Mesh relations in $C_{(T,F)}$.

$$\Omega : C_{(T,F)} \to \text{mod}_{sp}(k\mathcal{P})$$

from the category of sp-diagonals to the category of finitely generated socle-projective $k\mathcal{P}$-modules such that for any sp-diagonal $\gamma$ we have $\Omega(\gamma) = M^\gamma = (M^\gamma_x, y h^\gamma_x)$ where $M^\gamma$ is defined by the following identities:

$$M^\gamma_x = \begin{cases} k & \text{if } x \in \text{Supp} \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

and if $x \leq y \in \mathcal{P}$ then $y h^\gamma_x = \begin{cases} \text{id}_k & \text{if } x, y \in \text{supp} \gamma, \\ 0 & \text{otherwise.} \end{cases}$

Now, we define the functor $\Omega$ on morphisms. By additivity, it is sufficient to define the functor on morphisms between sp-diagonals. Our strategy is to define the functor on pivoting sp-moves and then check that the mesh relations in $C_{(T,F)}$ hold. For any pivoting sp-move $P : \gamma \to \gamma'$, we define the morphism $\Omega(P) = (\Omega(P)_x)_{x \in \mathcal{P}} : (M^\gamma_x, y h^\gamma_x) \to (M'^\gamma_x, y h'^\gamma_x)$ by

$$\Omega(P)_x = \begin{cases} \text{id}_k, & \text{if } M^\gamma_x = M'^\gamma_x = k, \\ 0, & \text{otherwise.} \end{cases}$$

By definition, $\Omega$ maps compositions of pivoting sp-moves to composition of the images of the pivoting sp-moves. Note that if $\mathcal{P}$ is the poset $\mathcal{P}_Q$ associated to a Dynkin quiver $Q$ of type $A$ then the functor $\Omega$ is the restriction of the functor $\Theta$ defined in 2.2 to the category of sp-diagonals $C_{(T,F)}$. Now, we prove that the functor $\Omega$ is well defined and that it is a equivalence of categories.

**Theorem 2.9.** $\Omega$ is a equivalence of categories.

**Proof.** Recall here that $\mathcal{P}$ and $\mathcal{P}_Q$ are two different posets, that they have the same vertices and that $\mathcal{P}$ is obtained from $\mathcal{P}_Q$ by adding edges to the Hasse diagram corresponding to the alien arrows in $F$. In particular if $x \prec y$ in $\mathcal{P}_Q$ then $x \prec y$ in $\mathcal{P}$.

In order to prove that $M^\gamma \in \text{mod}_{sp}(k\mathcal{P})$ we show that the conditions (a) and (b) in Remark 1.3 hold. To prove (a) it is enough to consider the non trivial situation when $x \prec y \prec w$ in $\mathcal{P}$ such that $x, w \in \text{supp} \gamma$ and $y \notin \text{supp} \gamma$. First we note that, by Lemma 2.6 if $x \prec y \prec w$ in $\mathcal{P}_Q$ then $y \in \text{supp} \gamma$ which is contradictory. Therefore, we have that $x \nless y$ or $y \nless w$ in $\mathcal{P}_Q$. We consider the following cases (we recall that we always suppose $x \prec y \prec w$ in $\mathcal{P}$ such that $x, w \in \text{supp} \gamma$ and $y \notin \text{supp} \gamma$).

**Case 1** $y \prec w$ in $\mathcal{P}_Q$ and $x \nless y$ in $\mathcal{P}_Q$. In this case, there exists an alien arrow $\alpha : x' \to y'$ on vertices of a $z$-subquiver $Q^{(z)}$ of $Q$ such that $x \preceq x' \prec z$ and $y' \preceq y \prec w \prec z$ in
$P_Q$. Since $w \in \text{supp} \gamma$ and $\tau_z$ is the maximal diagonal in the unique fan of $\tau_w$ then $\gamma$ crosses $\tau_z$. Thus, since $x \in \text{supp} \gamma$ Lemma 2.6 implies that $\gamma$ crosses $\tau_y$, and since $\gamma$ is non-frozen, then $\gamma$ crosses $\tau_y'$. Again using Lemma 2.6 we obtain that $\gamma$ crosses $\tau_y$, that is, $y \in \text{supp} \gamma$ which is contradictory.

**Case 2** $x \prec y$ in $P_Q$ and $y \nprec w$ in $P_Q$. In this case, there exists an alien arrow $\alpha : y' \rightarrow w'$ on vertices of a z-subquiver $Q^{(z)}$ such that $x \prec y \preceq y' \prec z$ and $w' \preceq w < z$ in $P_Q$. Since $w \in \text{supp} \gamma$ and $\tau_z$ is the maximal diagonal in the unique fan of $\tau_w$, then $\gamma$ crosses $\tau_z$. Thus, since $x \in \text{supp} \gamma$ Lemma 2.6 implies that $\gamma$ crosses $\tau_y$, in other words, $y \in \text{supp} \gamma$ which cannot be.

**Case 3** $x \nprec y$ and $y \nprec w$ in $P_Q$. In this case, there exist two alien arrows $\alpha : x' \rightarrow y'$ and $\alpha : y'' \rightarrow w'$ on vertices of a z-subquiver $Q^{(z)}$ of $Q$ such that $x \preceq x' < w' \preceq w < z$ and $y' \preceq y \preceq y'' < z$ in $P_Q$. Since $w \in \text{supp} \gamma$ and $\tau_z$ is the maximal diagonal in the unique fan of $\tau_w$, then $\gamma$ crosses $\tau_z$. Thus, Lemma 2.6 implies that $\gamma$ crosses $\tau_y'$, and since $\gamma$ is non frozen we conclude that $\gamma$ crosses $\tau_y$. Again using Lemma 2.6 we obtain that $\gamma$ crosses $\tau_y$, that is, $y \in \text{supp} \gamma$ which is contradictory.

We have shown that if $x \prec y < w$ in $P = P_{Q^F}$ such that $x, w \in \text{supp} \gamma$ then $y \in \text{supp} \gamma$. Thus, $w h^y_x = y h^y_x = w h^y_x = \text{id}_k$ and condition (a) holds. To prove condition (b), let $x$ be an element of $P^- = P \setminus \text{max} P$. If $x \notin \text{supp} \gamma$ then clearly ker $h^y_x = 0$ for all $z \in \text{max} P$ such that $x \prec z$. If $x \in \text{supp} \gamma$ then $h^y_x = \text{id}_k$ for some $z' \in \text{max} P$, where $\tau_{z'}$ is the peak-diagonal in some fan containing $\tau_x$. Thus, $\bigcap_{z \in \text{max} P} \ker h^y_x = 0$ for all $x \in P^-$ such that $x \prec z$. This shows that $\Omega(\gamma) = M^\gamma$ is indeed an object in $\text{mod}_{sp}(kP)$.

Let us now check that $\Omega(P)$ is well defined for every pivoting sp-move $P : \gamma \rightarrow \gamma'$. Indeed, it is enough to show that for any relation $x \prec y$ such that $y$ covers $x$ in $P$ the diagram

\[
\begin{array}{ccc}
M^x \xrightarrow{y h^y_x} M^y \\
\downarrow \Omega(P)_x \downarrow \Omega(P)_y \\
M'^x \xrightarrow{y h'^y_x} M'^y
\end{array}
\]

commutes. Note that the result holds if $M^x_y = 0$ or $M'^x_y = 0$ and also if both $M^x_y$ and $M'^x_y$ are null spaces. Suppose now that $M^x_y = M'^x_y = k$. If $M^y_y = M'^y_y = k$, then all four maps are id$_k$ and the diagram commutes. The only remaining case is if exactly one of $M^x_y$, $M'^x_y$ is nonzero. We will show that this cannot happen. Suppose that $M^x_y = 0$ and $M^y_y = k$, that is, $x, y \in \text{supp} \gamma$, $y \in \text{supp} \gamma'$ and $x \notin \text{supp} \gamma'$. Since $y$ covers $x$ in $P$, there exists an arrow $\alpha : x \rightarrow y$ in $Q^F$. If $x \prec y$ in $P_Q$ then $\alpha$ is an arrow in $Q$, that is, $\tau_x$ and $\tau_y$ share a vertex of the polygon and are connected by a pivoting elementary move. Since $P : \gamma \rightarrow \gamma'$ is a pivoting sp-move we get that $\tau_x$ crosses $\gamma$, that $\tau_x$ and $\gamma'$ have a common point on the boundary of the polygon and that $\tau_y$ crosses $\gamma$ and $\gamma'$. This implies that $\tau_y$ is clockwise from $\tau_x$ and that contradicts the orientation $x \rightarrow y$ in the quiver $Q$ (see Figure 2.3). Next, we suppose that $x \nprec y$ in $P_Q$, then $\alpha : x \rightarrow y$ is an alien arrow in $F$ with $x$ and $y$ in Supp $I_z$ for some sink vertex $z$ in $Q$. Now, by Definition 2.2 part (b), $y$
is not a source vertex in $Q$ unless $y$ is an extremal vertex in $Q$. Thus, there is at most one arrow in $Q$ with starting point $y$, and therefore there is exactly on fan $\Sigma$ containing $\tau_y$ and $\tau_z$ is its peak-diagonal. By Definition 2.3 both $\gamma$ and $\gamma'$ cross $\tau_z$, because they are $\star$-diagonals crossing $\tau_y$.

On the other hand, there is a pivoting path from $\tau_z$ to $\tau_x$ in $\Pi_{n+3}$, since $x$ belongs to $\text{Supp } I_2$. But this is impossible, because if $\tau \rightarrow \tau_x$ is a pivot, then $\tau$ does not cross $\gamma'$. The other case where $M_{x'}^\gamma = k$ and $M_{y'}^\gamma = 0$ is proved in a similar way.

To show that the functor $\Omega$ is well defined, it only remains to check the mesh relations. Indeed, let $\gamma \xrightarrow{P_1} \beta, \beta \xrightarrow{P_2} \gamma', \gamma \xrightarrow{P_3} \beta', \beta' \xrightarrow{P_4} \gamma'$ be pivoting sp-moves as in Figure 2.6 with $\gamma, \gamma'$ sp-diagonals and $\beta \neq \beta'$ sp-diagonals, diagonals in $T$ or boundary edges. Note that, we can exclude the case where $\beta$ and $\beta'$ are both diagonals in the triangulation $T$ or both boundary edges because in this case either $\gamma$ or $\gamma'$ is a diagonal in $T$. Without loss of generality, we may assume from now on that $\beta$ is an sp-diagonal. Suppose first that $\beta'$ is an sp-diagonal; then one has to check the commutativity of the following diagram

$$
\begin{array}{ccc}
M_x^\gamma & \xrightarrow{\Omega(P_1)_x} & M_x^\beta \\
\downarrow{\Omega(P_3)_x} & & \downarrow{\Omega(P_2)_x} \\
M_x^\beta & \xrightarrow{\Omega(P_4)_x} & M_x^\gamma'
\end{array}
$$

for all $x \in P$. Again, the only non trivial case happens when $M_x^\beta = M_x^\gamma = k$. In this case we also have $M_x^\beta = M_x^\gamma' = k$ because any diagonal crossing both $\gamma$ and $\gamma'$ must also crosses $\beta$ and $\beta'$. Thus all maps are $\text{id}_k$ and the diagram commutes. Suppose now that $\beta'$ is a boundary edge or diagonal in $T$. Then we have to show that the composition $M_x^\beta \xrightarrow{\Omega(P_1)} M_x^\beta \xrightarrow{\Omega(P_2)} M_x^\gamma'$ is zero for all $x \in P$. Clearly if $\beta'$ is a boundary edge or diagonal in $T$ then no diagonal $\tau \in T$ can cross both $\gamma$ and $\gamma'$ then $\text{Hom}(\Omega(\gamma), \Omega(\gamma')) = 0$.

In order to prove that $\Omega$ is dense we fix an indecomposable $M \in \text{mod}_{sp}(kP)$. Then by Lemma 2.4 part (b), Lemma 2.6 and Theorem 2.7 part (a) there exists a diagonal $\gamma \notin T$ such that $\text{supp } \gamma = \text{supp } M$. We show that $\gamma$ is an sp-diagonal. Indeed, since the socle of $M$ is projective, Lemma 2.8 implies that $\gamma$ is a $\star$-diagonal. Moreover, given an alien arrow $\alpha : x \rightarrow y$ in $F$, with $x$ and $y$ in $\text{Supp } I_2$ for some sink vertex $z$ in $Q_0$ such that $x, z \in \text{supp } M$ then $z h_x = \text{id}_k$. By Proposition 1.3 part (a), we have that $z h_x = z h_y \cdot y h_x$, thus $y \in \text{supp } M$. Therefore $\gamma$ crosses $\tau_y$ and thus $\gamma$ is a non-frozen diagonal. We conclude that $\gamma$ is an sp-diagonal and that $\Omega(\gamma) = M$. 

![Diagram](image-url)
To show that $\Omega$ is faithful, it is enough to prove that the image of a nonzero morphism between sp-diagonals is a nonzero morphism in $\text{mod}_{sp}(kP)$. Indeed, let $P \in \text{Hom}_{C(T,F)}(\gamma, \gamma')$ be a nonzero morphism in $C(T,F)$. Then $P$ also is a nonzero morphism in $C_T$. Lemma 2.5 implies that there exists a diagonal $\tau_x \in T$ crossing $\gamma$ and $\gamma'$ as in Figure 2.2. In particular, $M^2 = M^2' = k$, and therefore $\Omega(P)_x = \text{id}_k \neq 0$.

Finally, we show that functor $\Omega$ is full. To do so, let $\Omega(\gamma) \to \Omega(\gamma')$ be a nonzero morphism in $\text{mod}_{sp}(kP)$. Then $g = (g_x)_{x \in Q_0}$, where $g_x$ is a linear map from $\Omega(\gamma)_x$ to $\Omega(\gamma')_x$. The map $\hat{g} = (\hat{g}_x)_{x \in Q_0}$ from $\Theta(\gamma)$ to $\Theta(\gamma')$ such that $\hat{g}_x = g_x$ is a morphism of representations in $\text{mod}_{sp}(kQ)$. Indeed, for each arrow $\alpha : x \to y$ in $Q_1$, we have $x \prec y$ in $P$. Since $g$ is morphism in $\text{mod}_{sp} kP$, then the diagram

$$
\begin{array}{ccc}
\Omega(\gamma)_x & \xrightarrow{y h^\gamma_x} & \Omega(\gamma)_y \\
\downarrow g_x & & \downarrow g_y \\
\Omega(\gamma')_x & \xrightarrow{y h^\gamma'_x} & \Omega(\gamma')_y
\end{array}
$$

commutes. Note that the elements in $P$ are the vertices in $Q_0$. Moreover, if $\gamma$ is an sp-diagonal then the representations $\Theta(\gamma) = (\Theta(\gamma)_x, f^\alpha)$ in $\text{mod}_{kQ}$ and $\Omega(\gamma) = (\Omega(\gamma)_x, y h^\gamma_x)$ in $\text{mod}_{sp}(kP)$ have the same $k$-vector spaces $\Omega(\gamma)_x = \Theta(\gamma)_x$ for all $x \in P$ and the same maps $y h^\gamma_x = f^\alpha$ for each $\alpha : x \to y$ in $Q_1$ (the map $f^\alpha$ is not defined when $\alpha$ is an alien arrow for $Q$). Thus we have a commutative diagram

$$
\begin{array}{ccc}
\Theta(\gamma)_x & \xrightarrow{f^\gamma} & \Theta(\gamma)_y \\
\downarrow \hat{g}_x & & \downarrow \hat{g}_y \\
\Theta(\gamma')_x & \xrightarrow{f^\gamma'} & \Theta(\gamma')_y
\end{array}
$$

and hence the map $\hat{g}$ is a morphism in $\text{mod}_{kQ}$. Under the equivalence of categories $\Theta : C_T \to \text{mod}_{kQ_T}$ of Theorem 2.7, the morphism $\hat{g}$ corresponds to a morphism $P \in \text{Hom}_{C_T}(\gamma, \gamma')$, with $\Theta(P) = \hat{g}$. Since $\gamma$ and $\gamma'$ are sp-diagonals in $C_T$, $P$ also is a morphism in the full subcategory $C_{(T,F)}$ of $C_T$. The definition of the functors $\Theta$ and $\Omega$ on morphisms implies that $\Omega(P) = g$. \hfill $\square$

The following corollary is an direct consequence of the arguments used in Theorem 2.9 and section 2.3.

**Corollary 2.10.** Let $P$ be a poset of type $A$ associated to the quiver $Q^F$ as in Proposition 2.6 and let $C_{(T,F)}$ be the corresponding category of sp-diagonals. Then

(a) The irreducible morphisms of $C_{(T,F)}$ are direct sums of the generating morphisms given by pivoting sp-moves.

(b) Let $\gamma \xrightarrow{P_1} \beta \xrightarrow{P_2} \gamma'$ be a composition of two pivoting sp-moves as in Figure 2.6, where $\gamma$, $\gamma'$, and $\beta$ are sp-diagonals. Then

(i) The sequence $0 \to \gamma \to \beta \oplus \beta' \to \gamma' \to 0$ is an AR-sequence if $\beta'$ is a sp-diagonal.
(ii) The sequence $0 \rightarrow \gamma \rightarrow \beta \rightarrow \gamma' \rightarrow 0$ is an AR-sequence if $\beta'$ is either a boundary edge or a diagonal in $T$.

(iii) If $\beta' \notin E(T, F)$ then $\gamma'$ is an indecomposable projective in $\mathcal{C}(T, F)$ and $\gamma$ is an indecomposable injective in $\mathcal{C}(T, F)$.

Example 2.5. The poset $\mathcal{P}$ of Example 1.11 is the poset $\mathcal{P}_{Q^F}$ of type $A$ associated to the quiver $Q^F$ given in Example 2.2. The Auslander Reiten quiver $\Gamma(\mathcal{C}(T, F))$ of the category of sp-diagonals $\mathcal{C}(T, F)$ is as follows

![Diagram of Auslander Reiten quiver]

Example 2.6. Let $F = \{1 \xrightarrow{\alpha} 4, 2 \xrightarrow{\beta} 3\}$ be an alien set for the Dynkin quiver

$$Q : 1 \rightarrow 3 \rightarrow 5 \leftarrow 4 \leftarrow 2$$

of type $A_5$. Then $Q^F$ has the form

![Diagram of quiver Q^F]

and the poset $\mathcal{P} = \mathcal{P}_{Q^F}$ associated to the quiver $Q^F$ has the form

![Diagram of poset P]

It is well known that the category $\text{mod}(k\mathcal{P})$ is of infinite representation type (see [55]) but the AR-quiver $\Gamma(\text{mod}_{sp}(k\mathcal{P}))$ of the category $\text{mod}_{sp}(k\mathcal{P})$ has the form

![Diagram of AR-quiver]
Note that, we have drawn dotted lines to describe the action of the AR translation. Let $T$ be the triangulation $\tau_1 \tau_2 \tau_5 \tau_4 \tau_3$ associated to $Q$, then the AR-quiver $\Gamma(C(T,F))$ of the category of sp-diagonals $C(T,F)$ has the form

2.4 What is a cluster algebra?

Cluster algebras were introduced by Fomin and Zelevinsky [33] in 2002. Their original motivation came from canonical bases in Lie Theory; however, the theory of cluster algebras is connected to various fields of mathematics, including the representation theory of finite-dimensional algebras.

Cluster algebras associated with quivers. A starting point is to present Fomin-Zelevinsky’s definition, in its simplest form, of cluster algebras associated with quivers. For that, we fix a natural number $n \geq 1$. Furthermore, although there is a more general setup, we stick to the case when the base field $k = \mathbb{Q}$ is the field of rational numbers. First of all, let $\mathcal{F}$ be a field extension of $\mathbb{Q}$. Typically we have that $\mathcal{F} = \mathbb{Q}(u_1, \ldots, u_n)$ is the field of rational functions in variables $u_1, \ldots, u_n$ with coefficients in $k$, where $u_1, \ldots, u_n$ are algebraically independent variables over $k$, it means that, there is no a polynomial $f$ in the polynomial algebra $k[X_1, \ldots, X_n]$ with coefficients in $k$ such that $f(u_1, \ldots, u_n) = 0$. The field $\mathcal{F}$ is called the ambient field.

A cluster is a sequence $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{F}^n$ of algebraically independent elements of length $n$. We refer to the elements in a cluster $\in \mathcal{F}^n$ as cluster variables. Note that, if $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{F}^n$ is a cluster, then the field $\mathcal{F}$ must contain the field $\mathbb{Q}(x_1, \ldots, x_n)$. Thus, if we have a distinguished cluster $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{F}^n$, then the smallest possible
field, namely $Q(x_1, \ldots, x_n)$, is a natural choice of an ambient field.

A seed is a pair $(x, Q)$, where $x \in \mathcal{F}^n$ is a cluster and $Q$ is a quiver with vertices $Q_0 = \{1, \ldots, n\}$ without loops $\bullet \xrightarrow{} \bullet$ and 2-cycles $\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$. Moreover, the mutation of the quiver $Q$ at vertex $k$ is a new quiver $\mu_k(Q)$ constructed from $Q$ using the following procedure.

1. We reverse all arrows starting or ending at $k$.
2. For each path $i \rightarrow k \rightarrow j$ in $Q$ we add a new arrow $i \rightarrow j$.
3. We remove one possibly created 2-cycle after the other until the quiver does not contain 2-cycles anymore.

Note that, the vertices of $\mu_k(Q)$ are the same vertices of $Q$. Moreover, by definition, the quiver $\mu_k(Q)$ again contains neither loops nor 2-cycles.

Let $(x, Q)$ be a seed and $k \in \{1, \ldots, n\}$ an index. The mutation of the seed $(x, Q)$ at $k$ is a seed $\mu_k(x, Q) = (\mu_k(x), \mu_k(Q))$ where $\mu_k(Q)$ is the mutation of the quiver $Q$ at vertex $k$ and $\mu_k(x) = (x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_n)$, where $x_k'$ is given by the so-called exchange relation

$$x_k' x_k = p_k^- + p_k^+,$$

(2.1)

defined for any vertex $k$ in the quiver $Q$, where $p_k^- = \prod_{\alpha, r \rightarrow k} x_r$ and $p_k^+ = \prod_{\beta, k \rightarrow r} x_r$. Here the product $p_k^-$ (resp. $p_k^+$) is taken over all arrows $\alpha \in Q_1$ (resp. $\beta \in Q_1$) that terminate (resp. start) in vertex $k$, counted possibly with multiplicity. Of course, the product is understood to be 1 if there are no such arrows. We say that two seeds $(x, Q)$ and $(x', Q')$ are isomorphic if they are obtained from each other by a simultaneous reordering of cluster variables and quiver vertices, in other words, if there exists a quiver isomorphism given by two bijections $Q_0 \xrightarrow{\sigma} Q_0'$ and $Q_1 \xrightarrow{\sigma} Q_1'$ such that $x_i = x'_\sigma(i)$ for all indices $i = 1, 2, \ldots, n$. Moreover $(x, Q)$ and $(x', Q')$ are mutation equivalent if there exists a sequence $(k_1, \ldots, k_r)$ in $Q_0'$ of indices of length $r \geq 0$ such that the seed $\mu_{k_1} \circ \cdots \circ \mu_{k_r}(x, Q)$ is isomorphic to $(x', Q')$. In this case, we write $(x, Q) \sim (x', Q')$.

Now, we are ready to define cluster algebras. Let $(x, Q)$ be a seed. The cluster algebra $\mathcal{A}_{(x, Q)}$ attached to the seed is the subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables in all seeds that are mutation equivalent to the given seed. Thus, the cluster algebra consists of all $\mathcal{Q}$-linear combinations of monomials in the cluster variables.

**Remark 2.1.** Since all cluster variables of a cluster algebra $\mathcal{A}_{(x, Q)}$ lie in the subfield $\mathcal{Q}(x_1, \ldots, x_n) \subset \mathcal{F}$, the definition is independent of the choice of the ambient field. Of course, if the seeds $(x, Q)$ and $(x', Q')$ are mutation equivalent, then the cluster algebras $\mathcal{A}_{(x, Q)} = \mathcal{A}_{(x', Q')}$ are the same. Moreover, if $y \in \mathcal{G}^n$ is other cluster of the same length (in an another ambient field $\mathcal{G}$), then the cluster algebras $\mathcal{A}_{(x, Q)} \equiv \mathcal{A}_{(y, Q)}$ are isomorphic algebras.

Henceforth, we write $\mathcal{A}_Q$ instead of $\mathcal{A}_{(x, Q)}$. If we think of the cluster algebras as being associated with a distinguished seed $(x, Q)$, then we will refer to this seed as the initial
seed and to the cluster variables in \( x \) as initial cluster variables. It is well known that the initial cluster variables correspond to the shift of indecomposable projectives in the cluster category (see [17]). Moreover, we will refer to the natural number \( n \in \mathbb{N} \) as the rank of the cluster algebra \( \mathcal{A}_Q \).

**Example 2.7.** Let \( Q \) be the quiver \( 1 \rightarrow 2 \rightarrow 3 \) and \( x = (x_1, x_2, x_3) \), where \( x_1, x_2, x_3 \) are indeterminates, and \( \mathcal{F} = \mathbb{Q}(x_1, x_2, x_3) \). We have \( \mu_1(x, Q) \), where \( \mu_1(Q) \) is the quiver \( 1 \leftarrow 2 \rightarrow 3 \) and \( \mu_1(x) = (x_1', x_2', x_3) \), where \( x_1 x_1' = 1 + x_2 \), so that \( x_1' = \frac{1 + x_2}{x_1} \). Further, we have \( \mu_2(x, Q) \), where \( \mu_2(Q) \) is the quiver \( 1 \rightarrow 2 \rightarrow 3 \), and \( \mu_2(x) = (x_1, x_2', x_3) \), where \( x_2 x_2' = x_1 + x_3 \), so that \( x_2' = \frac{x_1 + x_3}{x_2} \). Continuing, we have \( \mu_3(x, Q) \), where \( \mu_3(Q) \) is the quiver \( 1 \rightarrow 2 \leftarrow 3 \) and \( \mu_3(x) = (x_1, x_2, x_3') \), where \( x_3 x_3' = x_2 + 1 \), so that \( x_3' = \frac{x_2 + 1}{x_3} \). We continue by applying \( \mu_1, \mu_2, \mu_3 \) to the new seeds, keeping in mind that \( \mu_2^2 \) is the identity. In this example we get only a finite number of seeds up isomorphism, namely 14. The clusters are: \((x_1, x_2, x_3), (\frac{1 + x_2}{x_1}, x_2, x_3), (x_1, x_2, \frac{1 + x_2}{x_3}), (\frac{1 + x_2}{x_1}, x_1 + (1 + x_2) x_3), (x_1, \frac{1 + x_2}{x_1}, x_2, x_3), (\frac{1 + x_2}{x_1}, x_2, \frac{1 + x_2}{x_3}), (x_1 + (1 + x_2) x_2 x_3, x_1 + x_3), (\frac{1 + x_2}{x_1}, x_1 + x_2, x_2, x_3), (\frac{1 + x_2}{x_1}, x_1 + x_2, x_2, \frac{1 + x_2}{x_3}), (1 + x_2 x_1 + (1 + x_2) x_3, x_1, x_2, x_3), (1 + x_2 x_1 + (1 + x_2) x_3, x_1, \frac{1 + x_2}{x_3}), (1 + x_2 x_1 + (1 + x_2) x_3, \frac{1 + x_2}{x_1}, x_2, x_3), (1 + x_2 x_1 + (1 + x_2) x_3, \frac{1 + x_2}{x_1}, x_2, \frac{1 + x_2}{x_3}), (1 + x_2 x_1 + (1 + x_2) x_3, \frac{1 + x_2}{x_1}, \frac{1 + x_2}{x_3}, x_2, x_3), (1 + x_2 x_1 + (1 + x_2) x_3, \frac{1 + x_2}{x_1}, \frac{1 + x_2}{x_3}, \frac{1 + x_2}{x_1}, x_2, x_3), (1 + x_2 x_1 + (1 + x_2) x_3, \frac{1 + x_2}{x_1}, \frac{1 + x_2}{x_3}, \frac{1 + x_2}{x_1}, \frac{1 + x_2}{x_3}, x_2, x_3), (1 + x_2 x_1 + (1 + x_2) x_3, \frac{1 + x_2}{x_1}, \frac{1 + x_2}{x_3}, \frac{1 + x_2}{x_1}, \frac{1 + x_2}{x_3}, \frac{1 + x_2}{x_1}, x_2, x_3)\).

The set of cluster variables in a cluster algebra may be finite or infinite. The first important result of the theory is the classification of those cluster algebras where it is finite: the cluster algebras of finite type. This is equivalent to saying that there is only a finite number of seeds. But it is not equivalent to having only a finite number of quivers in the mutation class.

**Theorem 2.11.** [Fomin-Zelevinsky] Let \( Q \) be a quiver without loops and 2-cycles. Then the cluster algebra \( \mathcal{A}_Q \) is of finite type if and only if \( Q \) is mutation equivalent to a Dynkin quiver.

Thus, for each Dynkin diagram \( \Delta \), there is a canonical cluster algebra \( \mathcal{A}_\Delta \).

**The knitting algorithm.** Following [51], one can directly construct the cluster variables without first constructing the cluster. We describe an algorithm that produces all of the cluster variables (no clusters) in \( \mathcal{A}_Q \); when \( Q \) is a Dynkin quiver. The computation is periodic and we always get only finitely many cluster variables (as expected). Furthermore, the algorithm can easily be extended from Dynkin-diagrams to arbitrary trees, and then one can characterize the cluster algebra by the periodicity of the algorithm. The general algorithm will become clear from the following examples.

**Example 2.8.** We start with the simplest non trivial Dynkin diagram: \( A_2 = \bullet \rightarrow \bullet \). We first choose a numbering of its vertices and an orientation of its edges, for example \( Q : 1 \rightarrow 2. \) Now, we construct its repetition \( \mathbb{Z}Q \). First, we draw the product \( \mathbb{Z} \times \mathbb{Q} \). Then we draw for each arrow \( i \rightarrow j \in \mathcal{Q} \) and each \( k \in \mathbb{Z} \) the arrows \( (k, i) \rightarrow (k, j) \) and \( (k, j) \rightarrow (k + 1, i) \) in \( (\mathbb{Z}Q)_1 \).
We shall now assign a cluster variable to each vertex of the repetition. We start by assigning $x_1$ to $(0,2)$ and $x_2$ to $(0,1)$. In general, when $Q$ has $n$ vertices, we assign the initial variables $x_1, \ldots, x_n$ to the vertices of the 0-th copy of $Q$ in such a way that if $i \rightarrow j$ is an arrow in $Q$ then $x_j \rightarrow x_i$ is an arrow in the quiver of repetition $\mathbb{Z}Q$.

Next, we construct new variables $x'_1, x'_2, x''_1, \ldots$ by knitting from the left to the right (an analogous procedure can be used to go from the right to the left).

To compute $x'_2$, we divide the sum of the immediate predecessor of $x'_2$ and 1 by the left translate of $x'_2$. That gives us $x'_2 = \frac{x_1 + 1}{x_2}$. Similarly, we compute $x'_1$ by adding 1 to its predecessor $x'_2$ and dividing the result by the left translate $x_1$. That gives us $x'_1 = \frac{1 + 2x'}{x_1} = \frac{x_2 + 1}{x_1 + 1}$. Using the same rule we obtain $x''_2 = \frac{x_1 + 1}{x_2}, x''_1 = \frac{1 + x''_2}{x'_1} = x_2, x''_1 = x_2$. Clearly, from here on, the whole thing will repeat. In conclusion, there are 5 cluster variables $x_1, x_2, x'_1, x'_2$ and $x''_1$ and the cluster algebras $A_Q$ is a $Q$-subalgebra of $Q(x_1, x_2)$ generated by the cluster variables above.

We can see that all the denominators of all cluster variables are monomials. This hold for all cluster algebras. It is called **Laurent phenomenon**.

**Example 2.9.** Let us now consider the example $A_3$. We choose a quiver $Q: 1 \rightarrow 2 \rightarrow 3$. The associated repetition looks as follows.

Thus, $x'_1 = \frac{1 + x'_2}{x_1}$. However, to compute $x'_2$, we have to modify the rule, since $x'_2$ has two immediate predecessors with associated variables $x_1$ and $x'_3$. In the formula, we simply take the product over all immediate predecessors. Hence, $x'_2 = \frac{1 + x_1 x'_1}{x_2} = \frac{x_2 + x_1}{x_2 x_3}$. In the same way, we have $x'_1 = \frac{x_2 + x_1}{x_1}, x'_3 = \frac{x_2 + x_1}{x_2}, x''_2 = \frac{x_1 + x_2 + x_3}{x_1 x_2}, x''_1 = x_3$, and $x''_3 = \frac{1 + x_2}{x_1}$. Since the pattern shown in the diagram above is periodic, in total, we find 9 = 3 + 6 cluster variables, namely $x_1, x_2, x_3, \frac{1 + x_2}{x_1}, 1 + x_2, x_1 + x_2, x_1 + x_2, x_1 + x_2, x_1 + x_2, x_1 + x_2$, and $1 + x_2$. Thus, the cluster algebras $A_Q$ is a $Q$-subalgebra of $Q(x_1, x_2, x_3)$ generated by the cluster variables above.

**Cluster variables from quiver representations.** We can observe in Example 2.8 that there are 5 = 2 + 3 cluster variables. Two initial cluster variables $x_1, x_2$ and three non initial ones $x'_1, x'_2$ and $x''_1$. The non initial ones are in natural bijection with the isoclasses of indecomposables representations in $\text{rep} Q$. To see that, it suffices to look at the denominators of the three variables: The denominator $x'_1 x''_2$ corresponds to the dimension $\dim M = (d_1, d_2)$. It was proved by Fomin and Zelevinsky [34] that this generalizes to arbitrary Dynkin diagrams. In other words, there is a canonical bijection between positive roots of the quadratic form $q_Q$ of a Dynkin quiver $Q$ and the non trivial cluster variables of the corresponding cluster algebra. More precisely, we have the following.

**Theorem 2.12.** The map taking an indecomposable representation $M$ with dimension vector $\dim M = (d_1, \ldots, d_n)$ of a Dynkin quiver $Q$ to the unique non initial cluster variable
Moreover, all arrows \( w \to \beta \) following the knitting algorithm, we have that
\[ p_2.1 \). In case \((i) \alpha \in \mathbb{A} \) such that \( p \) is a sp-diagonal in the category \( C_{(T,F)} \) together with the cluster variables in the initial cluster \( x \). It is natural to ask under which conditions we have \( A(\mathcal{P}) = A_Q \). A partial answer is given in Theorem 2.14.

For an acyclic quiver \( Q \) with \( n \) vertices, we can describe the cluster algebra \( A_Q \) associated to \( Q \) using less cluster variables as follows. The lower bound cluster algebra associated to \( Q \) is the subring \( L_Q \) of \( \mathbb{Q}(x_1, \ldots, x_n) \) generated by the cluster variables \( x_1, \ldots, x_n \), where \( x_k, k = 1, \ldots, n \) is defined in Equation 2.1. It is well known that if \( Q \) is an acyclic quiver then \( L_Q = A_Q \) [10 Corollary 1.21]. In the same way, we introduce the subring \( R_Q \) of \( \mathbb{Q}(x_1, \ldots, x_n) \) which is generated by the cluster variables \( x_1, \ldots, x_n \), where, for all \( i = 1, \ldots, n \), \( x_P \) is the cluster variable in \( A_Q \) associated to the indecomposable projective \( kQ \)-module \( P_i \) in mod \( kQ \). Thus, we have the following lemma.

**Lemma 2.13.** If \( Q \) is tree quiver then \( R_Q = A_Q \).

*Proof.* We proceed by induction on the number \( n \) of vertices in \( Q \). The case \( n = 1 \) is trivial. Now, let us consider \( Q \) a tree quiver with \( n \) vertices, then \( Q \) has \( n - 1 \) arrows. Let \( w \) be a leaf of \( Q \) and define \( Q' \) to be the full subquiver of \( Q \) whose vertices are \( Q_0 \setminus \{ w \} \). Then \( Q \) is obtained from \( Q' \) by adding one vertex \( t \) and one arrow \( \alpha_w : t \to w \) for some \( t \in Q_0 \).

We shall prove that the variables \( x'_w = p_w + p_w^t \) and \( x'_t = p_t + p_t^w \) belong to \( R_Q \) (see Equation 2.1). In case \( (i) \), \( w \) is a sink vertex and then \( x'_w = x_{P_w} \). Hence, \( x'_w \in R_Q \). Additonally, following the knitting algorithm, we have that
\[ x_{P_t}x_t = 1 + p_t^{-1} \prod_{\beta : t \to r} x_{P_r}, \tag{2.2} \]
where the product is taken over all arrows \( \beta \in Q_1 \) that start in vertex \( t \). We multiply \( p_t^+ \) by \( p_t^\alpha \) and we obtain \( x_{P_t}x_t p_t^\alpha = p_t^t + p_t^i \prod_{\beta : t \to r} x_{P_r} \). Since \( \alpha_w \) is an arrow from \( t \) to \( w \) then \( x_{P_t}x_t p_t^\alpha = p_t^t + p_t^i x_{P_{w}} \delta \) where the product \( \delta = \prod_{\beta : t \to r} x_{P_r} \) is taken over all arrows \( \beta \in Q_1 \) that start in vertex \( t \) and terminate in a vertex \( r \) (note that \( r \neq w \)). Moreover, \( x_{P_t}x_t p_t^\alpha = p_t^t + p_t^i (1 + x_{t}) \delta \) because \( x_{w} x_{P_w} = x_{x_{w}} x_{w} = 1 + x_{t} \). Since \( x_{P_t}, p_t^\alpha \in R_Q \) we have
\[ x_{P_t}p_t^\alpha = p_t^t + p_t^i (1 + x_{t}) \delta = \frac{p_t^t + p_t^i}{x_{t}} + p_t^i \delta \in R_Q. \]
Since $p_t \delta$ belongs to $R_Q$ and by (2.1) we conclude that $x'_t \in R_Q$. In case (ii), we have that
\[ x_w x_{P_w} = 1 + x_{P_t}. \] (2.3)
Multiplying (2.3) by $x_t$ and using (2.2) we deduce that
\[ x_w x_{P_w} x_t = x_t + 1 + p_t^{-} \prod_{\beta:t \rightarrow r} x_{P_r}. \]
Since $x_{P_w}, x_t \in R_Q$ then
\[ x_{P_w} x_t = \frac{x_t + 1 + p_t^{-} \prod_{\beta:t \rightarrow r} x_{P_r}}{x_w} \in R_Q. \]
Since there is an arrow $\alpha_w$ from $w$ to $t$ then $x_w$ is a factor of $p_t^{-}$; thus,
\[ x'_w = \frac{1 + x_t}{x_w} \in R_Q. \]
Analogous to the proof of the case (i), we can prove that $x'_t \in R_Q$. As a consequence of the hypothesis of induction on the quiver $Q'$ the variables $x'_s$ with vertex $s \neq t$ in $Q'_0$ belong to $R_Q$. Thus, the lower cluster algebra $L_Q$ is contained in the subalgebra $R_Q$. Since $L_Q = A_Q$ then $R_Q = A_Q$.

**Theorem 2.14.** Let $\mathcal{P}$ be a poset of type $\hat{\mathbb{A}}$ associated to the quiver $Q^\emptyset$ as in Proposition 2.2 and let $A(\mathcal{P})$ be the subalgebra of $A_Q$ associated to $\mathcal{P}$. Then $A(\mathcal{P}) = A_Q$.

**Proof.** In this case, the poset $\mathcal{P}$ is viewed as the quiver $Q$ of type $\hat{\mathbb{A}}$. Then, the subcategory $C_{(T,F)}$ of $C_T$ is given by $\star$-diagonals because $F = \emptyset$ and it is equivalent to the category $\text{mod}_{sp} kQ$ of socle-projective $kQ$-modules (see Theorem 2.9). By Theorem 2.7 part (e) the indecomposable projectives in $\text{mod} kQ$ can be identified with diagonals $r^{-}(T)$ in the category $C_T$ which are clearly $\star$-diagonals. Hence, $R_Q \subseteq A(\mathcal{P})$. Moreover, since $Q$ is a tree quiver then Lemma above implies that $A(\mathcal{P}) = A_Q$. □
CHAPTER 3

Finitely generated Zavadskij modules over finite-dimensional algebras

Zavadskij modules were introduced by Rump in 2000 [69]. They arose from a generalization of the famous Zavadskij’s algorithm of differentiation with respect to a suitable pair of points [91]. For instance, it was used to classify tiled orders of finite representation type and posets of finite growth type [61, 88]. Rump gave a module-theoretic approach to the known versions of such an algorithm, for tiled orders [88], representations of posets [87, 91], and vector spaces categories [77–79]. These were unified and extended to a part of representation theory of general orders.

In this chapter we characterize in a combinatorial way the finitely generated Zavadskij modules over a finite-dimensional algebra \( A \). Precisely, we prove that the indecomposable uniserial \( A \)-modules are Zavadskij \( A \)-modules. Furthermore, since the Zavadskij property of modules carries over to direct summands, but it is not invariant under formation of direct sums, we use the mast associated to each uniserial indecomposable direct summand \( A \)-module in order to give a criterion that determinate when the direct sum of them inherits the property. Finally, we find formulas for the number of indecomposable Zavadskij \( A \)-modules over a Dynkin algebra.

3.1 Zavadskij modules and its relation to uniserial modules

For simplicity of the exposition, although the results in [69] are more general, we will restrict ourselves to the case when \( A \) is a finite-dimensional \( k \)-algebra.

Let \( M \) be an \( A \)-module. A module \( L \) over \( A \) is said to be \( M \)-projective if for any epimorphism \( g : M \rightarrow N \) of \( A \)-modules the corresponding induced homomorphism \( \text{Hom}_A(L,M) \xrightarrow{g^*} \text{Hom}_A(L,N) \) is surjective. In other words, if \( f : L \rightarrow N \) is any morphism, then there exists a morphism \( h : L \rightarrow M \) such that the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
M & \xrightarrow{h^*} & N
\end{array}
\]

is commutative. If \( M \) is projective, then it is two-sided projective. These definitions extend to complexes of projective modules.
A module $L$ over $A$ is said to be $M$-injective if for any monomorphism $g : N \hookrightarrow M$ the induced homomorphism $g^* : \text{Hom}_A(M, L) \rightarrow \text{Hom}_A(N, L)$ is surjective. In other words, if $f : N \rightarrow L$ is any morphism, then there exists a morphism $h : M \rightarrow L$ such that the diagram

\[
\begin{array}{ccc}
N & \xhookrightarrow{g} & M \\
\downarrow{f} & & \downarrow{h} \\
L & \xhookleftarrow{h} & \end{array}
\]

commutes, that is, $f = h \circ g = g^*(h)$. Moreover, if $M$ itself is $M$-injective then $M$ is said to be quasi-injective. As an example, a projective (resp. injective) $A$-module is $M$-projective (resp. $M$-injective) for each $A$-module $M$, whereas any injective module is quasi-injective.

**Definition 3.1.** An $A$-module $M$ is a Zavadskij module if all of its submodules are $M$-projectives and each factor module is $M$-injective. In other words, this says that for each homomorphism $f : U \rightarrow W$ from a submodule $U$ to a factor module $W$ of $M$, the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & W \\
\downarrow{h} & & & \\
\end{array}
\]

can be completed by an endomorphism $f$ of $M$. Thus, for $M$ to be a Zavadskij module, it suffices to see that each submodule is $M$-projective and that $M$ itself is quasi-injective.

**Example 3.1.** Let $Q$ be the quiver $1 \rightarrow 2$ then the path algebra $A = \mathbb{k}Q$ is the matrix ring $[\begin{smallmatrix} \mathbb{k} & 0 \\ 0 & \mathbb{k} \end{smallmatrix}]$ and the indecomposable projective $A$-modules are Zavadskij modules, whereas the module $A_A$ is not a Zavadskij module because it is not a quasi-injective module. Thus, in general a direct sum of Zavadskij modules is not a Zavadskij module.

**Relation to uniserial modules.** A composition series of an $A$-module $M$ is a chain of submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ such that $M_{i+1}/M_i$ is an $A$-module simple for $i = 0, \ldots, n - 1$. The modules $M_{i+1}/M_i$ are called the composition factors of $M$. The Jordan-Hölder theorem tells us that any two composition series $(M_i)_{i=1}^n$ and $(N_j)_{j=1}^l$ have the property that $n = l$ and that there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $M_{i+1}/M_i$ is isomorphic to $N_{\sigma(i+1)}/N_{\sigma(i)}$ for each $i = 0, \ldots, n - 1$. In particular, the number $n$ of modules in a composition series is called the length of $M$. 
Remark 3.1. The finitely generated modules over a finite-dimensional algebra are finite-dimensional modules, i.e. its dimension as a $k$-vector space is finite and also, they are length-finite modules.

A nonzero module $M$ over a finite-dimensional $k$-algebra $A$ is uniserial if the lattice of its submodules forms a chain, i.e. if every two submodules of $M$ are comparable. Clearly, all subfactors of a uniserial module are again uniserial. The lattice of proper nonzero submodules of a length-finite $A$-module $M$ is then a finite chain with maximal element $\text{rad} M$ and minimal element $\text{soc} M$, where these are the notations for Jacobson radical and socle of $M$ respectively. Note that, if $J = \text{rad} A$ then the submodules of $M$ are given by $J^l M$, with $l = 0, \ldots, \text{length}(M)$ (see [2]).

If $M$ is a uniserial $A$-module with submodules $0 = M_0 \subset M_1 \subset M_2 \cdots \subset M_n = M$ and composition factors $U_i = M_{i+1}/M_i$ then there are ring homomorphisms

$$\text{End}_A M \rightarrow \text{End}_A U_i$$

which are all injective if the $U_i$ are pairwise nonisomorphic. In this case, the maps (3.1) are monomorphisms between skewfields. In this setting we recall that a uniserial $A$-module $M$ is tame if the $U_i$ are mutually nonisomorphic and the maps (3.1) are isomorphisms.

In relation to uniserial modules, a characterization of Zavadskij modules was achieved by Rump. Let us first consider the indecomposables.

**Theorem 3.1.** [69, Proposition 3] Let $M$ be a module over a finite-dimensional $k$-algebra. $M$ is an indecomposable Zavadskij module if and only if $M$ is a tame uniserial module.

Now, let consider us the general situation, in that case, an $A$-module $M$ has a finite decomposition of the form

$$M = M_1^{n_1} \oplus \ldots \oplus M_r^{n_r}$$

(3.2)

where all of $M_i$ are indecomposable pairwise non isomorphic, the module $M_1 \oplus \ldots \oplus M_r$ is called the reduced part of $M$.

The following results regarding the characterization of Zavadskij $A$-modules are due to Rump.

**Theorem 3.2.** [69, Proposition 4] An $A$-module is a Zavadskij module if and only if its reduced part is a Zavadskij module.

**Theorem 3.3.** [69, Theorem 1] An $A$-module $M$ is a Zavadskij module if and only if it satisfies the following conditions.

1. $M$ can be decomposed into tame uniserial modules.
2. Two indecomposable direct summands of $M$ with a common composition factor are isomorphic.
3.2 A combinatorial characterization

In this section, we prove theorems 3.5 and 3.7 and Corollary 3.6 in order to characterize Zavadskij modules over finite-dimensional algebras using the mast associated to each indecomposable uniserial module.

Via Theorem 1.6, it is suffice to consider a bound quiver algebra $A = kQ/I$, where $Q$ is a connected quiver and $I$ is an admissible ideal of the path algebra $kQ$. A path $p$ in the quiver $Q$ is said to be a mast of the uniserial $A$-module $M$ if $\text{length}(p) = \text{length}(M) - 1$ and $pM \neq 0$. We let $\mu(x)$ denote the multiplicity of a vertex $x$ in a mast $p$ which is the number of times that the vertex $x$ occurs in $p$.

**Example 3.2.** Let $A = kQ/I$ be, where $Q$ is the quiver

$$
\begin{array}{c}
\text{1} \\
\downarrow \gamma \\
\text{3}
\end{array}
\quad
\begin{array}{c}
\text{2} \\
\downarrow \beta \\
\text{3}
\end{array}
\quad
\begin{array}{c}
\text{1} \\
\downarrow \gamma \\
\text{4}
\end{array}
\quad
\begin{array}{c}
\text{1} \\
\downarrow \delta \\
\text{3}
\end{array}
$$

and $I = \langle \beta \alpha - \delta \gamma \rangle$. Then, the paths of length one $\alpha, \beta, \gamma$, and $\delta$ are masts of the uniserial modules

$$
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array},
\begin{array}{c}
k \\
\downarrow 0 \\
0
\end{array}
$$

respectively. Of course, the simple modules $S_1 = \langle e_1 \rangle$, $S_2 = \langle e_2 \rangle$, $S_3 = \langle e_3 \rangle$, and $S_4 = \langle e_4 \rangle$ are uniserial whose masts are the stationary paths $e_1, e_2, e_3$, and $e_4$ respectively. However, neither $\beta \alpha$ nor $\delta \gamma$ is a mast of a uniserial $A$-module.

According to [13,93], we have some elementary observations.

(1) Not every path in $kQ$ with nonzero image in $A$ needs to occur as a mast of a uniserial module.

(2) If the quiver $Q$ has no double arrows, then each uniserial $A$-module has a unique mast. Conversely, the uniqueness of masts in all uniserial modules implies absence of double arrows.

(3) For each uniserial $A$-module $M$ with sequence $(S_1, \ldots, S_{l+1})$ of consecutive composition factors, there exists at least one path $p$ of length $l$ in $kQ$ such that $pM \neq 0$ (see [13]); necessarily $p$ passes in order through the sequence $(1, \ldots, l+1)$ of those vertices in $Q_0$ which represent the simple modules $S_i$. Moreover, $0 \neq \text{soc } M = J^lM$ and $J^l$ is generated by the images of the paths of length $l$. 

Let \( p \) be a path in a quiver \( Q \) then a path \( w \) is said to be a **right subpath** of \( p \) if there exists a path \( r \) with \( p = rw \). Moreover, a **detour** on the path \( p \) is a pair \((\alpha, w)\) with \( \alpha \) an arrow and \( w \) a right subpath of \( p \), where \( \alpha w \) is a path in \( Q \) which is not a right subpath of \( p \), but there exists a right subpath \( w' \) of \( p \) with \( \text{length}(w') \geq \text{length}(w)+1 \) such that the endpoint of \( w' \) coincides with the endpoint of \( \alpha \).

**Theorem 3.4.** [Lemma 2.1.1] The finite-dimensional hereditary algebra \( A = \mathbb{k}Q \) admits infinitely many non isomorphic uniserial modules if and only if \(|\mathbb{k}| = \infty \) and \( Q \) contains a subquiver of the form

\[
\circ \rightarrow \circ \rightarrow \cdots \rightarrow \circ.
\]

**Theorem 3.5.** Let \( Q \) be a finite connected quiver and let \( A = \mathbb{k}Q/I \) be the algebra bounded by an admissible ideal \( I \). Then an indecomposable \( A \)-module \( M \) is a Zavadskij module if and only if it is a uniserial and for any mast \( p \) of \( M \) it holds that \( \mu(x) = 1 \) for every vertex \( x \) that \( p \) passes through. In particular, if the algebra \( A \) is hereditary then the indecomposable \( A \)-module \( M \) is a Zavadskij module if and only if it is uniserial.

**Proof.** First we consider the case where \( A \) is a hereditary algebra. The necessary condition follows immediately from Theorem 3.1. On the other hand, we consider \( M \) as a representation \( M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1} \) of the quiver \( Q \). We suppose that the mast \( p \) has the form \( 1 \overset{\alpha_1}{\longrightarrow} 2 \overset{\alpha_2}{\longrightarrow} \ldots \overset{\alpha_{n-1}}{\longrightarrow} n \) thus, \( M \) is defined by the formulas

\[
\varphi_\alpha = \begin{cases} 
1_\mathbb{k}, & \text{if } \alpha \in \{\alpha_1, \ldots, \alpha_{n-1}\}, \\
0, & \text{otherwise.}
\end{cases}
\]

and \( M_i = \begin{cases} \mathbb{k}, & \text{if } i \in \{1, \ldots, n\}, \\
0, & \text{otherwise.}
\end{cases} \)

Clearly the rings of endomorphisms \( \text{End} M \) and \( \text{End} (M_{i+1}/M_i) \) are isomorphic to \( \mathbb{k} \). Thus, the maps (3.1) are isomorphism. Moreover, the composition factors are mutually nonisomorphic. In indeed, we assume that \( \dim_{\mathbb{k}}(\text{Hom}_A(P_i, M)) > 1 \) for some \( i \). Since \( M \) is finitely generated then there exists a projective resolution of \( M \) of the form

\[
0 \longrightarrow P' \longrightarrow P \longrightarrow M \longrightarrow 0,
\]

with \( P = \bigoplus_{i \in Q_0} d_i P_i \) and \( P' = \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} P_{t(\alpha)} \), where \( d_i \) denotes the dimension of \( M_i \) and \( d_i P_i \) stands for the direct sum of \( d_i \) copies of \( P_i \). It is clear that the indecomposable projective module \( P_i \) appears once, thus \( \dim_{\mathbb{k}}(\text{Hom}_A(P_i, M)) = 1 \). Therefore, the simple module \( S_i \) associated to the vertex \( i \) appears only once as factor composition of \( M \). This is enough to prove that \( M \) is a tame module. Theorem 3.1 guarantees the result.

In the general case, let \( M \) be an indecomposable Zavadskij \( A \)-module over a bound quiver algebra \( A = \mathbb{k}Q/I \). Theorem 3.1 implies that \( M \) is a tame uniserial module of length \( l + 1 \geq 1 \), therefore indecomposable simple modules \( S_i, 1 \leq i \leq l+1 \) in the sequence of composition factors \((S_1, \ldots, S_{l+1})\) are pairwise nonisomorphic. Additionally, if \( p \) is a mast of \( M \) then we note that \( p \) is a path in \( \mathbb{k}Q \) of length \( l \) such that \( p \notin I \) and \( pM \neq 0 \); necessarily \( p \) passes in order through the sequence \((1, \ldots, l+1)\) of those vertices in \( Q_0 \) which represent the simple modules \( S_i \). Thus, \( \mu(x) = 1 \) for any \( x \in p \). On the other hand, let us suppose that \( M \) is uniserial and that \( p \) is a mast of \( M \) with \( \mu(x) = 1 \) for any \( x \in p_0 \). Therefore, factors composition of \( M \) are precisely the indecomposable simple modules \( S_i \) corresponding bijectively to the set \( p_0 = \{1, \ldots, l, l+1\} \) of vertices in \( p \), therefore \( S_i \) is not
isomorphic to \( S_j \) if \( i \neq j \). Since End \( M \cong \mathbb{k} \) we have that \( M \) is a tame \( A \)-module. Thus, Theorem\textsuperscript{3.1} allows to conclude that \( M \) is an indecomposable Zavadskij module. We are done. \( \square \)

**Corollary 3.6.** If \( Q \) is tree quiver then the following bijection holds

\[
\{ \text{paths in } Q \} \cong \{ \text{Indecomposable Zavadskij } \mathbb{k}Q\text{-modules} \}.
\]

**Proof.** It suffices to note that each indecomposable uniserial module determines a unique mast in \( Q \) because there are not double arrows in \( Q \). Furthermore, each path in \( Q \) determines exactly one uniserial \( \mathbb{k}Q\)-module because there are not detours in \( Q \). We are done. \( \square \)

**Theorem 3.7.** Let \( Q \) be a finite connected quiver and let \( A = \mathbb{k}Q/I \) be the algebra bounded by an admissible ideal \( I \) then a direct sum \( M \) of indecomposable Zavadskij \( A \)-modules defined as in formula \textsuperscript{1.2} is a Zavadskij module if and only if for each pair \( i \neq j \) the masts \( p_i \) and \( p_j \) associated to \( M_i \) and \( M_j \) respectively do not have common vertices.

**Proof.** Let \( p_i : a_0 \rightarrow a_1 \rightarrow \ldots \rightarrow a_s \) and \( p_j : a'_0 \rightarrow a'_1 \rightarrow \ldots \rightarrow a'_s \) be the masts associated to the indecomposable \( A \)-modules \( M_i \) and \( M_j \) respectively, where \( i \neq j \). First, we suppose that \( p_i \) and \( p_j \) share the vertex \( a'_k = a_k \). Since \((M_i)_{a_j} = \mathbb{k}\) for all \( 1 \leq j \leq s \), then \((M_i)_{a_j} = \mathbb{k}\). Thus, the simple \( S_{a_k} \) is a composition factor of \( M_i \) and of \( M_j \). Then \( M_i \cong M_j \) which is contradiction (see Theorem\textsuperscript{3.3}). On the other hand, we assume that \( p_i \) and \( p_j \) do not have a vertex in common. Then \( M \) is of finite length because the \( \mathbb{k}\)-algebra \( A \) is finite-dimensional, thus the reduced part \( \mathcal{M} = \bigoplus_{i=1}^{n} M_i \) of \( M \) can be decomposed in tame uniserial \( A \)-modules provided that each module \( M_i \) is an indecomposable Zavadskij \( A \)-module. Now, if the simple \( S_{a} \) is a composition factor of \( M_i \) and \( M_j \) with \( i \neq j \), then \((M_i)_{a_j} \cong \mathbb{k}\) and \((M_j)_{a_j} \cong \mathbb{k}\). Therefore \( a \) is a vertex in \( p_i \) and \( p_j \) which contradicts the hypothesis, so \( M_i \) and \( M_j \) do not have a common composition factor. Since theorems\textsuperscript{3.2} and \textsuperscript{3.3} establish that \( \mathcal{M} \) and \( M \) are Zavadskij modules. \( \square \)

### 3.3 On the number of indecomposable Zavadskij modules over Dynkin algebras

In this section, we give explicit formulas for the number of Zavadskij modules over Dynkin algebras of type \( A_n, D_n, E_6, E_7, E_8 \).

An algebra \( A \) is said to be **right serial** if every indecomposable projective right \( A \)-module is uniserial. An algebra \( A \) is called **left serial** if every indecomposable projective left \( A \)-module is uniserial. It is well known that a basic \( \mathbb{k}\)-algebra \( A \) is right serial if and only if, for every vertex \( x \) of its ordinary quiver \( Q_A \), there exists at most one arrow of source \( x \) (see \textsuperscript{[2]}). We suppose that \( A = \mathbb{k}Q \) is any right serial \( \mathbb{k}\)-algebra associated to a Dynkin quiver \( Q \) of type \( \Delta \in \{A_n, D_n, E_6, E_7, E_8\} \), where \( Q_0 = \{1, \ldots, n\} \) and \( 1 \leq i \leq n \) is the unique sink in \( Q \). For any \( i \geq 1 \), we define the function

\[
Z_i^\Delta : \mathbb{N}_{\geq i} \rightarrow \mathbb{N}
\]
where $Z^A_i(n)$ is the number of indecomposable Zavadskij $A$-modules and $\mathbb{N}_{\geq i}$ denotes the set $\{i, i+1, \ldots\}$. Note that the path algebra $A = \mathbb{k}Q$ of a tree quiver $Q$ is a right serial algebra if and only if $Q$ has only one sink vertex $i \in Q_0$.

### 3.3.1 Case $\mathbb{A}_n$

Now, the number of Zavadskij modules over a Dynkin algebra $\mathbb{k}Q$ of type $\mathbb{A}_n$ is considered. In this subsection, we suppose that the underlying graph $\overline{Q}$ of $Q$ has the numbering of vertices and edges as in Figure 3.1.

![Figure 3.1. Numbering of vertices and edges in $\overline{Q}$ ($\mathbb{A}_n$ case)](image)

First, we consider that the corresponding algebra $A$ is associated to a quiver with just one sink vertex.

**Lemma 3.8.** For $n > 1$ fixed, let $A = \mathbb{k}Q$ be the right serial Dynkin algebra of type $\mathbb{A}_n$ and $i$ the unique sink vertex in $Q$. Then the number of indecomposable Zavadskij $A$-modules is given by $Z^A_i(n) = t_{i-1} + t_{n-i} + n$, where $t_j$ denotes the $j$-th triangular number.

**Proof.** It suffices to consider the number of paths in $Q$ (see Corollary 3.6). Note that if $j \in \{1, n\}$ then $Z^A_i(n) = t_n$. Actually, there are $n-i$ paths of length $i$ for all $i = 0, \ldots, n-1$.

Since the quiver $Q$ has the shape $1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} i \xrightarrow{\alpha_i} \cdots \xrightarrow{\alpha_{n-1}} n$ then $t_i$ is the number of paths in $Q$, which are subpaths of the path $\alpha_1 \cdots \alpha_{i-1}$. Analogously, $t_{n-(i-1)}$ is the number of paths in $Q$, which are subpaths of the path $\alpha_{n-1} \cdots \alpha_i$. Since the stationary path $e_i$ has been counted twice then we have that $Z^A_i(n) = t_i + t_{n-(i-1)} - 1$, in other words, $Z^A_i(n) = t_{i-1} + t_{n-i} + n$. \qed

**Theorem 3.9.** Let $A = \mathbb{k}Q$ be a Dynkin algebra of type $\mathbb{A}_n$ and $p$ the number of sink vertices in $Q$, then the number of indecomposable Zavadskij modules is given by

$$m_Q = n + \sum_{k=1}^{p} (t_{l_k-l_{k-1}} + t_{l_k-l_k}) ,$$

where $l_0 = 1$, $l_p = n$, $i_1 < l_1 < i_2 < \cdots < i_{p-1} < l_{p-1} < i_p$, $\{i_1, \ldots, i_p\}$ is the set of sink vertices in $Q_0$, and the $l_k$ are source vertices.

**Proof.** The structure of the quiver $Q$ allows us to conclude that there exists exactly one source vertex between two consecutive sink vertices $i_k$ and $i_{k+1}$. Thus we can build the sequence $i_1 < l_1 < i_2 < \cdots < i_{p-1} < l_{p-1} < i_p$ of vertices in $Q_0$, where the $l_k$ are source vertices in $Q_0$. For $1 \leq k \leq p$, let $Q^{(k)}$ be the full subquiver of $Q$ whose vertices are in the support of the indecomposable injective representation $I_{i_k}$ at vertex $i_k$. We re-enumerated vertices of $Q^{(k)}$ by using numbers $1, \ldots, l_k - l_{k-1} + 1$. The quiver $Q^{(k)}$ is of type $\mathbb{A}_{l_k - l_{k-1} + 1}$ with a unique sink vertex $i_k - l_{k-1} + 1$. According to Lemma 3.8 and Corollary 3.6 we
observe that the number of masts in $Q^{(k)}$ is given by

$$Z_{i_k-l_{k-1}+1}^k(l_k - l_{k-1} + 1) = t_{i_k-l_{k-1}} + t_{l_k-i_k} + l_k - l_{k-1} + 1.$$  

Thus, the number of indecomposable Zavadskij $A$-modules is given by the formula

$$m_Q = \sum_{k=1}^{p} (t_{i_k-l_{k-1}} + t_{l_k-i_k} + l_k - l_{k-1} + 1) - (p - 1),$$

where number $p - 1$ corresponds to $A$- simple modules $S_1, \ldots, S_{p-1}$ which have been counted twice. Since $l_0 = 1$ and $l_p = n$ we have that $m_Q = n + \sum_{k=1}^{p} (t_{i_k-l_{k-1}} + t_{l_k-i_k})$. □

Table 3.1 shows the values of the formula $Z_{i_1}^i(n) = t_{i-1} + t_{n-i} + n$, for $3 \leq n \leq 10$ and a unique sink at vertex $i$. Note that, it holds that $a_{i,j+1} + a_{i+1,j} = a_{i,j} + a_{i+1,j+1} + 1$. In particular, sequences $a_{i+1,j} + a_{i,j+1}$, $i = 1, 2$ correspond respectively to subsequences of the sequences A027688 and A122793 in the OEIS.

<table>
<thead>
<tr>
<th>$i$ \ $n$</th>
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</table>

Table 3.1. Values of the formula $Z_{i_1}^i(n)$

### 3.3.2 Case $\mathbb{D}_n$

Now, the number of Zavadskij modules over a Dynkin algebra $\mathbb{A}Q$ of type $\mathbb{D}_n$ is considered. In this subsection, we suppose that the underlying graph of $Q$ has the numbering of vertices and edges as in Figure 3.2.

![Figure 3.2. Numbering of vertices and edges in $Q$ (D case)](image)

First, let us consider that $A$ is a right serial Dynkin algebra of type $\mathbb{D}_n$.

**Lemma 3.10.** Let $A = \mathbb{Q}Q$ be a right serial Dynkin algebra of type $\mathbb{D}_n$ and $i$ the unique sink vertex in $Q$. Then the number of indecomposable Zavadskij $A$-modules is given by

$$Z_{i_1}^D(n) = t_{i-1} + t_{n-i} + n - 1.$$
Proof. Let us consider the number of paths in \( Q \). We do not loss generality if we suppose that \( 1 \leq i \leq n - 1 \). Since the subquiver \( Q' \) of \( Q \) obtained from \( Q \) by removing the vertex \( n \) is of type \( A_n \), Lemma 3.8 allows us conclude that the number of paths in \( Q \) which are paths in \( Q' \) is given by

\[
Z^h_i(n - 1) = t_{i-1} + t_{n-1-i} + n - 1
\]

Further, the number of subpaths of the path \( \alpha_{n-1} \ldots \alpha_i \) in \( Q \) starting at the vertex \( n \) is given by \( n - i \). Thus, we have that

\[
Z^p_i(n) = t_{i-1} + t_{n-1-i} + n - 1 + n - i = t_{i-1} + t_{n-i} + n - 1
\]

\[
\square
\]

Corollary 3.11. Let \( A = kQ \) be a Dynkin algebra of type \( \mathbb{D}_n \), and \( p \) the number of sink vertices in \( Q_0 \setminus \{n - 1, n\} \) then the number of indecomposable Zavadskij modules is given by

\[
m_Q = \left\{ \sum_{k=1}^{p} \left( t_{i_k-1} + t_{l_k-1} \right) + n + 2 \mid \begin{array}{l}
\text{if } s(\alpha_{n-1}) = s(\alpha_{n-2}) = n - 2 \text{ and } i_p \neq n - 2, \\
\text{if } \alpha_{n-1} = s(\alpha_{n-2}) = n - 2 \text{ and } i_p = n - 2, \\
\text{if } t(\alpha_{n-1}) = t(\alpha_{n-2}) = n - 2, \\
\text{if } s(\alpha_{n-1}) = t(\alpha_{n-2}) = n - 2 \text{ and } i_p \neq n - 2, \\
\text{if } \alpha_{n-1} = t(\alpha_{n-2}) = n - 2 \text{ and } i_p \neq n - 2, \\
\text{if } \alpha_{n-1} = t(\alpha_{n-2}) = n - 2 \text{ and } i_p = n - 2, \\
\text{if } s(\alpha_{n-1}) = t(\alpha_{n-2}) = n - 2 \text{ and } i_p = n - 2 \end{array} \right. 
\]

where \( l_0 = 1, l_p = n - 2, i_1 < l_1 < i_2 < \cdots < i_{p-1} < l_{p-1} < i_p, \{i_1, \ldots, i_p\} \) is the set of sink vertices in \( Q_0 \setminus \{n - 1, n\} \), and the \( l_k \) are source vertices.

Proof. It suffices to enumerate the paths in \( Q \) (see Corollary 3.6). Let \( Q' \) be the full subquiver of \( Q \) such that \( Q'_0 = Q_0 \setminus \{n - 1, n\} \). Theorem 3.9 implies that the number of paths in \( Q' \) is given by

\[
m_{Q'} = \sum_{k=1}^{p} \left( t_{i_k-1} + t_{l_k-1} \right) + n - 2.
\]

Moreover, we consider the following cases.

Case 1. \( s(\alpha_{n-1}) = s(\alpha_{n-2}) = n - 2 \) and \( i_p \neq n - 2 \). Clearly, \( Q \) has 4 paths that are not paths in \( Q' \) which are determined by the vertices \( n-1 \) and \( n \) and the arrows \( \alpha_{n-1} \) and \( \alpha_{n-2} \).

Case 2. \( s(\alpha_{n-1}) = s(\alpha_{n-2}) = n - 2 \) and \( i_p = n - 2 \). In this case, \( Q \) has \( 2(n - l_{p-1}) \) paths that are not paths in \( Q' \). Indeed, the number of subpaths of the mast \( \alpha_{l_{p-1}} \ldots \alpha_{n-2} \) in \( Q \) ending at vertex \( n - 1 \) is \( n - l_{p-1} \). Analogously, the number subpaths of the mast \( \alpha_{l_{p-1}} \ldots \alpha_{n-1} \) in \( Q \) ending at vertex \( n \) is \( n - l_{p-1} \).
Chapter 3. Zavadskij Modules over Finite-Dimensional Algebras

Case 3. \( t(\alpha_{n-1}) = t(\alpha_{n-2}) = n - 2 \). In this case, \( Q \) has \( 2(n - i_p) \) paths that are not paths in \( Q' \). Indeed, the number of subpaths of the mast \( \alpha_{n-2} \ldots \alpha_{i_p} \) in \( Q \) starting at vertex \( n - 1 \) is \( n - i_p \). Analogously, we have the same result for the number of subpaths of the mast \( \alpha_{n-1} \ldots \alpha_{i_p} \) in \( Q \) starting at vertex \( n \).

Case 4. \( s(\alpha_{n-1}) = t(\alpha_{n-2}) = n - 2 \) and \( i_p \neq n - 2 \). In this case, \( Q \) has \( n - i_p + 3 \) paths that are not paths in the quiver \( Q' \). Indeed, the number of subpaths of the path \( \alpha_{n-2} \ldots \alpha_{i_p} \) in \( Q \) starting at vertex \( n - 1 \) is \( n - i_p \). We also count paths \( \alpha_{n-2} \alpha_{n-1}, \alpha_{n-1} \) and the stationary path at vertex \( n \).

Case 5. \( s(\alpha_{n-1}) = t(\alpha_{n-2}) = n - 2 \) and \( i_p = n - 2 \). In this case, \( Q \) has \( n - l_{p-1} + 3 \) paths that are not paths in the quiver \( Q' \).

Since all possible cases have been considered, we are done. \( \square \)

3.3.3 Case \( E_6, E_7, \) and \( E_8 \)

Finally, let us turn our attention to Dynkin algebras \( kQ \) of type \( E \). In this subsection, we suppose that the underlying graph \( \overline{Q} \) of \( Q \) has the numbering of vertices and edges as in Figure 3.3.

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node[draw, circle] (1) at (0,0) {1};
  \node[draw, circle] (2) at (1,0) {2};
  \node[draw, circle] (3) at (2,0) {3};
  \node[draw, circle] (4) at (3,0) {4};
  \node[draw, circle] (5) at (4,0) {5};
  \node[draw, circle] (6) at (2,1) {6};
  \node[draw, circle] (7) at (1,1) {7};
  \node[draw, circle] (8) at (0,1) {8};
  \draw[-latex] (1) -- (2);
  \draw[-latex] (2) -- (3);
  \draw[-latex] (3) -- (4);
  \draw[-latex] (4) -- (5);
  \draw[-latex] (5) -- (6);
  \draw[-latex] (6) -- (7);
  \draw[-latex] (7) -- (8);
\end{tikzpicture}
\caption{Numbering of vertices and edges in \( \overline{Q} \) (\( E \) case)}
\end{figure}

Corollary 3.12. Let \( A = kQ \) be a Dynkin algebra of type \( E_n, n = 6, 7, 8, \) and \( p \) the number of sink vertices in \( Q_0 \setminus \{n\} \) then the number of indecomposable Zavadskij modules...
is given by

\[ m_Q = \begin{cases} 
\sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n + l_f - l_s + 1, & \text{if } s(\alpha_{n-1}) = t(\alpha_2) = t(\alpha_3) = 3, \\
\sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n + l_f - 2, & \text{if } s(\alpha_{n-1}) = s(\alpha_2) = t(\alpha_3) = 3, \\
\sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n - i_s + 4, & \text{if } t(\alpha_{n-1}) = s(\alpha_2) = t(\alpha_3) = 3, \\
\sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n - l_s + 4, & \text{if } s(\alpha_{n-1}) = t(\alpha_2) = s(\alpha_3) = 3, \\
\sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n + i_f - 2, & \text{if } t(\alpha_{n-1}) = t(\alpha_2) = s(\alpha_3) = 3, \\
\sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n + i_f - i_s + 1, & \text{if } t(\alpha_{n-1}) = s(\alpha_2) = s(\alpha_3) = 3. 
\end{cases} \]

where \( l_0 = 1, l_p = n - 1, i_1 < l_1 < i_2 < \cdots < i_{p-1} < l_{p-1} < i_p, \{i_1, \ldots, i_p\} \) is the set of sink vertices in \( Q_0 \setminus \{n-1, n\} \), the \( i_k \) are source vertices, \( i_s \) (resp. \( l_s \)) is the predecessor-sink (resp. predecessor-source) of vertex \( 3 \), and \( i_f \) (resp. \( l_f \)) is the successor-sink (resp. successor-source) of vertex \( 3 \).

**Proof.** Let \( Q' \) be the full subquiver of \( Q \) such that \( Q'_0 = Q_0 \setminus \{n\} \). Theorem 3.9 implies that the number of paths in \( Q' \) that are also paths in \( Q \) is given by the identity

\[ m_{Q'} = \sum_{k=1}^{p} \left( t_{i_k-l_{k-1}} + t_{l_k-i_k} \right) + n - 1. \]

Now, we consider the following cases.

**Case 1.** \( s(\alpha_{n-1}) = t(\alpha_2) = t(\alpha_3) = 3, n = 6, 7, 8 \). \( Q \) has \( 2 + l_f - l_s \) paths that are not paths in \( Q' \) which are determined by the vertex \( n \) and the arrow \( \alpha_{n-1} \). Moreover, the number of paths in \( Q \) containing properly the arrow \( \alpha_{n-1} \) is \( l_f - l_s \).

**Case 2.** \( s(\alpha_{n-1}) = s(\alpha_2) = t(\alpha_3) = 3, n = 6, 7, 8 \). \( Q \) has \( l_f - 1 \) paths that are not paths in \( Q' \) which are determined by the vertex \( n \) and the arrow \( \alpha_{n-1} \). In this case, the number of paths in \( Q \) containing properly the arrow \( \alpha_{n-1} \) is \( l_f - 3 \).

**Case 3.** \( t(\alpha_{n-1}) = s(\alpha_2) = t(\alpha_3) = 3, n = 6, 7, 8 \). \( Q \) has \( 5 - i_s \) paths that are not paths in \( Q' \) which are determined by the vertex \( n \) and the arrow \( \alpha_{n-1} \). In this case, the number of paths in \( Q \) containing properly the arrow \( \alpha_{n-1} \) is \( 3 - i_s \).

**Case 4.** \( s(\alpha_{n-1}) = t(\alpha_2) = s(\alpha_3) = 3, n = 6, 7, 8 \). \( Q \) has \( 5 - l_s \) paths that are not paths in \( Q' \) which are determined by the vertex \( n \) and the arrow \( \alpha_{n-1} \). In this case, the number of paths in \( Q \) containing properly the arrow \( \alpha_{n-1} \) is \( 3 - l_s \).

**Case 5.** \( t(\alpha_{n-1}) = t(\alpha_2) = s(\alpha_3) = 3, n = 6, 7, 8 \). \( Q \) has \( i_f - 1 \) paths that are not paths in \( Q' \) which are determined by the vertex \( n \) and the arrow \( \alpha_{n-1} \). In this case, the number of paths in \( Q \) containing properly the arrow \( \alpha_{n-1} \) is \( i_f - 3 \).
**Case 6.** \( t(\alpha_{n-1}) = s(\alpha_2) = s(\alpha_3) = 3, \; n = 6, 7, 8. \) \( Q \) has 2 + \( i_f - i_s \) paths that are not paths in \( Q' \) which are determined by the vertex \( n \) and the arrow \( \alpha_{n-1} \). In this case, the number of paths in \( Q \) containing properly the arrow \( \alpha_{n-1} \) is \( i_f - i_s \). Note also that if 3 is a sink vertex (or source vertex) in \( Q \) then the number of indecomposable Zavadskij modules is given by

\[
m_Q = \sum_{k=1}^{p} (t_{i_k-l_{k-1}} + t_{l_k-i_k}) + n + 1.
\]

Since all possible cases have been considered, we are done. \( \square \)

Table 3.2 shows the number of indecomposable Zavadskij modules over some Dynkin algebras of type \( \mathbb{E} \).

<table>
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<th>( i ) ( \setminus ) ( n )</th>
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Table 3.2. Values of the formula \( Z_{i}^{\mathbb{E}}(n) \)
CHAPTER 4

Integer partitions arising from $\tau$-orbits in the AR quiver of a Dynkin algebra

In this Chapter, we introduce the $\tau$-orbit partitions of integer numbers which arise from the $\tau$-orbits in the module category of a Dynkin algebra $\mathbb{k}Q$, where $\tau$ is the Auslander-Reiten translation in $\text{mod}\mathbb{k}Q$. We give a description of such partitions obtaining explicit formulas for the number of them, when the Dynkin diagram is simply-laced. In that way, the number of $\tau$-orbit partitions of type $A_n$ and $D_n$ correspond to the sequences A016116 [83] and A000034 [84] respectively, in the Online Encyclopedia of Integer Sequences. Additionally, we introduce an algorithm to compute the length of the $\tau$-orbits in the $A_n$ case using tiled orders.

4.1 Combinatorial AR-quiver of a Dynkin algebra

In this section, we recall some concepts to give a combinatorial characterization of the Auslander-Reiten quiver of a Dynkin algebra of finite type. We fix a Dynkin quiver $Q$ of type $A$, $D$ or $E$, where $Q_0 = \{1, \ldots, n\}$.

The **Weyl group** $W = W_Q$ associated to the quiver $Q$ has a distinguished set of generators $S = \{s_1, \ldots, s_n\}$, with relations $s_i^2 = e$ (the identity element), $s_is_j = s_js_i$ if there is no arrow between $i$ and $j$, and $s_is_js_i = s_js_is_j$ if there is exactly one arrow between $i$ and $j$. Typically the element $s_i$ can be viewed as the **reflection** $s_i : \mathbb{Q}^n \longrightarrow \mathbb{Q}^n$ at vertex $i \in Q_0$, i.e. the $\mathbb{Q}$-linear map $s_i$ given by

$$s_i(x) = x - 2(x, e_i)e_i$$

for $x \in \mathbb{Q}$, where $(-,-)$ denotes the **symmetric bilinear form** corresponding to the quadratic form $q_Q(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)}x_{t(\alpha)}$ associated to $Q$. Thus,

$$(x,y) = \sum_{i \in Q_0} x_iy_i - \frac{1}{2} \sum_{\alpha \in Q_1} \{x_{s(\alpha)}y_{t(\alpha)} + x_{t(\alpha)}y_{s(\alpha)}\}.$$
Clearly \( q_Q(x) = (x, x) \) for all \( x \in \mathbb{Q}^n \) and \( (x, y) = \frac{1}{2} [q_Q(x + y) - q_Q(x - y)] \) for all \( x, y \in \mathbb{Q}^n \). In terms of the coordinates \( x_i \) of \( x \) in the canonical basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{Q}^n \), we see that \( y = s_i(x) \) has coordinates

\[
y_j = \begin{cases} x_j & \text{if } j \neq i, \\
-x_i + \sum_{k=1}^{j-1} x_k & \text{if } j = i,
\end{cases}
\]

where the sum is taken over all edges \( k \rightarrow i \) in the underlying graph \( \overline{Q} \).

**Example 4.1.** Let \( Q \) be a Dynkin quiver of type \( A_n \). The corresponding Weyl group is isomorphic to the symmetric group on \( n + 1 \) letters. We can take the isomorphism to send \( s_i \) to the adjacent transposition \((i, i + 1)\). Particularly, if \( Q \) is the quiver \( 1 \leftarrow 3 \rightarrow 2 \) whose underlying graph is the Dynkin diagram \( A_3 \), then the reflections \( s_1, s_2, s_3 \) are expressed by their matrices in the canonical basis as

\[
s_1 = \left(\begin{array}{ccc}-1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right), \quad s_2 = \left(\begin{array}{ccc}0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0\end{array}\right), \quad s_3 = \left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right).
\]

By definition, each element in \( W \) can be expressed as a product of elements in the distinguished set of generators. An expression \( \tilde{w} = s_{i_1} \ldots s_{i_l} \) of an element \( w \in W \) is said to be reduced if \( l \) is as small as possible. In this case, \( l = \text{length}(w) \) is the length of \( w \). Typically an element \( w \in W \) will have many reduced expressions. Particularly, the reduced expression \( \tilde{w} \) of a \( w \in W \) is called adapted to \( Q \) if \( i_k \) is a source vertex of \( s_{i_{k-1}} \ldots s_{i_2}s_{i_1}Q \) for all \( k = 1, \ldots, l \), where \( s_lQ \) is the quiver obtained from \( Q \) by reversing the orientation of each arrow that ends at \( i \) or starts at \( i \). A Coxeter element of \( W \) is a product of all the elements in \( S \). It is well known that there is a unique Coxeter element \( \tau \) called the Coxeter transformation of \( Q \) whose reduced expressions are adapted to \( Q \). Moreover, the Coxeter number is the order of any Coxeter element. Table 4.1 shows the Coxeter number in the Weyl groups of simply-laced Dynkin diagrams.

<table>
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<th>Coxeter number</th>
<th>Involution ( \star )</th>
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</thead>
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<td>( A_n )</td>
<td>( n + 1 )</td>
<td>( i^* = n - (i - 1) )</td>
</tr>
<tr>
<td>( B_n ) (n even)</td>
<td>( 2n - 2 )</td>
<td>identity</td>
</tr>
<tr>
<td>( D_n ) (n odd)</td>
<td>( 2n - 2 )</td>
<td>( i^* = i ) for all ( i = 1, \ldots, n - 2 ), ( (n - 1)^* = n )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>12</td>
<td>( 1^* = 5, \ 6^* = 6, \ 3^* = 3, \ 2^* = 4 )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>18</td>
<td>identity</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>30</td>
<td>identity</td>
</tr>
</tbody>
</table>

**Table 4.1.** Coxeter number and involution \( \star \) (simply-laced Dynkin diagrams).

**Example 4.2.** In the above example, the matrices

\[
\tau = s_3s_2s_1 = \left(\begin{array}{ccc}-1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1\end{array}\right) \quad \text{and} \quad \tau^{-1} = s_1s_2s_3 = \left(\begin{array}{ccc}1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1\end{array}\right)
\]

are the Coxeter transformation and the inverse Coxeter transformation respectively.

We denote \( \phi^+_n \) (resp. \( \phi^-_n \)) the set of positive (resp. negative) roots. By Gabriel’s theorem the map \( M \rightarrow \dim M \) gives a bijection from the set of isoclasses of indecomposable \( kQ \)-modules to the set of positive roots \( \phi^+_n \). Thus, up to isomorphism, each indecomposable
\[\mathsf{kQ}\text{-module in } \mathsf{mod} \mathsf{kQ} \text{ has the form } M_\beta, \text{ where } \beta \text{ is a positive root in } \phi_n^+ \text{ and } \text{dim } M_\beta = \beta.\]

A function \(Q_0 \overset{\xi}{\rightarrow} \mathbb{Z}\) such that \(\xi_j = \xi_i - 1\) for each arrow \(\alpha : i \rightarrow j \in Q_1\) is called a \textbf{height function on } \(Q\). Note that, connectedness of \(Q\) implies that any two height functions on \(Q\) differ by a constant. Now, we recall a generalization of the repetition quiver \(\mathbb{Z}Q\) of \(Q\) given in Example 2.8. The \textbf{repetition quiver} of \(Q\) associated to the height function \(\xi\) consists of the set of vertices \((\mathbb{Z}Q)_0 = \{(i, p) \in Q_0 \times \mathbb{Z} : p - \xi_i \in 2\mathbb{Z}\}\) and the set of arrows of the form \((i, p) \rightarrow (j, p + 1)\) and \((j, q) \rightarrow (i, q + 1)\) for any pair of connected vertices \(i, j \in Q_0\). Note that \(\mathbb{Z}Q\) does not depend on the orientation of the quiver \(Q\). It is well known that the quiver \(\mathbb{Z}Q\) itself has an isomorphism with the AR-quiver of the bounded derived category \(D^0(\mathsf{kQ})\) of the Dynkin algebra \(\mathsf{kQ}\) [40]. According to [62], the indecomposable injective \(\mathsf{kQ}\)-module \(I_i\) is located at the vertex \((i, \xi_i)\) of \(\mathbb{Z}Q\).

Let \(\hat{\phi}_n := \phi_n^+ \times \mathbb{Z}\). For \(i \in Q_0\), we define

\[
\gamma_i = \sum_{j \in B(i)} \alpha_j \quad \text{and} \quad \theta_i = \sum_{j \in C(i)} \alpha_j
\]

where \(B(i)\) (resp. \(C(i)\)) is the set of vertices \(j \in Q_0\) such that there exists a path from \(j\) to \(i\) (resp. from \(i\) to \(j\)). Following [41] the bijection \(\varphi : \mathbb{Z}Q \rightarrow \hat{\phi}_n\) is described combinatorially as follows.

1. \(\varphi(i, \xi_i) = (\gamma_i, 0)\).
2. For \(\beta \in \phi_n^+\) with \(\varphi(i, p) = (\beta, m)\) we have:
   
   (a) If \(\tau(\beta) \in \phi_n^+\) then \(\varphi(i, p - 2) = (\tau(\beta), m)\),
   
   (b) If \(\tau(\beta) \in \phi_n^-\) then \(\varphi(i, p - 2) = (-\tau(\beta), m - 1)\),
   
   (c) If \(\tau^{-1}(\beta) \in \phi_n^+\) then \(\varphi(i, p + 2) = (\tau^{-1}(\beta), m)\),
   
   (d) If \(\tau^{-1}(\beta) \in \phi_n^-\) then \(\varphi(i, p + 2) = (-\tau^{-1}(\beta), m + 1)\).

The Auslander-Reiten quiver \(\Gamma(\mathsf{mod} \mathsf{kQ})\) of the category \(\mathsf{mod} \mathsf{kQ}\) is the full subquiver of \(\mathbb{Z}Q\) whose set of vertices is \(\varphi^{-1}(\phi_n^+ \times \{0\})\). The vertex \(\varphi^{-1}(\beta, 0)\) corresponds to the indecomposable module \(M_\beta\) in \(\mathsf{mod} \mathsf{kQ}\) and the arrow \(\varphi^{-1}(\beta, 0) \rightarrow \varphi^{-1}(\beta', 0)\) is associated to the irreducible morphism from \(M_\beta\) to \(M_{\beta'}\). In particular, the injective envelope \(I_i\) of \(S_i\) corresponds to the vertex \(\varphi^{-1}(\gamma_i, 0)\) and the projective cover \(P_i\) of \(S_i\) is associated to the vertex \(\varphi^{-1}(\theta_i, 0)\), where the involution \(\star\) on \(Q_0\) is given by \(w_0\alpha_i = \alpha_i^*\) for the unique longest element \(w_0\) in the Weyl group \(W = W_Q\) associated to \(Q\) (see [15]). Table E.1 shows the involution \(\star\) for vertices of simply-laced Dynkin diagrams.

It is well known that

\[
\theta_i = \tau^{m_i}(\gamma_i^\star), \text{ where } m_i = \max\{k \geq 0 : \tau^k(\gamma_i) \in \phi_n^+\}
\]

and for \(\beta \in \phi_n^+\), \(\tau(\beta) \in \phi_n^-\) if and only if \(\beta = \theta_i\) for some \(i \in Q_0\).

Let \(\beta \in \phi_n^+\) be such that \(\tau(\beta) \in \phi_n^+\), the action of \(\tau\) on indecomposable \(\mathsf{kQ}\)-modules in \(\mathsf{mod} \mathsf{kQ}\) given by \(\tau M_\beta = M_{\tau(\beta)}\) defines the Auslander-Reiten translation in \(\Gamma(\mathsf{mod} \mathsf{kQ})\).
Moreover, the dimension vector is an additive function on $\Gamma(\text{mod } kQ)$ with respect to the map $\tau$; that is, for each vertex $X \in \Gamma(\text{mod } kQ)$ such that $X = \varphi^{-1}(\beta, 0)$ and $\tau(\beta) \in \phi_n^+$,

$$\dim X + \dim \tau X = \sum_{Z \in X^-} \dim Z,$$

where $X^-$ is the set of vertices $Z \in \Gamma(\text{mod } kQ)$ such that there exists an arrow from $Z$ to $X$.

In a combinatorial way, the full subquiver of $ZQ$ given by

$$\varphi^{-1}(\phi_n^+ \times \{0\}) = \{(i, p) \in ZQ : \xi_i - 2m_i \leq p \leq \xi_i\}$$

is the AR-quiver $\Gamma(\text{mod } kQ)$ of the category $kQ$. Moreover, Gabriel \[37\] introduced the Nakayama-permutation $\vartheta$ of $ZQ$ which is defined by

$$\vartheta(i, p) = (i^*, p + h_n - 2),$$

(4.2)

where $h_n$ is the coxeter number associated to $Q$. For all $i \in Q_0$, the Nakayama permutation $\vartheta$ sends vertices corresponding to $P_i$ to vertices corresponding to $I_i$; thus, $\vartheta(\varphi^{-1}(\dim P_i, 0)) = \varphi^{-1}(\dim I_i, 0)$. Formula (4.1) implies that

$$\vartheta(\varphi^{-1}(\tau^m_{i^*}(\gamma_{i^*}), 0)) = \varphi^{-1}(\gamma_i, 0) = (i, \xi_i).$$

Since $\tau^m_{i^*}(\gamma_{i^*}) \in \phi_n^+$ we obtain

$$(i, \xi_i) = \vartheta(i^*, \xi_i - 2m_i) = (i, \xi_i - 2m_i + h_n - 2).$$

Thus,

$$\xi_i = \xi_{i^*} - 2m_{i^*} + h_n - 2.$$  (4.3)

This formula allows us to know $m_{i^*}$ using the involution $\star$ associated to the Dynkin diagram and a suitable height function.

If $P_i$ is the projective cover of the simple representation $S_i$ in the category $\text{rep } Q$ then the set $O_i = \{M \in \text{Ind } Q : \tau^k P_i = M \text{ for some } k \in \mathbb{Z}\}$ is called the $\tau$-orbit of $P_i$. According to \[72\] each $\tau$-orbit in an AR-quiver of Dynkin type contains exactly one projective representation and one injective representation. It is well known that the injective envelope $I_{i^*}$ of the simple representation $S_{i^*}$ belongs to $O_i$. Formulas (4.1) and (4.3) allow to obtain the cardinality of the $\tau$-orbit $O_i$ as follows:

$$|O_i| = m_{i^*} + 1.$$  (4.4)

Let us stress that the maximal length $d$ of the $\tau$-orbits depends not only on the diagram Dynkin $\mathcal{Q}$, but on the given orientation. In fact, the (optimal) bounds $d' \leq d \leq d''$ for the length of the $\tau$-orbits of the simply laced cases can be viewed in Table 4.2.
### 4.2 τ-orbit partitions

A partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of a positive integer \( m \) is a finite nonincreasing sequence of positive integers \( \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \) such that \( m = \sum_{i=1}^{n} \lambda_i \). We use the length of the \( \tau \)-orbits in the AR-quiver of algebras of Dynkin type to define suitable integer partitions.

**Definition 4.1.** Let \( Q \) be a Dynkin quiver of finite type with vertices \( Q_0 = \{1, \ldots, n\} \). We call \( \tau \)-orbit partition associated to \( Q \) to the partition \( \lambda_Q = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i = |O_{\sigma(i)}| \) is the cardinality of the \( \tau \)-orbit \( O_{\sigma(i)} \) of the indecomposable projective \( P_{\sigma(i)} \) in mod \( kQ \) and \( \sigma \) is a permutation of the elements of \( Q_0 \) such that \( |O_{\sigma(1)}| \geq \cdots \geq |O_{\sigma(n)}| \).

**Example 4.3.** Let \( Q = 1 \xleftarrow{} 2 \xleftarrow{} 3 \rightarrow 4 \xleftarrow{} 5 \) be a quiver of type \( \mathbb{A}_5 \). The AR-quiver \( \Gamma(\text{mod} \, kQ) \) has the form

![Diagram](image)

In this case, the \( \tau \)-orbit partition \( \lambda_Q = (4, 3, 3, 3, 2) \) is a integer partition of the integer number 15. Note that each part in \( \lambda_Q \) is given by the cardinality of a \( \tau \)-orbit ordered in the natural way.

#### 4.2.1 Case \( \mathbb{A}_n \)

We introduce a height function which can help us to calculate the cardinality of a \( \tau \)-orbit in an easy way. In this subsection, we suppose that the underlying graph of \( \overline{Q} \) of a Dynkin quiver \( Q \) of type \( \mathbb{A}_n \) has the numbering of vertices and edges as in Figure 3.1

**Definition 4.2.** Let \( Q \) be a Dynkin quiver of type \( \mathbb{A}_n \). An arrow \( \alpha_i \in Q_1 \) is called a right arrow (resp. left arrow) if \( i \xleftarrow{\alpha_i} i+1 \) (resp. \( i+1 \xrightarrow{\alpha_i} i \)). Moreover, we call orientation vector of \( Q \) to the vector \( v_Q = \sum_{k=1}^{n} a_k e_k \in \mathbb{Z}^n \), where \( a_1 = 0 \), \( a_k = \sum_{i=1}^{k-1} v(\alpha_i) \) for \( k \geq 2 \) and

\[
v(\alpha_i) = \begin{cases} 
 1 & \text{if } \alpha_i \text{ is a right arrow,} \\
 0 & \text{if } \alpha_i \text{ is a left arrow.}
\end{cases}
\]
We obtain the following result.

**Lemma 4.1.** Let \( Q \) be a quiver of type \( A_n \) such that \( Q_0 = \{1, \ldots, n\} \) with orientation vector of the form \( v_Q = \sum_{k=1}^n a_k e_k \). Then \( |O_i| = a_i - a_i + i \).

**Proof.** Let \( Q^{\text{op}} \) be the opposite quiver of \( Q \). For \( i \) fixed, let \( \xi_i \) be such that \( \xi_i = a_i - a_i^{\text{op}} \) where \( v_Q^{\text{op}} = \sum_{k=1}^n a_k^{\text{op}} e_k \) and \( v_Q = \sum_{k=1}^n a_k e_k \). It is easy to see that the function \( \xi : Q_0 \rightarrow \mathbb{Z} \) with \( \xi(i) = \xi_i \) is a height function. According to the formula (4.3) we have that \( a_i - a_i - 2m_i + h_n - 2 = a_i^{\text{op}} - a_i^{\text{op}} \). Since the coxeter number of \( A_n \) is \( h_n = n+1 = i + i^* \) we have that, \( a_i - a_i = a_i^{\text{op}} - a_i^{\text{op}} + 2m_i - (i + i^*) + 2 \). Since \( a_i^{\text{op}} - a_i^{\text{op}} + i = i^* - (a_i - a_i) \) we obtain \( a_i - a_i + i = m_i + 1 = |O_i| \) and with this identity we are done. \( \square \)

So far, we have given an explicit formula for the cardinality of a \( \tau \)-orbit in the AR-quiver \( \Gamma(nQ) \) for a Dynkin quiver \( Q \) of type \( A_n \). Moreover we have defined the \( \tau \)-orbit partition of the triangular number \( t_n = \frac{n(n+1)}{2} \) of indecomposable isoclasses of \( kQ \)-modules into \( n \) parts. It is natural to ask under which conditions a partition \( \lambda \) of \( t_n \) is a \( \tau \)-orbit partition.

**Lemma 4.2.** A partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of the triangular number \( t_n \) is a \( \tau \)-orbit partition of type \( A_n \) if and only if it satisfies the following conditions:

(a) \( \lambda_i + \lambda_i^* = h_n \) for each integer \( i = 1, \ldots, n \),

(b) \( 0 \leq \lambda_i - \lambda_{i+1} \leq 1 \) for each integer \( i = 1, \ldots, n-1 \).

**Proof.** Suppose that \( \lambda_Q = (\lambda_1, \ldots, \lambda_n) \) is the \( \tau \)-orbit partition induced by the Dynkin quiver \( Q \) of type \( A_n \). Without loss of generality we can suppose that \( \lambda_i = |O_i| \) for each integer \( 1 \leq i \leq n \). Let \( v_Q = \sum_{k=1}^n a_k e_k \) be the orientation vector associated to \( Q \). Lemma 4.1 allows us to establish that

\[
\lambda_i + \lambda_i^* = |O_i| + |O_i^*| = (a_i - a_i + i) + (a_i - a_i + i^*) = i + i^* = h_n.
\]

Moreover, \( \lambda_i - \lambda_{i+1} = (a_i - a_i + i) - (a_{i+1} - a_{i+1} + i + 1) = (a_{i+1} - a_i) + (a_i - a_i - 1) \). Thus, \( a_{i+1} - a_i = 1 \) and \( a_i - a_{i-1} = 1 \), therefore \( \lambda_i - \lambda_{i+1} \leq 1 \). Since \( \lambda_i \geq \lambda_{i+1} \), it follows that \( \lambda_Q \) satisfies the conditions (a) and (b).

Now suppose that \( \lambda = (\lambda_1, \ldots, \lambda_n) \) satisfies (a) and (b), let \( a_n \) be such that \( a_n = n - \lambda_n \), we define \( a_{i-1} = a_i - 1 \) for each integer \( \lceil \frac{n}{2} \rceil \leq i \leq n \) and \( a_i = a_i^* - u_i \), for each integer \( 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1 \) with \( u_i = i - \lambda_i \). Given the vector \( v = \sum_{k=1}^n a_k e_k \) we define a quiver \( Q \) with orientation vector \( v_Q = v \). We set \( Q_0 = \{1, \ldots, n\} \) and \( Q_1 = \{a_1, \ldots, a_{n-1}\} \) where \( a_i \) is an arrow with vertices \( i \) and \( i + 1 \) oriented according to the case \( a_{i+1} - a_i = 0 \) or \( a_{i+1} - a_i = 1 \). By construction, it is easy to see that \( v_Q = v \). Finally, we see that the partition induced by the quiver \( Q \) is \( \lambda_Q = \lambda \). Moreover, for any integer \( i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1 \), we have that \( |O_i| = a_i - a_i + i = u_i + 1 = i^* - \lambda_{i^*} + i = h_n - \lambda_i = \lambda_i \), and with that identity we are done. \( \square \)

We denote the number of \( \tau \)-orbit partitions of type \( A_n \) by \( P_\tau(A_n) \). Then we have the following result.

**Theorem 4.3.** \( P_\tau(A_n) = 2^{\lfloor \frac{n}{2} \rfloor - 1} \).
Proof. First, let us to consider the case \( n = 2k - 1 \) for some \( k \geq 1 \). We proceed by induction on \( k \). If \( k = 2 \) then it is easy to see that there are two \( \tau \)-orbit partitions which are \((3, 2, 1)\) and \((2, 2, 2)\) of type \( A_3 \). Since \( 2^\left\lceil \frac{n}{2} \right\rceil - 1 = 2^k - 1 = 2 \) then theorem holds in this case. Now we suppose that the assertion is true for any \( s < k \) and \( j \) such that \( 2s - 1 = j \leq 2k - 1 \), then we will see that the theorem is true for \( N = n + 2 = 2(k + 1) - 1 \).

It is clear that a \( \tau \)-orbit partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}, \lambda_{n+2}) \) of the triangular number \( t_n \) arises from the \( \tau \)-orbit partition \( \overline{\lambda} = (\lambda_2 - 1, \lambda_3 - 1, \lambda_{n+1} - 1) \) of \( t_n \). On the other hand, if \( \lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) \) is an integer partition of \( t_n \) and \( \lambda \) is an integer partition of \( t_N \) such that \( \overline{\lambda} = \lambda' \) then \( \lambda = (\lambda_1', \lambda_2'+1, \ldots, \lambda_n'+1, \lambda_{n+2}) \). Lemma 4.2 part (b) implies that \( 0 \leq \lambda_1 - \lambda'_1 + 1 \leq 1 \). Thus, \( \lambda_1 = \lambda'_1 + 2 \) or \( \lambda_1 = \lambda'_1 + 1 \). If \( \lambda_1 = \lambda'_1 + 2 \) then Lemma 4.2 part (a) implies that \( \lambda_1 + \lambda_{n+2} = n + 3 \). Therefore, \( \lambda_{n+2} = \lambda'_n \). Then, \( \lambda = (\lambda'_1 + 2, \lambda'_2 + 1, \ldots, \lambda'_n + 1, \lambda'_n) \). If \( \lambda_1 = \lambda'_1 + 1 \) then via Lemma 4.2 part (a) we obtain \( \lambda_1 + \lambda_{n+2} = \lambda'_1 + 1 + \lambda_{n+2} = n + 3 \). Therefore, \( \lambda_{n+2} = n - (\lambda'_1 - 1) + 1 = \lambda'_1 + 1 = \lambda'_n + 1 \).

Then
\[
\lambda = (\lambda'_1 + 1, \lambda'_1 + 1, \lambda'_2 + 1, \ldots, \lambda'_n + 1, \lambda'_n + 1).
\]

Thus, each integer partition of \( t_n \) gives place to two partitions of \( t_{n+2} \), that is,
\[
P_\tau(A_{n+2}) = 2P_\tau(n) = 2(2^\left\lceil \frac{n}{2} \right\rceil - 1) = 2(2^k - 1) = 2(k+1) - 1 = 2^\left\lceil \frac{N}{2} \right\rceil - 1.
\]

Since the proof for the case \( n \) even follows in a similar way, we are done. 

The integer sequence \( (2^\left\lceil \frac{N}{2} \right\rceil - 1)_{n \geq 1} \) is encoded as A016116 in the On-line Encyclopedia of Integer Sequences [83].

Example 4.4. Table 4.3 shows the \( \tau \)-orbit partitions of type \( A_n \) for \( n = 1, \ldots, 8 \).

<table>
<thead>
<tr>
<th>Dynkin diagram</th>
<th>( \tau )-orbit partitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>(1)</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>(2, 1)</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>(3, 2, 1), (2, 2, 2)</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>(4, 3, 2, 1), (3, 3, 2, 2)</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>(5, 4, 3, 2, 1), (4, 4, 3, 2, 2), (4, 3, 3, 3, 2), (3, 3, 3, 3, 3)</td>
</tr>
<tr>
<td>( A_6 )</td>
<td>(6, 5, 4, 3, 2, 1), (5, 5, 4, 3, 2, 2), (5, 4, 4, 3, 3, 2), (4, 4, 4, 3, 3, 3)</td>
</tr>
<tr>
<td>( A_7 )</td>
<td>(7, 6, 5, 4, 3, 2, 1), (6, 6, 5, 4, 3, 2, 2), (6, 5, 5, 4, 3, 3, 2), (6, 5, 4, 4, 4, 3, 2)</td>
</tr>
<tr>
<td>( A_8 )</td>
<td>(8, 7, 6, 5, 4, 3, 2, 1), (7, 7, 6, 5, 4, 3, 2, 2), (7, 6, 6, 5, 4, 3, 3, 2)</td>
</tr>
<tr>
<td></td>
<td>(7, 6, 5, 5, 4, 4, 3, 2), (6, 6, 6, 5, 4, 3, 3, 3), (6, 6, 5, 5, 4, 4, 3, 3)</td>
</tr>
<tr>
<td></td>
<td>(6, 5, 5, 5, 4, 4, 4, 3), (5, 5, 5, 5, 4, 4, 4, 4)</td>
</tr>
</tbody>
</table>

Table 4.3. Examples of \( \tau \)-orbit partitions of type \( A \)

4.2.2 Case \( D_n \)

In this subsection, we suppose that the underlying graph of \( \overline{Q} \) of a Dynkin quiver \( Q \) of type \( D_n \) has the numbering of vertices and edges as in Figure 3.2.

Theorem 4.4. A \( \tau \)-orbit partition of type \( D_n \) is either \( (n - 1, n - 1, \ldots, n - 1) \) or \( (n, n - 1, \ldots, n - 1, n - 2) \) if \( n \) is odd, whereas it is \( (n - 1, n - 1, \ldots, n - 1) \) if \( n \) is even.
Proof. Suppose that $n$ is an even number, since $\xi_i = \xi_{i^*} - 2m_{i^*} + h_n - 2$ and $i^* = i$ we have that $2m_{i^*} = h_n - 2$, that is, $m_i = n - 2$ therefore $|O_i| = n - 1$. According to this fact, each quiver $D_n$ with $n$ even has associated the partition $(n - 1, n - 1, \ldots, n - 1)$ which does not depend on orientation. On the other hand, if $n$ is odd then we have that if $i \neq n, n - 1$ then $i^* = i$, thus $m_i = n - 2$, that is $|O_i| = n - 1$ for $1 \leq i \leq n - 2$. It remains to compute $|O_{n-1}|$ and $|O_n|$. Since $n^* = n - 1$ and $(n - 1)^* = n$ then $\xi_n = \xi_{n-1} - 2m_{n-1} + h_n - 2$ and $\xi_{n-1} = \xi_n - 2m_n + h_n - 2$. Definition of height function allows us to conclude that $|\xi_{n-1} - \xi_n| = 2$ or $0$. Indeed, if $n - 2 \xrightarrow{\alpha_{n-2}} n - 1$ and $n - 2 \xleftarrow{\alpha_{n-2}} n - 1$ and $n - 2 \xrightarrow{\alpha_{n-1}} n$ then $\xi_{n-1} = \xi_{n-2} - 1$ and $\xi_n = \xi_{n-2} - 1$ or $\xi_{n-2} = \xi_{n-1} - 1$ and $\xi_{n-2} = \xi_n - 1$. Therefore, $\xi_{n-1} = \xi_n$. Moreover, if $n - 2 \xrightarrow{\alpha_{n-2}} n - 1$ and $n \xrightarrow{\alpha_{n-1}} n - 2$ or if $n - 2 \xleftarrow{\alpha_{n-2}} n - 1$ and $n \xleftarrow{\alpha_{n-1}} n - 2$ then $\xi_n - \xi_{n-1} = 2$ or $\xi_n - \xi_{n-1} = 2$. Now, if $|\xi_{n-1} - \xi_n| = 0$ then since $\xi_n = \xi_{n-1} - 2m_{n-1} + (2n-2) - 2$ and $\xi_{n-1} = \xi_n - 2m_n + (2n-2) - 2$, we conclude that $m_n = m_{n-1} = n - 2$, that is, $|O_{n-1}| = |O_n| = n - 1$ thus the $\tau$-orbit partition induced is $(n - 1, \ldots, n - 1)$. Finally, if $|\xi_{n-1} - \xi_n| = 2$, then we take into account that $\xi_n = \xi_{n-1} - 2m_{n-1} + (2n-2) - 2$ and $\xi_{n-1} = \xi_n - 2m_n + (2n-2) - 2$ to observe that $\xi_{n-1} - \xi_n = 2$ or $-2$ thus $m_n = n - 1$ and $m_{n-1} = n - 3$ or $m_n = n - 3$ and $m_{n-1} = n - 1$, that is, $|O_{n-1}| = n$ and $|O_n| = n - 2$ or $|O_{n-1}| = n - 2$ and $|O_n| = n$ therefore the $\tau$-orbit partition induced has the form $(n, n - 1, \ldots, n - 1, n - 2)$. \hfill \Box

We denote the number of $\tau$-orbit partitions of type $D_n$ by $P_\tau(D_n)$. Theorem above allows us to conclude the following result.

Corollary 4.5. $P_\tau(D_n) = 1 + [n]_{\text{modulo } 2}$.

The integer sequence $P_\tau(D_n)$ is encoded as A000034 in the Online Encyclopedia of Integer Sequences \[84\].

4.2.3 Case $\mathbb{E}_6, \mathbb{E}_7,$ and $\mathbb{E}_8$

Finally, let us turn our attention to Dynkin algebras $kQ$ of type $\mathbb{E}$. In this subsection, we suppose that the underlying graph of $\mathcal{Q}$ of $Q$ has the number of vertices and edges as in Figure 3.3. We denote the number of $\tau$-orbit partitions of type $\mathbb{E}$ by $P_\tau(\mathbb{E}_n)$ for $n = 6, 7, 8$. Then, we have the following result.

Theorem 4.6. $P_\tau(\mathbb{E}_6) = 5$ and $P_\tau(\mathbb{E}_7) = P_\tau(\mathbb{E}_8) = 1$.

Proof. Suppose that $Q$ is a quiver of type $\mathbb{E}$ and $\xi$ is a height function defined on $Q_0$. In the cases where $\mathcal{Q}$ is $\mathbb{E}_7$ or $\mathbb{E}_8$ it suffices take into account that the involution $\star$ is the identity. Therefore, for any vertex $i$, we have that $\xi_i = \xi_{i^*}$. Thus, as a consequence of (4.3) it holds that $2m_i = 2m_{i^*} = h_n - 2$. Let $Q$ be a Dynkin quiver of type $\mathbb{E}_7$. By (4.4) we have that $|O_i| = 9$, for all $i = 1, \ldots, 7$. Therefore, the $\tau$-orbit partition has the form $\lambda_Q = (9, 9, 9, 9, 9, 9, 9)$ which does not depend on orientation. Analogously, if $Q$ is a Dynkin quiver of type $\mathbb{E}_8$ then the $\tau$-orbit partition induced has the form $\lambda_Q = (15, 15, 15, 15, 15, 15, 15)$ which does not depend on orientation.

On the other hand, if $\mathcal{Q} = \mathbb{E}_6$ we must compute cardinalities of $\tau$-orbits independently. Note that, if $i = 3$ or $i = 6$ then $\star$ is defined in such a way that $\xi_i = \xi_{i^*}$. Therefore, $2m_i = 2m_{i^*} = h_n - 2$. Formula (4.3) implies that $|O_3| = |O_6| = 6$. Moreover, if $i = 2$
or \(i = 4\) we note that \(2^* = 4\) and \(4^* = 2\), moreover \(|\xi_2 - \xi_4|\) is 0 or 2. In case \(\xi_2 = \xi_4\), formulas (4.3) and (4.4) imply that \(|O_2| = |O_4| = 6\). If \(|\xi_2 - \xi_4| = 2\) we observe that \(|O_4| = 5\) and \(|O_2| = 7\) or \(|O_2| = 7\) and \(|O_2| = 5\). Case \(i = 1\) or \(i = 5\) implies that the value of \(|\xi_1 - \xi_5|\) is 0, 2 or 4. \(\xi_1 = \xi_5\) implies that \(|O_1| = |O_5| = 6\). Case \(|\xi_1 - \xi_5| = 2\) implies that \(|O_1| = 5\) and \(|O_5| = 7\) or \(|O_1| = 7\) and \(|O_5| = 5\). Finally, case \(|\xi_1 - \xi_5| = 4\) implies that \(|O_1| = 8\) and \(|O_5| = 4\) or \(|O_1| = 4\) and \(|O_5| = 8\). Thus, \(\lambda_Q = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\) is such that for any \(i = 1, \ldots, 6\), \(\lambda_i = |O_{\sigma(i)}|\) where \(\sigma\) is the permutation which satisfies the condition \(|O_{\sigma(1)}| \geq \cdots \geq |O_{\sigma(6)}|\). \(|O_1| + |O_{\ast}| = h_m = 12\), \(|O_1| \in \{4, 5, 6, 7, 8\}\), \(|O_6| = |O_3| = 6\) and \(|O_2| \in \{5, 6, 7\}\).

\[\square\]

**Example 4.5.** The \(\tau\)-orbit partitions of type \(E_6\) are: \((6, 6, 6, 6, 6, 6), (7, 6, 6, 6, 6, 5), (7, 7, 6, 6, 5, 5), (8, 6, 6, 6, 6, 4)\) and \((8, 7, 6, 6, 5, 4)\).

### 4.3 Relation to tiled orders

A field \(k\) is said to be of **discrete norm** or **discrete valuation** if it is endowed with a surjective map \(\nu : k \rightarrow \mathbb{Z} \cup \{\infty\}\), which satisfies the following conditions:

1. \(\nu(x) = \infty\) if and only if \(x = 0\),
2. \(\nu(xy) = \nu(x) + \nu(y)\),
3. \(\nu(x + y) \geq \min\{\nu(x), \nu(y)\}\).

We let \(\mathcal{O}\) denote, the **normalization ring** of the field \(k\), such that \(\mathcal{O} = \{x \in k \mid \nu(x) \geq 0\}\). An element \(\pi \in \mathcal{O}\) such that \(\nu(\pi) = 1\) is a **prime element** of \(\mathcal{O}\). For each \(x \in \mathcal{O}\) we have that \(x \in \mathcal{O}\) if and only if \(x = \varepsilon \pi^m\), for some \(m \geq 0\) and \(\varepsilon \in \mathcal{O}^\times\). Moreover, \(x \in k\) if and only if \(x = \varepsilon \pi^m\) for some \(m \in \mathbb{Z}\) and \(\varepsilon \in \mathcal{O}^\times\). The ring \(\mathcal{O}\) is such that \(\mathcal{O} \supset \pi \mathcal{O}\), where \(\pi \mathcal{O}\) is the unique maximal ideal, therefore ideals of \(\mathcal{O}\) generate a chain of the form \(\mathcal{O} \supset \pi \mathcal{O} \supset \pi^2 \mathcal{O} \supset \cdots \supset \pi^m \mathcal{O} \supset \cdots\).

A **tiled order** \(\Lambda\) is a subring of the matrix algebra \(k^{n \times n}\) with the form

\[
\Lambda = \sum_{i,j=1}^{n} e_{ij} \pi^{\lambda_{ij}} \mathcal{O} = \begin{pmatrix}
\mathcal{O} & \pi^{\lambda_{12}} \mathcal{O} & \cdots & \pi^{\lambda_{1n}} \mathcal{O} \\
\pi^{\lambda_{21}} \mathcal{O} & \mathcal{O} & \cdots & \pi^{\lambda_{2n}} \mathcal{O} \\
\vdots & \vdots & \ddots & \vdots \\
\pi^{\lambda_{n1}} \mathcal{O} & \pi^{\lambda_{n2}} \mathcal{O} & \cdots & \mathcal{O}
\end{pmatrix}.
\]

In other words, \(\Lambda\) consists of all matrices whose entries \(ij\) belong to \(\pi^{\lambda_{ij}} \mathcal{O}\), in this case the \(e_{ij} \in k^{n \times n}\) are unit matrices such that \(e_{ij} e_{kl} = \delta_{jk} e_{il}\), where \(\delta_{jk} = 1\) if \(j = k\) and \(\delta_{jk} = 0\) otherwise. Moreover, the numbers \(\lambda_{ij}\) are integers which satisfy the following conditions.

(a) \(\lambda_{ii} = 0\), for each \(i\),
(b) \(\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}\) for all \(i, j, k\).

An order \(\Lambda\) is said to be **Morita reduced** or **reduced** if it satisfies the additional condition (c) \(\lambda_{ij} + \lambda_{ji} > 0\), for each \(i \neq j\). In such a case, projective modules are pairwise nonisomorphic, that is, in the decomposition of \(\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n\) via indecomposable
projective modules all of the summands are pairwise nonisomorphic. The following result characterizes isomorphic orders via matrix problems.

**Theorem 4.7.** [86, Assertion 2.3] Two orders \( \Lambda \) and \( \Lambda' \) are isomorphic if the corresponding exponents matrices \( \lambda_{ij} \) and \( \lambda'_{ij} \) can be turned into each other with the help of the following admissible \( t \)-transformations:

(i) To add an integer \( n \) to each entry of a given row \( i \) and simultaneously subtract \( n \) to each entry of the column \( i \).

(ii) To transpose simultaneously rows \( i \) and \( j \) and columns \( i \) and \( j \).

Let \( \mathcal{O} \) be a discrete valuation ring with prime element \( \pi \). We define the reduced tiled order \( \Lambda_{A_n} = \bigoplus_{k=1}^{\left\lceil \frac{n}{2} \right\rceil} \Lambda_k \) associated to the Dynkin diagram \( A_n \), where \( \Lambda_k \) is the matrix ring \( \left( \begin{array}{cc} \pi^{k_{ij}} & 0 \\ 0 & \pi^{k_{ji}} \end{array} \right) \) whose adjacency matrix is \( \Lambda_k = \left( \begin{array}{cc} 0 & k \\ k & 0 \end{array} \right) \) and \( * \) is the involution associated to the underlying graph \( \mathcal{Q} \) (see Table 4.1). Lemma 4.1 and Theorem 4.7 define the following algorithm to calculate the cardinality of \( \tau \)-orbits of type \( A_n \).

**Algorithm 4.8.**

*Input:* A Dynkin quiver \( Q \) of type \( A_n \)

*Output:* Cardinality of the \( \tau \)-orbits \( \mathcal{O}_k \) for each \( k = 1, 2, \ldots, n \).

**Step 1:** To find the vector orientation \( v_Q = (a_1, \ldots, a_n) \) introduced in Definition 4.2.

**Step 2:** To apply the admissible transformation (i) in Theorem 4.7 with integer \( a_k - a_k \) on row and column one of the matrix \( \Lambda_k \), \( k = 1, \ldots, \left\lceil \frac{n}{2} \right\rceil \), to obtain an isomorphic tiled order \( \Lambda_Q = (\lambda_{ij}^Q) \).

**Step 3:** To calculate \( |\mathcal{O}_k| = \lambda_{21}^Q \) and \( |\mathcal{O}_{k^*}| = \lambda_{21}^Q \), for each \( k = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil \).

**Example 4.6.** Let \( Q \) and \( Q' \) be the quivers \( 1 \leftarrow 2 \rightarrow 3 \rightarrow 4 \) and \( 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \) respectively, whose orientation vectors are \( v_Q = (0, 0, 1, 1) \) and \( v_{Q'} = (0, 0, 0, 1) \). Now, \( \Lambda_{A_4} = \Lambda_1 \oplus \Lambda_2 \), where \( \Lambda_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) and \( \Lambda_2 = \left( \begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array} \right) \). Firt, we consider the quiver \( Q \). Then, applying admissible \( t \)-transformation (i) in Theorem 4.7 to row and column one of the matrices \( \Lambda_1 \) and \( \Lambda_2 \) with integer number \( -1 \) and \( 1 \) respectively, we obtain the tiled orders given by the matrices \( \Lambda_1^Q = \left( \begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array} \right) \) and \( \Lambda_2^Q = \left( \begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array} \right) \) respectively. Thus, the \( \tau \)-orbit partition associated to \( Q \) is the particion \( \lambda_Q = (3, 3, 2, 2) \) of the number 10. In the same way, we consider the quiver \( Q' \). It implies that the integers to apply the \( t \)-transformation (i) change, they are \( 1 \) and \( 0 \) respectively. Thus, we have \( \Lambda_1^{Q'} = \left( \begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array} \right) \) and \( \Lambda_2^{Q'} = \left( \begin{array}{cc} 0 & 2 \\ 3 & 0 \end{array} \right) \). We conclude that the quivers \( Q \) and \( Q' \) have associated the same partition \( \lambda_{Q'} = \lambda_Q \). However \( Q \) and \( Q' \) are not isomorphic as quivers.

**Remark 4.1.** Note that if \( Q \) and \( Q' \) are isomorphic quivers then the corresponding partitions are equal. Furthermore, by Example above the reciprocal statement is not true.
Matrix problems induced by visual cryptography schemes

In 1994, Naor and Shamir [57] proposed a new cryptography technique called visual cryptography (also called visual secret sharing) which is different from the other mainstream symmetric cryptosystems. Their \( k \) out \( n \) or \( (k,n) \)-threshold visual cryptography scheme (in short, TVS) allows visual information (pictures, text, etc.) to be encrypted in such a way that the decrypted information appears as a visual image.

Although a lot of the investigations in the field have been focused on the new schemes, the improving of the contrast, the reducing of the pixel expansion, the new applications, among others; there is a recent line of work that considers algebraic approaches to the visual cryptography schemes (in short, VCS). In fact, some visual secret sharing schemes (in short, VSSS) have been introduced by using some algebraic structures, for instance, Cañadas et al. [20] introduced a visual cryptography scheme with a special share \( T_0 \) containing sets of nested images, all secrets can be revealed by superimposing some transparencies to this fixed share. These authors also have used some properties of \( k \)-linear maps to generate schemes of multiple secret sharing.

In that direction, we consider the lattice-based TVS for color images proposed by Koga and Yamamoto in 1998 [54]. In this case, pixels are treated as elements of a suitable lattice \( S \) and the stacking process is defined as an operation between elements of the lattice, where the commutative and associative laws of such operation allow to \( (k,n) \) VSSS to decrypt \( k \) shares by stacking up of all them in an arbitrary order. Permitting the existence of inverses for all \( s \in S \) leads to pathological VSSS for example, stacking a black subpixel with another subpixel yield to a white or transparent subpixel, finite lattices are one of the simplest structures that meet these requirements. Under these circumstances we generalize this kind of TVS by using \( k \)-linear representations in such a way that the VSSS is completely defined by the orbits defined by some admissible transformations between columns and rows of a matrix representation of a color-lattice.
5.1 What is visual cryptography?

Visual cryptography is a secret sharing technique which allows the encryption of a secret image among a number of participants. The decryption of the secret image requires neither cryptography knowledge nor complex computation; actually, it only requires the use of the human visual system. Compared to the traditional secret sharing schemes, it encrypts a large amount of secret information, i.e., an entire image where its content is versatile. Thus, the characteristics of visual cryptography are:

(a) Simple to implement.
(b) Perfect security.
(c) The encryption does not need complex computations.
(d) The decryption can be done by human visual system (human eye) without any complex computations and without the assistance of a computing device.

In the basic model, a black and white secret image is divided into a number $n$ of shares with no information about the secret. Such images are printed on transparencies (also called shadows). The transparencies are usually shared by $n$ participants so that each participant is expected to keep one transparency. The beauty of this is that the only way to recover the secret is by stacking the transparencies all together \cite{19,20,57}.

In the study of visual cryptography, two primary factors are important to determine the quality of the recovered image: the contrast and the pixel expansion. The contrast refers to the difference in contrast between the original and the recovered image, whereas the pixel expansion refers to the number of sub-pixels in a share required to represent a single pixel in the original image. Naor and Shamir \cite{57} suggested the contrast is preferred to be as large as possible and the pixel expansion to be as small as possible. Many studies have focused on visual cryptography methods, with improvement in the contrast and reducing the pixel expansion.

The visual cryptosystems are based on the threshold visual cryptography scheme of Naor and Shamir \cite{57}. In general, it is a method to encode a secret image into $n$ shadows such that it can be observed if any $k$ or more of them are stacked together. However, the secret image is totally invisible if fewer than $k$ transparencies are stacked. In this model, the original secret image consists of a set of black and white pixels and each pixel is encrypted separately. Each pixel in the original image is represented by $n$ blocks, which appear as $n$ shares, one block for each share. Each block consists of a set of $m$ black and white sub-pixels. Thus, the size of each share is $m$ times as large as that of the secret image. White and black sub-pixels are represented with 0’s and 1’s, respectively.

In particular, when $k = n$, an original pixel in the secret image is encrypted into $m$ sub-pixels, where $2^{n-1} \leq m \leq 2^n$ and $m$ is preferable to be as small as possible. The scheme is based on two $n \times m$ boolean matrices; $S^0$ and $S^1$ (called basis matrices). Each one of the $S^0$ and $S^1$ has a particular number of 1’s (black pixels) in its columns and rows. To maintain the same tone of gray throughout all the transparencies whether the image pixel is white or black, both $S^0$ and $S^1$ must have an equal number of 1’s and 0’s on each row.
Thus, the scheme requires $S^0$ to have an even number of 1's in each column, whereas $S^1$ are required to have odd number of 1's in each column with no two rows in $S^1$ are to be identical; where each row of the $S^0$ and $S^1$ matrices represents a separate share. There are two sets, the white set $C_0$, and black set $C_1$. Therefore sets which could be defined as the collections of all matrices obtained by all possible permutations of columns in $S^0$ and $S^1$ respectively. It must be noted that performing a column permutation on the basis of $S^0$ and $S^1$ matrices only changes the positions of the sub-pixels in the shares and not the number of 1's of the resulting $m$-vectors. It does not matter which matrix is selected at random from the collection $C_0$ for encoding white pixel and from the collection $C_1$ for encoding black pixel.

Example 5.1 (Naor and Shamir’s (2,2)-visual cryptography scheme). The basis matrices $S^0$ and $S^1$ and the collections of the encoding matrices $C_0$ and $C_1$ in this scheme could be written as $S^0 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $S^1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$, where $C_0$ is the set of matrices $\{ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \}$ and $\{ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \}$, whereas $C_1$ is the set of matrices $\{ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \}$ and $\{ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \}$. Every row in each matrix in the collections $C_0$ and $C_1$ above corresponds to four sub-pixels ($m = 4$). Moreover, the rows in each matrix in the collections present the shares. To encrypt a white pixel, one matrix from $C_0$ randomly selected and the sub-pixels in each row of the selected matrix are put into the respective shares. If the two shares are superimposed, the output would be 2 white and 2 black sub-pixels, arranged in a square block and producing a gray medium. To encrypt a black pixel, one matrix from $C_1$ is randomly selected and the sub-pixels in each row of the selected matrix are put into the respective shares. If the two shares are superimposed, the output would be 4 black sub-pixels arranged in a square block, producing black pixel. This scheme was used with a black and white picture in Figure 5.1.

Soon afterwards, this set up of Naor and Shamir was generalized in several directions. Here, we mention some examples:

(1) Some works have been devoted to the multiple secret sharing schemes, where two or more secret images can be secured at a time in two shares (see [25, 76]).

(2) An extended visual cryptography (EVC) provide techniques to create meaningful shares instead of random shares of traditional visual cryptography and help to avoid the possible problems, which may arise by noise-like shares in traditional visual cryptography (see [56]). An example is given in Figure 5.2.

(3) In $(k,n)$ visual cryptography scheme, all $n$ shares have equal importance. Any $k$ out of $n$ shares can reveal the secret information. It may compromise the security of system. To overcome this problem, G. Ateniese, C. Blundo, A.DeSantis, and D. R. Stinson extended $(k,n)$ visual cryptography model to general access structure [3].

5.2 Visual Cryptography Schemes

In this section, we introduce basic definitions, and notations to be used throughout the chapter, we follow the exposition of [11, 19, 20, 26].
A visual cryptography scheme (VCS) is based on the fact that each pixel of an image is divided into a certain number \( m \) of subpixels. This number \( m \) is called the pixel expansion of the image. If the number of black subpixels needed to represent a white pixel in an image is \( l \), and the number of black subpixels needed to represent a black pixel is \( h \), then we call the number \( \alpha = \frac{h-l}{m} \) the contrast of the image.

Here we present the definition of a VSSS according to [28]. Let \( P = \{1, 2, \ldots, n\} \) be a set of elements called participants, and let \( 2^P \) denote the set of all subsets of \( P \). Let \( \Gamma_{\text{Qual}} \subseteq 2^P \) and \( \Gamma_{\text{Forb}} \subseteq 2^P \), where \( \Gamma_{\text{Qual}} \cap \Gamma_{\text{Forb}} = \emptyset \). We will refer to members of \( \Gamma_{\text{Qual}} \) (resp. \( \Gamma_{\text{Forb}} \)) as qualified sets (resp. forbidden sets). The pair \( (\Gamma_{\text{Qual}}, \Gamma_{\text{Forb}}) \) is called the access structure of the scheme.

**Definition 5.1.** Let \( (\Gamma_{\text{Qual}}, \Gamma_{\text{Forb}}) \) be an access structure on a set of \( n \) participants. Two collections (multisets) of \( n \times m \) Boolean matrices \( C_0, C_1 \) constitute a visual cryptography scheme \( (\Gamma_{\text{Qual}}, \Gamma_{\text{Forb}}, m) \)-VCS with pixel expansion \( m \) if there exists values \( \alpha(m) \) and \( \{t_X\}_{X \in \Gamma_{\text{Qual}}} \) satisfying:

1. Any qualified set \( X = \{i_1, i_2, \ldots, i_p\} \in \Gamma_{\text{Qual}} \) can recover the shared image by stacking their transparencies. Formally, for any \( M \in C_0 \), the "or" \( V \) of rows \( i_1, i_2, \ldots, i_p \) satisfies \( w(V) \leq t_X - \alpha(m) \cdot \hat{m} \), whereas for any \( M \in C_1 \) it results that \( w(V) \geq t_X \), where \( w(V) \) is the Hamming weight of \( V \).

2. Any forbidden set \( X = \{i_1, i_2, \ldots, i_p\} \in \Gamma_{\text{Forb}} \) has no information on the shared image, i.e. the two collections of \( p \times m \) matrices \( D_t \), with \( t \in \{0, 1\} \), obtained by

![Figure 5.1](image_url)

**Figure 5.1.** The concept of Naor and Shamir’s \((2, 2)\) visual cryptography scheme with four subpixels: (a) the original secret image, (b) the first share, (c) the second share, and (d) the recovered image by superimposing (b) and (c).
restricting each $n \times m$ matrix in $C_t$ to rows $i_1, i_2, \ldots, i_p$ are indistinguishable in the sense that they contain the same matrices with the same frequencies.

Instead of working with the collections $C_0$ and $C_1$, it is convenient to consider only two $n \times m$ boolean matrices $S^0$ and $S^1$ called basis matrices which satisfy the following definition.

**Definition 5.2.** Let $(\Gamma_{\text{Qual}}, \Gamma_{\text{Forb}})$ be an access structure on a set $\mathcal{P}$ of $n$ participants. A $(\Gamma_{\text{Qual}}, \Gamma_{\text{Forb}}, m)$-VCS with relative difference $\alpha(m)$ and a set of thresholds $\{t_X\}_{X \in \Gamma_{\text{Qual}}}$ is realized using the $n \times m$ basis matrices $S^0$ and $S^1$ if the following two conditions hold:

1. If $X = \{i_1, i_2, \ldots, i_p\} \in \Gamma_{\text{Qual}}$, then the "or" $V$ of the rows $i_1, i_2, \ldots, i_p$ of $S^0$ satisfies $w(V) \leq t_X - \alpha(m)\bar{m}$, whereas for $S^1$ it results that $w(V) \geq t_X$.
2. If $X = \{i_1, i_2, \ldots, i_p\} \in \Gamma_{\text{Forb}}$, the two $p \times m$ matrices obtained by restricting $S^0$ and $S^1$ to rows $i_1, i_2, \ldots, i_p$ are equal up to a column permutation.

The collections $C_0$ and $C_1$ are obtained by permuting the columns of the corresponding basis matrix in all possible ways. Note that, in this case, the sizes of the collections $C_0$ and $C_1$ are the same.

**Share distribution algorithm:** The encryption process is described as follows. For each pixel $P$ in the secret image, do the following:
(a) Generate a random permutation $\pi$ of the set $\{1, \ldots, m\}$.

(b) If $P$ is a black pixel, then apply $\pi$ to the columns of $S^0$; else apply $\pi$ to the columns of $S^1$. Call the resulting matrix $T$.

(c) For all $i = 1, \ldots, n$, the row $i$ of $T$ comprises the $m$ sub-pixels of $P$ in the $i$th share.

In [28] it is described a VCS with perfect reconstruction of black pixels (where all the subpixels associated in a reconstructed image with a black pixel are black), in this case, for $i = 1, 2, \ldots, q$, let $(\Gamma^i_{\text{Qual}}, \Gamma^i_{\text{Forb}})$ be an access structure on a set $\mathcal{P}$ of $n$ participants. If a participant $j \in \mathcal{P}$ is not essential for the $i$-th structure, it is assumed that $j \notin \Gamma^i_{\text{Forb}}$ and that $j$ does not receive any share. Suppose there exists a $(\Gamma^i_{\text{Qual}}, \Gamma^i_{\text{Forb}})$-VCS with a pixel expansion $m_i$ and basis matrices $S_i^0$ and $S_i^1$, for $i = 1, 2, \ldots, q$. The basis matrix $S^0$ (resp) of a VCS for the access structure $(\Gamma^i_{\text{Qual}}, \Gamma^i_{\text{Forb}})$ where $\Gamma^i_{\text{Qual}} = \sum_{i=1}^{q} \Gamma^i_{\text{Qual}}$ and $\Gamma^i_{\text{Forb}} = \bigcap_{i=1}^{q} \Gamma^i_{\text{Forb}}$ is constructed as the concatenation of some auxiliary matrices $\hat{T}^0_i$ (resp), for each $i = 1, 2, \ldots, q$. Such matrices are obtained as follows: for each $j = 1, 2, \ldots, n$, the $j$-th row of $\hat{T}^0_i$ (resp) has all ones as entries if the participant $j$ is not essential for $(\Gamma^i_{\text{Qual}}, \Gamma^i_{\text{Forb}})$, otherwise it is the row of $S^0_i$ (resp) corresponding to participant $j$. Hence, $S^0 = \hat{T}^0_1 \oplus \hat{T}^0_2 \oplus \cdots \oplus \hat{T}^0_q$ and $S^1 = \hat{T}^1_1 \oplus \hat{T}^1_2 \oplus \cdots \oplus \hat{T}^1_q$ where $\oplus$ denotes the concatenation of matrices. The resulting VCS has a pixel expansion $m = \sum_{i=1}^{q} m_i$.

**Lattice-Based VSSS.** Now, we present the definition of a lattice-based TVS in accordance with Koga and Yamamoto [54]. Let $m > 0$ be given and $\mathcal{L}$ a finite lattice of a finite number of colors that can be physically realized. Suppose that $\mathcal{C} = \{c_1, c_2, \ldots, c_J\}$ is a subset of elements in $\mathcal{L}$, which is not necessarily a sublattice of $\mathcal{L}$. For all $q$ satisfying $1 \leq q \leq k$ and distinct $i_1, i_2, \ldots, i_q \subseteq \{1, 2, \ldots, n\}$ define a mapping $h^{(i_1, i_2, \ldots, i_q)} : (\mathcal{L}^m)^n \to \mathcal{L}^m$ by

$$h^{(i_1, i_2, \ldots, i_q)}(x) = x_{i_1} \lor x_{i_2} \lor \cdots \lor x_{i_q}, \quad (5.1)$$

where $x = (x_1, x_2, \ldots, x_n) \in (\mathcal{L}^m)^n$. If there exists $(X_{c_j}, Y_{c_j})_{1 \leq j \leq J}$ is called the lattice-based $(k, n)$ VSSS with colors $\mathcal{C}$.

1. For all $j = 1, 2, \ldots, J$ and distinct $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$, all $x \in X_{c_j}$ satisfy $h^{(i_1, i_2, \ldots, i_k)}(x) \in Y_{c_j}$.

2. For all $q < k$ and $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ define

$$X^{(i_1, i_2, \ldots, i_q)}_{c_j} = \{(x_{i_1}, x_{i_2}, \ldots, x_{i_q}) : (x_1, x_2, \ldots, x_n) \in X_{c_j}\}$$

Then $X^{(i_1, i_2, \ldots, i_q)}_{c_j}$, $j = 1, 2, \ldots, J$ are indistinguishable in the sense that they contain the same elements with the same frequencies.

3. For all $c_j \in \mathcal{C}$ satisfying $c_j \neq 1 \in \mathcal{L}$ all the elements in $Y_{c_j}$ are composed by 1’s and at least one $c_j$. In case that $c_j = 1$, $Y_{c_j}$ has only one element composed by $m$ 1’s.

As an example, see the Figure 5.3.
5.3 A Matrix Problem Induced by a Lattice-Based VSSS

In this section, we interpret the Koga-Yamamoto scheme as a matrix problem. To do that, we consider matrix representations \((d, A)\) of a lattice \(L\) induced by a finite number of colors. In this case, for a \((k, n)\) VSSS we have that \(A \in \mathcal{L}_{d_0 \times d}\) with \(d_0 = m\) is the pixel expansion, \(d_i = d_j = n\) for any \(i, j \in L\) is the size of the set of participants and \(k\) is the number of qualified participants. In other words, each vertical stripe consists of \(n\) generators (of the corresponding module \(U_{c_i}^j\) for each color \(c_j \in L\)) with \(k < n\) linear independent columns. Besides a lattice-color matrix representation of \(L\) is an \(m \times d\) rectangular matrix separated into vertical stripes with the same number \(c\) of columns. In this case, columns in each stripe \(M_x\) are indistinguishable (i.e. they have the same elements appearing with the same frequency) and constitute a composition (i.e. a partition where the order matters) of a given vector \(F_x \in \mathcal{L}_m^m\) for any \(x \in \mathcal{C}\).

For \(\{x_1, x_2, \ldots, x_j\} = \mathcal{C}\), let \(M\) and \(M'\) be lattice-color matrix representations with associated vectors \(F_{x_1} \cdots F_{x_j}\) and \(F'_{x_1} \cdots F'_{x_j}\) respectively then \(M\) and \(M'\) are called equivalent if and only if there exists some permutation \(\pi \in S_m\) such that if \(F'_{x_j} = (x_{j}^{\pi(1)}, \ldots, x_{j}^{\pi(m)})\) then \(F_{x_j} = (x_{j}^{1}, \ldots, x_{j}^{m})\) for each chosen \(x_j \in \mathcal{L}\).

The following result establishes the existence of matrix representations whose columns within each vertical stripe \(M_x\) constitute compositions of a given set of fixed vectors \(F_x\).

**Theorem 5.1.** If \(x \in \mathcal{C}\) and \(F_x\) consists of \(k_1\) 1’s and \(k_x\) \(x\)'s with \(k_1 + k_x = m\) then there exist \(n\) indistinguishable vectors \(g^1_x, \ldots, g^n_x\) such that \(F(x) = \sum_{i=1}^{n} g^i_x\).

**Proof.** Let \(F(x)\) be such that, \(F_x = (a_1, \ldots, a_m) \in \mathcal{L}_m^m\) with \(k_1\) 1’s, \(k_x\) \(x\)'s and \(k_1 + k_x = m\), and a permutation \(\pi \in S_m\) such that \(F_x = (a_{\pi(1)}, \ldots, a_{\pi(m)}) = (x, \ldots, x, 1, \ldots, 1)\). We fix points \(y \in \mathcal{L} \cap x_{\Delta}\), \(z \in \mathcal{L}\), and an \(m \times n\) matrix \(R_x\) such that the entries of the first \(k(x)\) rows are \(y\)'s and the entries of the remain rows are \(z\)'s. Then there exist integers \(q_1\) and \(r_1\) such that \(k_1 = nq_1 + r_1\) with \(0 \leq r_1 < n\), \(k_x = nq_x + r_x\) with \(0 \leq r_x < n\). Let us consider...
the matrix block

\[
\overline{M}_x = R_x - \begin{array}{cc}
yI_n & xI_n \\
\vdots & \vdots \\
yI_n & xI_n \\
yA & A \\
zI_n & I_n \\
\vdots & \vdots \\
zI_n & I_n \\
zB & B
\end{array}
\]

where \(I_n\) denotes an \(n \times n\) matrix, the number of matrix blocks \(xI_n\) (\(I_n\)) in \(\overline{M}_x\) is given respectively by \(nq_x\) (\(nq_1\)). In this case, \(A\) is an \(r_x \times n\) matrix such that \(A \lor (-A) = 0\) with the form

\[
A = \begin{bmatrix} I_{r_x} \\ x, \ldots, x \end{bmatrix}
\]

and the block \(B\) is an \(r_1 \times n\) matrix with the form

\[
B = \begin{bmatrix} I_{r_1} \\ 1, \ldots, 1 \end{bmatrix}
\]

It is worth noting that empty blocks in \(A\) and \(B\) denote matrices whose entries are all zeroes. By construction columns of matrix \(\overline{M}_x\) constitute an \(n\)-elements set of indistinguishable vectors associated to the fixed vector \(F_x\). If matrix \(M_x\) is obtained from \(\overline{M}_x\) by applying permutation \(\pi\) to the rows then columns of \(M_x\) correspond to \(n\) indistinguishable vectors which define a composition of the fixed vector \(F_x\). □

The following result defines admissible transformations which guarantees the existence of equivalent lattice-color matrix representations. Therefore, it guarantees the construction of different types of \((k, n)\) lattice-based VSSS.

**Theorem 5.2.** Let \(M\) and \(M'\) be two lattice matrix representations of a given lattice \(\mathcal{L}\) then \(M\) and \(M'\) are equivalent if \(M\) and \(M'\) can be turned one into each other by applying the following transformations:

(a) row permutations of the whole matrix.

(b) column permutations within a given vertical stripe.

(c) multiplication of a given column \(j\) in the stripe \(M_x\) by some scalar \(z \in (\lambda^2_j)^\vee\), where \(\lambda^2_j\) is the maximum of all entries in such a column.
(d) addition of a given \( j \)-th column in the stripe \( M_x \) to the \( j \)-th column in the stripe \( M_y \) with coefficients in \((\delta^y_j)_\Delta\), where \( \delta^y_j \) is the minimum of all entries in the column of \( M_y \). If \( x \leq y \) in \( \mathcal{L} \).

Proof. Row permutations of \( M \) determine same indistinguishable vectors up to permutations. That is, if \( F_{x_1}, \ldots, F_{x_t} \) are fixed vectors attached to the representation \( M \) and \( F'_{x_1}, \ldots, F'_{x_t} \) are corresponding fixed vectors attached to the matrix \( M' \) obtained from \( M \) by row permutations then there exists a permutation \( \pi \in S_t \) such that

\[
F_{x_i} = (x^1_{j_1}, \ldots, x^m_{j_m}) \quad \text{then} \quad F'_{x_j} = (x^\pi(1)_{j_1}, \ldots, x^\pi(m)_{j_m}),
\]

thus \( M \) is equivalent to \( M' \). Besides, column permutations in a given vertical stripe \( M_x \) keep invariant vectors \( F_{x_1}, \ldots, F_{x_t} \). Therefore, if \( M' \) is obtained from \( M \) by transformations of type (a) then \( M \) is equivalent to \( M' \).

On the other hand, if \( \lambda^x_j \) is the maximum of all entries in a column \( j \in M_x \) and \( z \in (\lambda^x_j)\textsuperscript{y} \) then \( z \geq \lambda^x_j \) and \( \lambda^x_j \geq g^x_{kj} \) where \( g^x_{kj} = (g^x_{kj_1}, \ldots, g^x_{kj_m}) \) is the \( j \)-th column of the stripe \( M_x \) then \( zg^x_{kj} = g^x_j \). Therefore if \( M' \) has attached fixed vectors as defined above and \( M \) is obtained via transformations of type (b) then \( F'_{x_i} = F_{x'i} \) for any \( 1 \leq i \leq t \) therefore \( M \) is equivalent to \( M' \).

Finally, let us suppose that \( \delta^y_j \) is the minimum of the set of entries of the \( j \)-th column in a vertical stripe \( M_y \) \( (g^y_j = (g^y_{j1}, \ldots, g^y_{jm})) \), that is, \( \delta^y_j \leq g^y_{kj} \) for all \( 1 \leq k \leq m \) and if \( g^x_{ki} = (g^x_{ki1}, \ldots, g^x_{kim}) \) is the \( i \)-th column in \( M_x \) and we add \( z \land g^x_{ki} \) to the column \( g^y_{kj} \) with \( z \in (\delta^y_j)_\Delta \) then \( z \leq g^y_{kj} \) for all \( 1 \leq k \leq m \) thus \( (z \land g^x_{ki}) \lor g^y_{kj} = g^y_{kj} \) which means that \( (z \land g^x_{ki}) \lor g^y_{kj} \). Therefore if \( M' \) is obtained from \( M \) via transformations of type (c) then \( M = M' \). \( \square \)

The following result establishes the structure of vectors \( F_x \) with \( x \in \mathcal{C} \).

**Theorem 5.3.** If \( x \in \mathcal{C} \) with \( x \neq a \lor b \) for any \( x \neq a \) and \( x \neq b \) and \( F_x \) consists of \( k_1 \) \( 1 \)'s and \( k_x \) \( x \)'s with \( k_1 + k_x = m \) then an indistinguishable vector \( g_x \) consists of at least \( \lceil \frac{k_1}{n} \rceil \) \( x \)'s and at least \( \lceil \frac{k_x}{n} \rceil \) \( 1 \)'s where \( n \) is the number of generators in \( M_x \). Moreover if \( m_x \) is the number of \( x \)'s in \( g_x \) and \( m_1 \) is the number of \( 1 \)'s in \( g_x \) then there exist \( m - (m_1 + m_x) \) elements in \( x_\bullet \) in \( g_x \).

Proof. Let us suppose that \( F_x = (x, \ldots, x, 1, \ldots, 1) \) without loss of generality, where \( k_1 \) is the number of \( 1 \)'s and \( k_x \) is the number of \( x \)'s with \( k_1 + k_x = m \) and \( x \in \mathcal{C} \). Since \( x \) cannot be obtained as a supremum of two points \( y \) and \( z \) with \( y \neq x \) and \( z \neq x \) then the number of \( x \)'s must be at least \( \lceil \frac{k_x}{n} \rceil \). Indeed, for each occurrence of \( x \) each part of the partition of the vector \( F_x \) contains at least an \( x \) if they are ordered in the last \( k_x \) rows of the vertical stripe the result is obtained by using as few \( x \)'s as possible. Similarly, the result can be obtained for a minimal number of \( 1 \)'s if it is considered that \( 1 \in \mathcal{C} \). That there exist \( m - (m_1 + m_x) \) elements in \( x_\bullet \) follows from arguments used in Theorem 1. \( \square \)

We note that structure of the form \((\Gamma^i_{\text{Quad}}, \Gamma^i_{\text{Forb}})\) can be interpreted from this point of view as indecomposable lattice-color matrix representations (see Figure 5.4).
CHAPTER 5. MATRIX PROBLEMS INDUCED BY VISUAL CRYPTOGRAPHY SCHEMES

Remark 5.1. Lattice-based VSSS can be seen as particular cases of some matrix problems. Since permutations are part of the corresponding admissible transformations then matrix representations allow to define multiple schemes of visual cryptography.

Figure 5.4. Example of a matrix representation induced by a (2,2) lattice-based VSSS.

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    Y \\
\end{bmatrix}
\begin{bmatrix}
    C \\
    1 \\
    C \\
\end{bmatrix}
\begin{bmatrix}
    G \\
    1 \\
    G \\
\end{bmatrix}
\]

\[
F_Y \quad F_C \quad F_G
\]
APPENDIX A

Minimal posets of infinite prinjective type

The infinite series $\mathcal{P}_{1,n}, \mathcal{P}_{2,n+1}, \mathcal{P}'_{2,n}, \mathcal{P}_{3,n}, \mathcal{P}'_{3,n}, \mathcal{P}''_{3,n+1}, n \geq 0$.

Here, we had drawn infinite series of finite posets which are defined from el subposet with blue lines. Moreover, the posets $\mathcal{P}_{2,0}$ and $\mathcal{P}'_{3,0}$ are defined for the value $n = -1$. 

1Here, we had drawn infinite series of finite posets which are defined from el subposet with blue lines. Moreover, the posets $\mathcal{P}_{2,0}$ and $\mathcal{P}'_{3,0}$ are defined for the value $n = -1$. 

76
One-peak enlargements $\mathcal{P}_4 - \mathcal{P}_8$ of Kleiner’s posets.

Two-peak posets $\mathcal{P}_9 - \mathcal{P}_{31}$.
Posets \( P_{32}, \ldots, P_{110} \) with \( 3 \leq \max |P_i| \leq 5 \).

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Future work

We give a summary of some questions and open problems regarding the topics investigated in this thesis.

In relation to the posets of type \(\mathcal{A}\) and the geometric model for its categories of socle-projective representations given in Chapter 2, we could consider the following.

- What would be the posets of type \(\mathcal{D}\)? and how can \([71]\) be used to give a geometric model to the categories of socle-projective representations of those posets?
- In which cases is the category of socle-projective representations of a poset of type \(\mathcal{A}\) either a torsion class or a torsion-free class in the module category of a Dynkin algebra of type \(\mathcal{A}\)?
- To give a solution to the matrix problems associated to posets of type \(\mathcal{A}\) in the sense of \([82]\).

Regarding Zavadskij modules over finite-dimensional algebras studied in Chapter 3, we could consider the following.

- What can we say about the projective dimension of the Zavadskij modules over finite-dimensional algebras?
- What can we say about Zavadskij modules over another kind of rings?

In respect of \(\tau\)-partitions defined in Chapter 4, we could consider:

- The enumeration of the preprojective partitions in the Auslander-Reiten quiver of a Dynkin algebra.
- Graphs arising partitions of integer numbers whose parts are exactly three triangular numbers and their relation to the representation theory of quivers.

Finally, about visual cryptography schemes, we could consider others schemes and its possible relation to different kind of multiserial algebras.
Bibliography


