Associated prime ideals of noncommutative rings of polynomial type

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Title

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Título

Ideales primos asociados de anillos no conmutativos de tipo polinomial

Abstract: In this work we study the associated prime ideals of some noncommutative rings of polynomial type. In the literature we find that these ideals were characterized in a first work by Brewer and Heinzer [BH74], where they shown that the associated prime ideals of a polynomial ring over a ring R can be extended from the associated prime ideals of R. From that, different authors have extended this result to other structures as Annin did in [Ann04] developing his work over Ore extensions. Another work that we highlight be the one carried out by Bhat in [Bha10a] where he characterized the associated prime ideals over weak σ-rigid rings. From the results found in the literature, in this work we extend these works for the skew PBW extensions introduced by Gallego and Lezama [GL11]. We develop our work in two parts: first, we extend the results of [Ann04] for skew PBW extensions. With this objective in mind, we present some properties of this structure under the condition of (Σ, Δ)-compatibility (defined in [HKA17] and [RS18a]), and we define the notion of annihilator-compliant (notion defined by Annin in [Ann04] for Ore extensions) for the context of skew PBW extensions. As a second part, we extend the results of [Bha10a] for the skew PBW extensions over weak Σ-rigid rings introduced in [RS18b].

Resumen: En el presente trabajo estudiamos los ideales primos asociados de algunos anillos no conmutativos de tipo polinomial. En la literatura encontramos que estos ideales fueron caracterizados en un primer trabajo por Brewer y Heinzer [BH74], donde ellos muestran que los ideales primos asociados de un anillo de polinomios sobre un anillo R pueden ser extendidos a partir de los ideales primos asociados de R. A partir de esto, diferentes autores han extendido este resultado para otras estructuras como lo hizo Annin en [Ann04] desarrollando su trabajo sobre las extensiones de Ore. Otro trabajo que resaltamos es el realizado por Bhat en [Bha10a] en donde él caracterizó los ideales primos asociados sobre anillos σ-rígidos débiles. A partir de los resultados encontrados en la literatura, en este trabajo extendemos estos trabajos para las extensiones PBW torcidas introducidas por Gallego y Lezama [GL11]. Nosotros desarrollamos nuestro trabajo en dos partes: primero, extendemos los resultados de [Ann04] para las extensiones PBW torcidas. Con este objetivo en mente, presentamos algunas propiedades de esta estructura bajo la condición de (Σ, Δ)-compatibilidad (definida en [HKA17] y [RS18a]) y definimos la noción de anulador complaciente (noción definida por Annin en [Ann04] para extensiones de Ore) sobre las extensiones PBW torcidas. Como una segunda parte, extendemos los resultados de [Bha10a], para las extensiones PBW torcidas sobre anillos Σ-rígidos débiles introducidos en [RS18b].

Keywords: *Associated prime ideal; noncommutative ring; skew PBW extension; compatible ring; weak Σ-rigid ring.*

Palabras clave: *Ideal primo asociado; anillo no conmutativo; extensión PBW torcida; anillo compatible; anillo Σ-rígido débil.*

Dedicatory

To my parents.

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INTRODUCTION

The importance of associated prime ideals can be appreciated in works such as [BH74], [Fai00], [Ann02a], [Ann02b], [Ann04], [Bha10a] and [Nor12]. The associated prime ideals in commutative algebra are the center in the theory of primary decomposition [AM69]. In addition, the associated prime ideals facilitate the study of uniform modules over Noetherian rings [GW04].

Given the importance of these ideals, some results have developed not only focusing on commutative algebra: Brewer and Heinzer [BH74] and Faith [Fai00] shown that if R is a commutative ring, then the associated prime ideals of the commutative polynomial ring $R[x]$ (viewed as module over itself) are the ideals of the form $P[x]$, where P is an associated prime ideal of R. Annin in his PhD thesis [Ann02a], extended the result presented by them. Annin focus his work on three types of ring extensions: general polynomial modules, noncommutative rings, and to Ore polynomial extensions [Ore33]. He generalized the results obtained by Brewer, Heinzer and Faith proving that the result still holds in the more general setting of a polynomial module $M[x]$ over a skew polynomial ring $R[x;\sigma,\delta]$, with possibly noncommutative base ring R. For this, he defined two concepts over the right module M_R: the σ-compatibility and annihilator-compliant. On the other hand, Bhat [Bha08] investigated associated prime ideals of Ore extensions over a Noetherian ring R and an automorphism σ of R. Next, Bhat [Bha10a] studied associated prime ideals of weak σ-rigid rings and their extensions.

Our object of study in this work are the skew PBW extensions. These extensions were introduced by Gallego and Lezama [GL11] and are more general than Ore extensions of injective type [Ore33] and PBW extensions defined by Bell and Goodearl [BG88]. We want to continue to the study of ideals about these extensions as in [LAR15], [RS18c], [NR20] and [RS19a]. Our aim in this work is to characterize the associated prime ideals of these noncommutative rings extending the results obtained by Annin [Ann04] and Bhat [Bha10a], to the context of the skew PBW extensions and skew PBW extension over weak Σ-rigid rings, respectively. The notion of weak Σ-rigid ring was defined and studied by Reyes and Suárez [RS18b], generalizing some properties of the weak σ-rigid rings in [Ouy08].

We establish the structure of our work. In Chapter 1, Section 1.1, we recall some definitions about associated prime ideals. In Section 1.2 we present the results obtained in commutative algebra, and in Section 1.3 we consider the results presented by Annin and Bhat about the associated prime ideals in the noncommutative structures in which they developed their results. In Chapter 2 we present the main results of this work. First in Section 2.1, we focus on skew PBW extensions. We also present two definitions: the (Σ,Δ)-compatibility studied previously by

Hashemi et al. [HKA17] and Reyes and Suárez in [RS18a], and we define annihilator-compliant over skew PBW extensions; this is introduced with the objetive of extending the definition presented by Annin [Ann04]. These definitions are very important for the construction of the proof of the main result (Theorem 2.17) that is an extension of main theorem of Annin, which characterizes the associated prime ideals over skew PBW extensions. In Section 2.2 we present the notions of Σ-rigid ring and weak Σ-rigid ring; these notions have been studied by Reyes in [Rey15] and [RS18b]; over this structure we extend the main result of Bhat [Bha10a]. We present the characterization of the associated prime ideals of skew PBW extension over weak Σ-rigid rings (see the Theorems 2.23 and 2.24). Finally, we present some conclusions and a possible work of research.

CHAPTER 1

ASSOCIATED PRIME IDEALS

In this chapter we propose to address some basic terms related to the associated prime ideals and their most recent results presented by different authors. For this, in Section 1.1 we talk about some properties basic of these ideals in the context of commutative and noncommutative algebra. In Section 1.2 we describe the associated prime ideals with results found in the literature in commutative algebra; these correspond to those shown by Brewer and Heinzer [BH74] and Faith [Fai00]. Next, in Section 1.3 we describe the structures on which Annin en [Ann04] developed his work, followed by Annin's characterization of these ideals, describing the structure of his work and highlighting the most important proofs. In the same way, we show the result of Bhat in [Bha10a] obtained for the σ-rigid rings and the weak σ-rigid rings.

1.1 BASIC PROPERTIES OF ASSOCIATED PRIME IDEALS

We start by presenting the definition of associated prime ideal introduced in the literature.

DEFINITION 1.1 ([GP08], DEFINITION 1.3.10). Let R be a commutative ring and $I \subseteq R$ be an ideal.

1. I is a *prime ideal*, if $I \neq R$ and if for each $a, b \in R$, $ab \in I \Longrightarrow a \in I$ or $b \in I$.
2. The set of prime ideals is denoted by $\operatorname{Spec}(R)$.

DEFINITION 1.2 ([GW04], PAGE 39 AND 53). A *minimal prime ideal* in a ring R is any prime ideal on R that does not properly contain any other prime ideal. The *prime radical* of R is the intersection of all prime ideals of R.

DEFINITION 1.3 ([BOU72], PAGE 261 AND DEFINITION IV.1, [LAM98], LEMMA 3.56). Let R be a commutative ring and let M_R be a right R-module. For each $x \in M_R$, we define the set $\operatorname{Ann}(x)$ as the elements $a \in R$ such that $ax = 0$. Each element of $\operatorname{Ann}(x)$ is said to be an *annihilator* of x. A prime ideal P is said to be *associated* with M_R, if there exists $x \in M_R$ such that P is equal to be the annihilator of x. The set of associated prime ideals with M is denoted by $\operatorname{Ass}(M_R)$. Thus,

$$\operatorname{Ass}(M_R) := \{P \in \operatorname{Spec}(R) \mid \text{ there exists } x \in M \text{ such that } P = \operatorname{Ann}(x)\}.$$

It should be noted that since the annihilator of 0 is R, an element $x \in M_R$ whose annihilator is a prime ideal is necessarily nonzero. If an R-module M_R is the union of a family (M_j), $(j \in J)$, of submodules, then

$$\mathrm{Ass}(M_R) = \bigcup_{j \in J} \mathrm{Ass}(M_j).$$

If we want to formulate a generalization of the set of associated prime ideals, then we suppose R a noncommutative ring and let M_R be a right R-module; for any element $m \in M$, $\mathrm{Ann}(m)$ refers to the right annihilator of m in R, and it is only right ideal in general. Thus, $\mathrm{Ann}(m)$ is not necessarily a two-sided ideal and for that the commutative definition of associated prime is untenable in the noncommutative theater. That is the reason to formulate the Definition (1.5) that extends the associated prime ideals from the commutative case to the noncommutative case, by replacing elements of the module by submodules.

Definition 1.4 ([Lam98], Page 85). We say that an R-module N_R is *prime*, if $N_R \neq 0$ and $\mathrm{Ann}(N_R) = \mathrm{Ann}(N'_R)$, for every nonzero submodule $N'_R \subseteq N_R$.

Definition 1.5 ([Lam98], Page 86). Let M_R be an R-module. A prime ideal P of R is called an *associated prime* of M, if there exists a prime submodule $N_R \subseteq M_R$ such that $P = \mathrm{Ann}(N_R)$. The set of associated primes of M_R is denoted by $\mathrm{Ass}(M_R)$.

Note that if N is a prime module, then $\mathrm{Ass}(N) = \{\mathrm{Ann}(N)\}$. For this, let us see the following Remark.

Remark 1 ([Lam98], Example 3.55). Let P be an ideal in a ring R. Then $N = (R/P)_R$ is a prime module if, and only if, P is a prime ideal, in which case we have $\mathrm{Ass}(N) = \{P\}$. In fact, if N is a prime module, then, as we have observed, $\mathrm{Ann}(N) = P$ is a prime ideal. Conversely, assume P is a prime ideal and consider any nonzero submodule $N' = Q/P \subseteq R/P$, where $P \subset Q$ is a right ideal. We have for any $r \in R$, $N' \cdot r = 0$, thus $Q \cdot r \subseteq P$ and we have $Q \cdot (rR) \subseteq P$, whence we conclude that $r \in P$. Hence, $\mathrm{Ann}(N') = P = \mathrm{Ann}(N)$, so N is a prime module, with $\mathrm{Ass}(N) = \{P\}$.

Next, we show that Definition 1.5 is compatible with the definition over commutative algebra. We consider the proof presented by Lam [Lam98]. Let R be a commutative ring and let $P \in \mathrm{Ass}(M_R)$ (with M_R be an R-module). Then $P = \mathrm{Ann}(N)$ where $N \subseteq M$ is a prime module by Definition 1.5. If we choose any element $0 \neq m \in N$ arbitrary but fixed, we have that $P = \mathrm{Ann}(mR) = \mathrm{Ann}(m)$ by commutativity of R as in the definition in commutative case. Adding, if we consider P a prime ideal of R such that $P = \mathrm{Ann}(m)$ with $m \in M_R$, then $mR \cong R/P$ is a prime module by Remark 1, thus $P \in \mathrm{Ass}(M_R)$. Therefore Definition 1.3 is compatible with Definition 1.5. Next, we define one special case for associated prime ideals.

Definition 1.6 ([GW04], Page 95). Let R be a ring and M_R be an R-module. We say that M_R is an *uniform module*, if M_R is a nonzero module such that the intersection of any two nonzero submodules of M_R is nonzero.

Proposition 1.1 ([GW04], Lemma 5.26). *Let M_R be a uniform right module over a right Noetherian ring R. Then there is a unique prime ideal P in R such that P equals the annihilator of some nonzero submodule of M_R and P contains the annihilators of all nonzero submodules of M_R. Moreover, P is the unique associated prime of M_R.*

Definition 1.7 ([GW04], Page 102). Let R be a right Noetherian ring. If M is an uniform right module over R, the unique associated prime of M is called the *assassinator* of M. We consider the set set

$$\mathbb{A}(M_R) := \{\text{Assas}(J) \mid J \text{ is a uniform right } R\text{-submodule of } M\}.$$

1.2 Associated prime ideals: commutative algebra

The associated prime ideals have a first development in commutative algebra with Brewer and Heinzer in [BH74]. The main results of their paper show that for a commutative ring R, every $P \in \text{Ass}(R[x])$ (with $R[x]$ the commutative polynomial ring) is extended to $P = P_0[x]$, where $P_0 = P \cap R \in \text{Ass}(R)$. Brewer and Heinzer's proof uses localization theory and their proof is based on the following definition:

Definition 1.8 ([Laz69], Definition 1.1, page 92). Let R be a ring, M an R-module and P a prime ideal. We say that P is *associated* with M, if there exists $x \in M$ such that P is a minimal prime ideal containing the annihilator of x. We call *assasin* of M, and we denote $\text{Ass}_R(M)$ or $\text{Ass}(M)$, the set of ideals associated with M.

In their paper, they also prove that associated primes of regular elements, that is, nonzero divisors, behave well and that the associated primes of regular elements of $R[X]$ (with X be a collection of indeterminates) are related with associated primes of regular elements of R. More exactly

Proposition 1.2 ([BH74], Corollary 8). *Suppose that P is an associated prime of a regular element of $R[X]$ and let $Q = P \cap R$. If Q contains a regular element, then $P = Q[X]$ and Q is an associated prime of a regular element. Thus, if $R = D$ is an integral domain and P is an associated prime of a nonzero polynomial in $D[X]$, then $P \cap D = (0)$ or $P = (P \cap D)[X]$ and $P \cap D$ is an associated prime of a principal ideal.*

In addition to these results, they presented the following propositions related to the finitude of the associated prime ideals.

Proposition 1.3.

1. **[[BH74], Corollary 9]** *Let D be an integral domain and let X be a collection of indeterminates over D. If principal ideals of D have only finitely many associated primes, the same is true of $D[X]$.*

2. **[[BH74], Propositions 10]** *Let D be an integral domain having the property that principal ideals in D have only finitely many associated primes. If R is locally a polynomial ring over D and it is contained in a finitey generated ring extension of D, then R is a finitely generated ring extension of D.*

Brewer and Heinzer [BH74], p. 6, presented an example of a unique factorization domain D (that is, a domain in which principal ideals have only a finite number of associated primes) and a finite integral extension R of D such that some principal ideal of R has an infinite number of associated primes.

Followed by this, Faith in [Fai00] proved the same bijection between $\text{Ass}(R[X])$ and $\text{Ass}(R)$ from another way. He comment the following in his paper:

The aim of this paper is to give a new and direct proof of the theorem:

Proposition 1.4 (Brewer-Heinzer [BH74]). *If R is a commutative ring, then the restriction map*

$$\varphi : \mathrm{Ass}(R[x]) \to \mathrm{Ass}(R)$$

is a bijection. Thus, every $P \in \mathrm{Ass}(R[x])$ is **extended**, *that is, $P = P_0[x]$, where $P_0 = P \cap R \in \mathrm{Ass}(R)$.*

Faith's proof uses Definition 1.5 and results about uniform dimension, comparing the cardinality of the set $\mathrm{Ass}(R)$ and the uniform dimension of R. Furthermore, he needs two lemmas and a proposition from Shock [Sch72]:

Proposition 1.5. 1. **[[Sch72], First Lemma]** *If R is a ring (not necessarily commutative), and $a_1, \ldots a_n \in R$, then either:*

- $\mathrm{Ann}(a_i) = \mathrm{Ann}(a_j)$, *for every $i, j \leq n$;*
  or
- *There exists $b \in R$ and $j \leq n$ such that $a_j b \neq 0$ and $\mathrm{Ann}(a_j b) = \mathrm{Ann}(a_k b)$ whenever $a_k b \neq 0$.*

2. **[Sch72]** *Let R be a ring (not necessarily commutative). For a nonzero polynomial $f(X) \in R[X]$, there exists $b \in R$ so that $f(X)b \neq 0$ and the coefficients of $f(X)b$ have equal right annihilator.*

Remark 2 ([Fai00]). The referee of this paper add that: "if $P \in \mathrm{Ass} R[x]$ and $Q = P \cap R$, then $P = Q[x]$. Since, annihilators of finitely generated ideals behave well with respect to localization, e.g. by localizing at $R \backslash Q$ one sees this. Furthermore, since Q is the intersection of the coefficient of the polynomial $f \in R[x]$ such that $P = \mathrm{Ann}(f(x))$, and since Q is prime, then Q is the annihilator of one of the coefficients."

1.3 Associated prime ideals: noncommutative algebra

In this section we mention some results found in the literature for associated prime ideals of diferents authors in the context of noncommutative algebra. We emphasize that these results are very important in the study of ideals and all of them are a reference for research in this area. We start with one of the algebraic structures where the characterization of these ideals has been developed.

1.3.1 Ore extensions

These extensions were introduced by [Ore33] and cover a large class of noncommutative polynomials extensions. Ore extensions arise in several branches of mathematics, such as Lie theory, mathematical physics, as well important applications within noncommutative ring theory, (see [Ore33] or [Ros95] for more details).

Definition 1.9 ([MR01], Proposition 1.2.1). Let R be an associative ring with identity and

$\sigma : R \to R$ an endomorphism. An additive map $\delta : R \to R$ is called a *σ-derivation of R*, if

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s, \qquad \text{for all } r, s \in R.$$

Notice that, in particular, $\delta(1) = \delta(1 \cdot 1) = \sigma(1)\delta(1) + \delta(1)1 = 2\delta(1)$, whence $\delta(1) = 0$.

Definition 1.10 ([MR01], Proposition 1.2.3). Let R be an associative ring with identity, $\sigma : R \to R$ an endomorphism and $\delta : R \to R$ a σ-derivation of R. A ring S with the following properties

(O1) S contains R as a proper subring,

(O2) There is an element $x \in S$ such that S is a left free R-module with basis $\{1, x, x^2, x^3, \dots\}$,

(O3) $xr = \sigma(r)x + \delta(r)$, for all $r \in R$,

it is called an *Ore extension* or *skew polynomial ring over* R. In this case, we write $S := R[x;\sigma,\delta]$. Adding, the expression $R[x;\sigma]$ denotes the Ore extension when $\delta := 0$ and the expression $R[x;\delta]$ denotes the Ore extension when σ is the identity on R.

Definition 1.11 ([GW04], Page 36). Let $R[x;\sigma,\delta]$ be an Ore extension. Any nonzero element $p \in R[x;\sigma,\delta]$ can be uniquely expressed in the form $p = r_n x^n + r_{r-1}x^{n-1} + \cdots + r_1 x + r_0$, for some nonnegative integer n and some elements $r_i \in R$ with $r_n \neq 0$. The integer n is called the *degree* of p, abbreviated $\deg(p)$, and the element r_n is called the *leading coefficient* of p. The zero element of $R[x;\sigma,\delta]$ is defined to have degree $-\infty$ and leading coefficient 0.

Next, we show some examples of Ore extensions.

Example 1.1. Any polynomial ring $R[x_1,\dots,x_n]$ over a ring R. In this case, σ is the application identity in R and the σ-derivation is defined as $\delta := 0$.

We can iterate the construction of the Ore extensions and get the *iterated skew polynomial ring* $R[x;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$, with σ_i, δ_i defined on the ring $R[x;\sigma_1,\delta_1]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}]$, that is to say,

$$\sigma_i,\delta_i : R[x;\sigma_1,\delta_1]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}] \to R[x;\sigma_1,\delta_1]\cdots[x_{i-1};\sigma_{i-1},\delta_{i-1}].$$

Example 1.2 (Algebra of Ore, [Lez19], Example 1.3.3). An algebra of Ore is an extension of Ore in which the coefficient ring is $A := R[t_1,\dots,t_m]$, $m \geq 0$, with R a commutative ring and for $1 \leq i \leq n$, σ_1, δ_i are R-linear. This means that $\sigma_i(r) = r$ and $\delta_i(r) = 0$, for each $r \in R$. Thus, an algebra of Ore is an Ore extension $R[t_1,\dots,t_m][x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n]$, $m \geq 0$.

Example 1.3 (Weyl algebra, [LR14], Section 3.1). The first Weyl algebra $A_1(k)$ over k (with k a field) is defined to be the k-algebra generated by the indeterminates x, y subject to the relation $yx = xy + 1$. The nth Weyl algebra $A_n(\Bbbk)$ over k is the k-algebra generated by the $2n$ indeterminates $x_1,\dots,x_n,y_1,\dots,y_n$, where

$$x_j x_i = x_i x_j, \quad y_j y_i = y_i y_j, \qquad 1 \leq i, j \leq n,$$
$$y_j x_i = x_i y_j + \delta_{ij}, \quad \delta_{ij} \text{ is the Kroneckert's delta}, \quad 1 \leq i, j \leq n.$$

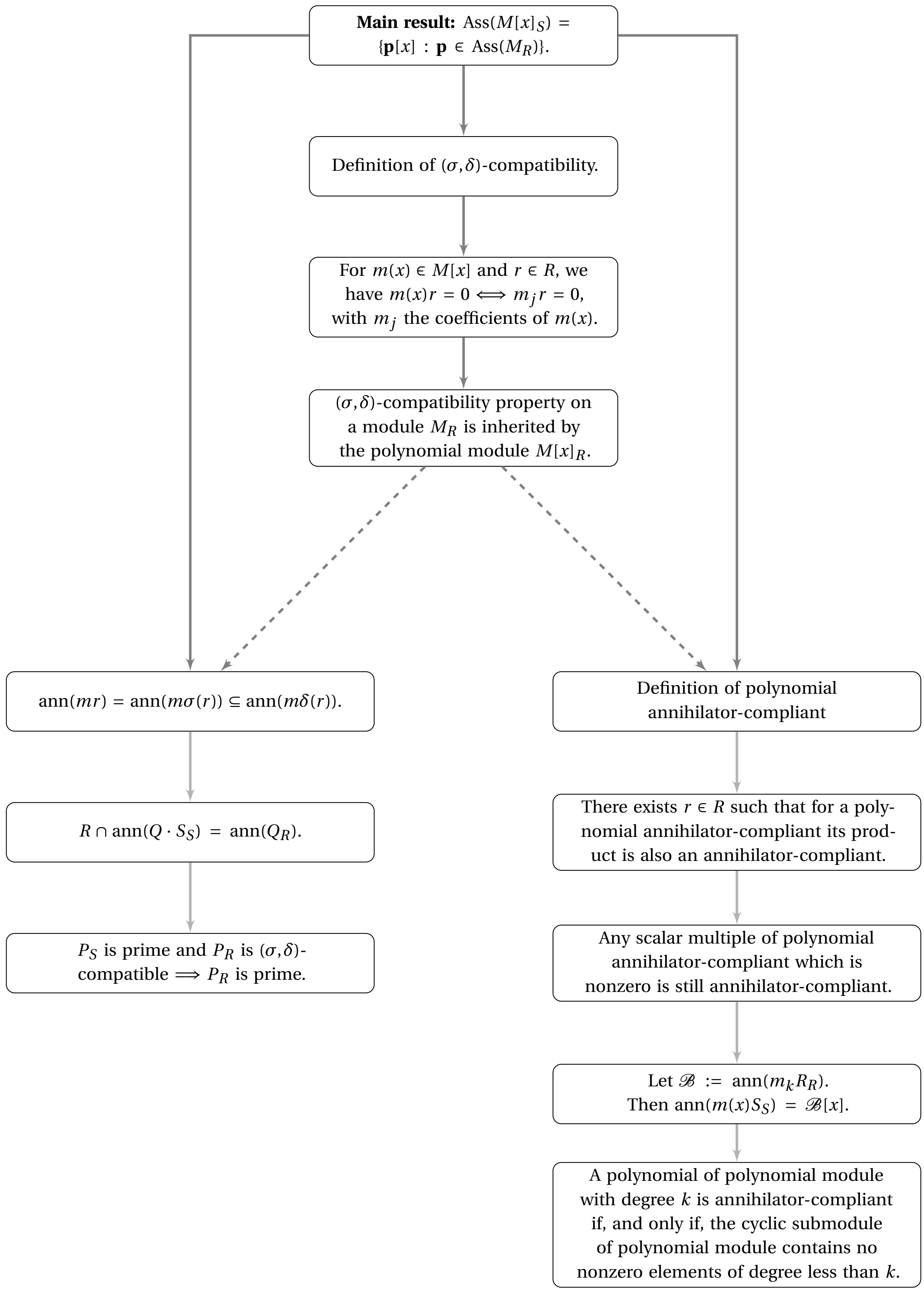
In the literature, the Weyl algebra is known as the first quantum algebra, and we can check that $A_n(\Bbbk) \cong A[x_1;\delta_1]\cdots[x_n;\delta_n]$.

Example 1.4 (The algebra of q-differential operators, [LR14], Section 3.2, (a)). Let $q, h \in k$ (with k a field), $q \neq 0$; consider $k[y][x;\sigma,\delta]$, $\sigma(y) := qy$ and $\delta(y) := h$. By definition of Ore extension, we have $xy = \sigma(y)x + \delta(y) = qyx + h$, and hence $xy - qyx = h$.

1.3.2 Associated prime ideals over Ore extensions

Annin, in his PhD thesis [Ann02a], asked the following question: "given any result on associated primes in the commutative setting, can the result of Faith be extended to noncommutative rings?" He was interested in solving this question for some ring extensions, therefore, he generalized the results obtained by Faith [Fai00] in three different directions: general polynomial modules, noncommutative rings, and Ore polynomial extensions. Throughout the section the letter S denotes $S := R[x;\sigma,\delta]$.

In this subsection, we show Annin's result that generalized Faith's result for Ore polynomial rings. Following Proposition 1.4, we have that $P[x]$ need not even be an ideal in $S := R[x;\sigma,\delta]$, as for P ideal of R, if $r \in P \subseteq P[x] \subseteq S$, $xr = \sigma(r)x + \delta(r)$ need not lie in $P[x]$, since there is no reason why $\sigma(r)$ or $\delta(r)$ need lie in P. Thus, we require restrictions to be imposed on the endomorphism σ and the σ-derivation δ. The following diagram explains the sequence of the elements that Annin used to prove his main result: the elements of $\mathrm{Ass}(M[x]_S)$ are in correspondence with the associated prime ideals of $\mathrm{Ass}(M_R)$. It should be emphasized that Annin introduced two definitions required for the proof of the main theorem; the definition of σ-compatibility (Definition 1.12) and that of polynomial annihilator-compliant (Definition 1.13).


Main result: Ass(M[x]_S) = {p[x] : p ∈ Ass(M_R)}.
Definition of (σ,δ)-compatibility.
For m(x) ∈ M[x] and r ∈ R, we have m(x)r = 0 ⟺ m_j r = 0, with m_j the coefficients of m(x).
(σ,δ)-compatibility property on a module M_R is inherited by the polynomial module M[x]_R.
ann(mr) = ann(mσ(r)) ⊆ ann(mδ(r)).
R ∩ ann(Q · S_S) = ann(Q_R).
P_S is prime and P_R is (σ,δ)-compatible ⟹ P_R is prime.
Definition of polynomial annihilator-compliant
There exists r ∈ R such that for a polynomial annihilator-compliant its product is also an annihilator-compliant.
Any scalar multiple of polynomial annihilator-compliant which is nonzero is still annihilator-compliant.
Let 𝓑 := ann(m_k R_R). Then ann(m(x)S_S) = 𝓑[x].
A polynomial of polynomial module with degree k is annihilator-compliant if, and only if, the cyclic submodule of polynomial module contains no nonzero elements of degree less than k.

Next, we present the results that Annin used to carry out the characterization described in diagram above.

Remark 3 ([Ann04], Notation 1.1). Given σ and δ where $\sigma : R \to R$ is an endomorphism and $\delta : R \to R$ is a σ-derivation, and integers $j \geq i \geq 0$, let us write f_i^j for the sum of all words in σ and δ in which there are i factors of σ and $j-i$ factors of δ. For intance, $f_i^j = \sigma^j$, $f_0^j = \delta^j$, and $f_{j-1}^j = \sigma^{j-1}\delta + \sigma^{j-2}\delta\sigma + \cdots + \delta\sigma^{j-1}$. Using recursive formulas for the f_i^j's and induction, as done in [Lam01], one can show that

$$x^j a = \sum_{i=0}^{j} f_i^j(a) x^i. \tag{1.1}$$

This formula determines uniquely a general product of (left) polynomials in S and it will be used in what follows.

Remark 4 (Polynomial with coefficients in a module, [LM04], Page 2745). Consider R an associative ring with identity and let M_R be a right R-module. Now we can define a polynomial module $M[x]$ over $S := R[x;\sigma,\delta]$ as follows. Since S is a free left R-module, the elements from $M[x]_S$ can be seen as polynomials in x with coefficients in M_R with natural additive and right S-module structure. Thus, the elements of $M[x]_S$ are of the form $\sum_{i=0}^{n} m_i x^i$, where $n \in \mathbb{N}$ and $m_i \in M$. In this polynomial module the addition of two elements in $M[x]_S$ is given by

$$\sum_{i=0}^{n} m_i x^i + \sum_{i=0}^{q} m_i' x^i = \sum_{j=0}^{p} (m_j + m_j') x^j \quad \text{with} \quad p = \max\{n, q\}.$$

The notion of degree of polynomials from $M[x]_S$ is defined similarly as in the case of polynomials in S.

Definition 1.12 ([Ann04], Definition 2.1). Let R be a ring and given an R-module M_R, an endomorphism $\sigma : R \to R$, and a σ-derivation $\delta : R \to R$, we say that M_R is σ-*compatible*, if for each $m \in M$, $r \in R$, we have $mr = 0 \Longleftrightarrow m\sigma(r) = 0$. Moreover, we say that M_R is δ-*compatible*, if for each $m \in M$, $r \in R$, we have $mr = 0 \Longrightarrow m\delta(r) = 0$. If M_R is both σ-compatible and δ-compatible, we say that M_R is (σ,δ)-*compatible*.

The condition of (σ,δ)-compatibility is necessary to establish the results that are required for the proof of the main theorem. From this definition, we present the following properties for the module M_R and the polynomial module $M[X]$.

Remark 5 ([Ann02a], Remark 46). We must emphasize the difference that there exists with the definition of σ-compatibility and δ-compatibility, since the first requires an if, and only if, while the other requires only one implication. The main reason is that in σ we need to consider the leading coefficient of an expression $m(x)r$, where $m(x) \in M[x]$ and $r \in R$. Observe that in the case where $\delta := 0$, one never has the reverse implication to the δ-compatibility condition for a nonzero module M_R; since, if $m\delta(r) = 0$ not necessarily $mr = 0$, so we certainly do not expect a two-sided implication for the condition on δ.

Proposition 1.6 ([Ann04], Remarks 2.1). *Given a module M_R, an endomorphism $\sigma : R \to R$, and a σ-derivation $\delta : R \to R$,*

(a) *If M_R is σ-compatible (resp. δ-compatible), then so is any submodule of M_R;*

(b) *If M_R is σ-compatible (resp. δ-compatible), then M_R is σ^i-compatible (resp. δ^i-compatible), for all $i \geq 0$.*

The following results were presented by Annin for the first part of his proof of Proposition 1.12.

Proposition 1.7. *1.* **([Ann04], Lemma 2.1).** *Assume that M_R is (σ,δ)-compatible. If $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ and $r \in R$, we have $m(x)r = 0 \Longleftrightarrow m_jr = 0$, for each $0 \leq j \leq k$.*

2. **([Ann04], Lemma 2.2).** *If M_R is (σ,δ)-compatible, then $M[x]_R$ is (σ,δ)-compatible.*

3. **([Ann04], Lemma 2.3).** *If M_R is (σ,δ)-compatible, then for all $m \in M$, $r \in R$, we have*

$$\mathrm{Ann}(mr) = \mathrm{Ann}(m\sigma(r)) \subseteq \mathrm{Ann}(m\delta(r)).$$

4. **([Ann04], Lemma 2.4).** *Let P_S be a right S-module. If Q_R is a R-submodule of P_S and P_R is (σ,δ)-compatible, then*

$$R \cap \mathrm{Ann}(Q \cdot S_S) = \mathrm{Ann}(Q_R).$$

5. **([Ann04], Lemma 2.5).** *If P_S is prime and P_R is (σ,δ)-compatible, then P_R is prime.*

Proof. The following proofs are a sketch of the results presented by Annin.

1. Using Remark 3, we have that

$$m(x)r = \sum_{i=0}^{k}\sum_{j=i}^{k} m_j f_i^j(r)x^i. \tag{1.2}$$

Assume that $m(x)r = 0$. We deduce from (1.2), that for each fixed $i \leq k$, we have

$$\sum_{j=i}^{k} m_j f_i^j(r) = 0. \tag{1.3}$$

We take $i = k$, from the equation (1.3) we have to $m_k\sigma^k(r) = 0$, so σ-compatibility of M_R yields $m_kr = 0$. Now we assume inductively that $m_jr = 0$, for each $j > i$. By (σ,δ)-compatibility of M_R, for $j > i$ we get $m_jf_i^j(r) = 0$. Using (1.3) again, we deduce that $m_i\sigma^i(r) = 0$, so $m_ir = 0$ as needed.
Conversely, note that the (σ,δ)-compatibility of M_R implies that $m_jf_i^j = 0$, for all i and j. Thus, (1.2) shows that $m(x)r = 0$.

2. Let $r \in R$ and $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]_R$. We must prove that $M[x]$ is σ-compatible. For this, we assume that $m(x)r = 0$. Thus, from part (1), $m_jr = 0$ for each $0 \leq j \leq k$. Fron (σ,δ)-compatibility of M_R we have that $m_jf_i^j(\sigma(r)) = m_jf_i^j(\delta(r)) = 0$, for every i and j with $i \leq j$. So that, the equation (1.2) with the scalar $\sigma(r)$ or $\delta(r)$ instead of r shows that $m(x)\sigma(r) = m(x)\delta(r) = 0$. So, we have $m(x)r = 0$ implies that $m(x)\sigma(r) = 0$,

and the converse follows using the same reasoning. Thus, $M[x]_R$ is (σ,δ)-compatible.

3. Let $a \in \mathrm{Ann}(m\sigma(r))$. Then $m\sigma(r)a = 0$. Thus, from σ-compatibility of M_R we have that for all $a \in R$,
$$m\sigma(r)a = 0 \Longleftrightarrow m\sigma(r)\sigma(a) = 0 \Longleftrightarrow m\sigma(ra) = 0 \Longleftrightarrow mra = 0.$$
We just need to show that $\mathrm{Ann}(mr) \subseteq \mathrm{Ann}(m\delta(r))$. For this, let $a \in \mathrm{Ann}(mr)$, thus $mra = 0$, using δ-compatibility, we get $m\delta(ra) = 0$, whence $0 = m(\sigma(r)\delta(a) + \delta(r)a)$. Since, we have that $m\sigma(r)a = 0$, then from δ-compatibility of M_R implies that $m\sigma(r)\delta(a) = 0$, and hence $m\delta(r)a = 0$, as needed.

4. We first prove that $R \cap \mathrm{Ann}(Q \cdot S_S) \subseteq \mathrm{Ann}(Q_R)$. Let $r \in R$, $q \in Q$ and $m \in S_S$, such that $qmr = 0$. Since we have $qm \in Q$, then $r \in \mathrm{Ann}(Q_R)$. Conversely, we assume that $Qr = 0$ and let $q \in Q$ and $g(x) = a_0 + a_1x + \cdots + a_lx^l \in S$. We must show that $qg(x)r = 0$. Using the commutation law, we compute that $qg(x)r = q\sum_{i=0}^{l}\sum_{j=i}^{l} a_j f_i^j(r)x^i$. By (σ,δ)-compatibility of Q_R, we have that $qa_j f_i^j(r) = 0$ for every $i \leq j$, so indeed, $qg(x)r = 0$.

5. Let $Q_R \neq 0$ and $Q_R \subseteq P_R$. We assume that $Qr = 0$ for some $r \in R$. By part (4), $Q \cdot Sr = 0$, so r annihilates a nonzero S-submodule of P_S, so $Pr = 0$ since P_S is prime.

□

The following definition was presented by Annin as a necessary condition to prove the other implication of its main theorem (Proposition 1.12). From this definition, it is possible to present some additional results for modules.

Definition 1.13 ([Ann04], Definition 3.1). We shall call a nonzero polynomial $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ (with $m_k \neq 0$) *annihilator-compliant*, if for each $i \leq k$, we have $\mathrm{Ann}(m_k) \subseteq \mathrm{Ann}(m_i)$.

Proposition 1.8 ([Ann04], Lemma 3.1). *If $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ is nonzero, then there exists $r \in R$ such that $m(x)r$ is annihilator-compliant.*

Proof. We follow the proof presented by Annin [Ann04]. We assume the result is false, and let $m(x) = m_0 + m_1x + \cdots + m_kx^k \in M[x]$ be a counterexample of minimal degree $k \geq 1$. We have that $m(x)$ is not annihilator-compliant. Therefore there exists $0 \leq i < k$ such that $\mathrm{Ann}(m_k) \nsubseteq \mathrm{Ann}(m_i)$. Thus, we can find $b \in R$ with $m_kb = 0$ but $m_ib \neq 0$. We see that the degree k, coefficient of $m(x)b$, is $m_k\sigma^k(b) = 0$, by σ-compatibility of M_R. Hence, $m(x)b$ has degree at most $k-1$. Moreover, since $m_ib \neq 0$, that Proposition 1.7 part (1), implies that $m(x)b \neq 0$. But, due to the minimality of k, we know that there exists $c \in R$ with $m(x)bc$ annihilator-compliant. This is a contradiction. □

The next proposition tells us that any scalar multiple of annihilator-compliant of it that is nonzero is still annihilator-compliant.

Proposition 1.9 ([Ann02a], Lemma 3.2). *Let $m(x) \in M[x]$ be annihilator-compliant. For all $r \in R$, if $m(x)r \neq 0$, then it is still annihilator-compliant.*

Proof. Following Annin [Ann04], let $m(x) = m_0 + m_1 x + \cdots + m_k x^k \in M[x]$ and assume that $m(x)r \neq 0$. By the equation (1.2), we have that the degree k, coefficient of $m(x)r$, is $m_k\sigma^k(r) \neq 0$. So $m_k\sigma^k(r)$ is the leading coefficient of $m(x)r$. Now, we suppose that $m_k\sigma^k(r)a = 0$. The degree i, coefficient of $m(x)r$, is $\sum_{j=i}^{k} m_j f_i^j(r)$. We note that a kills this, since by σ-compatibility, $m_k ra = 0$. Thus, for all $j \leq k$, $m_j ra = 0$. By the same iterative argument of Proposition 1.7 part (3), we find $m_j f_i^j(r)a = 0$, for all $i \leq j$, as desired. □

Lemma 1.10 ([Ann04], Lemma 3.3). *Let $m(x) \in M[x]$ be an annihilator-compliant polynomial. Write $m(x) = \sum_{i=0}^{k} m_i x^i$ with $m_k \neq 0$ and let $\mathscr{B} := \operatorname{Ann}(m_k R_R)$. We have*

$$\operatorname{Ann}(m(x)S_S) = \mathscr{B}[x].$$

Proof. First, we prove that $\operatorname{Ann}(m(x)S_S) \supseteq \mathscr{B}[x]$. For this, let $g(x) = a_0 + a_1 x + \cdots + a_l x^l \in \mathscr{B}[x]$ with $a_l \neq 0$. Since $\mathscr{B} := \operatorname{Ann}(m_k R_R)$, then $m_k R a_j = 0$, for every $0 \leq j \leq l$, and as $m(x)$ is annihilator-compliant, we have that $m_j R a_j = 0$, for all i and j. Suppose an element $m(x)h(x) \neq 0$ with $m(x)h(x) \in m(x)S_S$. For the product of $m(x)h(x)g(x)$, we get the terms all consist of an element of $m_i R_R$ times some $f_u^v(a_j)$. So, the terms must be zero since a_j kills $m_i R_R$. Thus, $g(x) \in \operatorname{Ann}(m(x)S_S)$.

Now, consider that $\operatorname{Ann}(m(x)S_S) \nsubseteq \mathscr{B}[x]$. Thus, there exists $g(x) \in \operatorname{Ann}(m(x)S_S) \setminus \mathscr{B}[x]$, and choose $g(x)$ so that it is minimal grade l. Let $g(x) = a_0 + a_1 x + \cdots + a_l x^l$. For any $h(x) \in S$ we have $0 = m(x)h(x)g(x)$. From this it follows by considering the highest-degree term of $m(x)h(x)g(x)$ that $m_k r a_j = 0$, for all $r \in R$. Thus, $m_i r a_l = 0$ by annihilator-compliance, for each $r \in R$ and each i. Thus, by compatibility and Proposition 1.7 part (4), $m(x)h(x)a_l = 0$. It follows that $m(x)S(a_0 + a_1 x + \cdots + a_{l-1}x^{l-1}) = 0$. Hence, $a_0 + a_1 x + \cdots + a_{l-1}x^{l-1} \in \operatorname{Ann}(m(x)S_S)$. But since $m_k R a_l = 0$, $a_l \in \mathscr{B}$. Thus, $a_0 + a_1 x + \cdots + a_{l-1}x^{l-1} \notin \mathscr{B}[x]$, which contradicts the minimality of l. □

Corollary 1.11 ([Ann04], Corollary 3.1). *A polynomial $m(x) \in M[x]$ of degree $k \geq 0$ is annihilator-compliant if and only if the cyclic submodule $m(x)S_S$ of $M[x]_S$ contains no nonzero elements of degree less than k.*

Proof. By contradiction, we assume that $m(x)$ is not annihilator-compliant. Thus, there exists $r \in R$ such that $m_k r = 0$ and $m_i r \neq 0$ for $1 \leq i < k$. Now, from Proposition 1.7 part (1), we have that $m(x)r \neq 0$, thus $m(x)r$ has degree less than k. The converse follows from the Proposition 1.10, since if $m(x)r \neq 0$ for $r \in R$, then $m(x)r$ has degree k. Therefore, if $g(x) \in S$ and $m(x)g(x) \neq 0$, then the degree of $m(x)g(x) = k + i$, where i is the largest number of the coefficient of $m(x)g(x)$ that is nonzero. □

These results give tools to prove Annin's main result.

PROPOSITION 1.12 ([ANN04], THEOREM 2.1, EXTENSION OF ASSOCIATED PRIMES). *Let M_R be a module over any ring R, let $\sigma : R \to R$ be an endomorphism, and let $\delta : R \to R$ be a σ-derivation. If M_R is (σ, δ)-compatible, then*

$$\mathrm{Ass}(M[x]_S) = \{P[x] : P \in \mathrm{Ass}(M_R)\}.$$

In fact, every $Q \in \mathrm{Ass}(M[x]_S)$ is extended; that is $Q = P[x]$, where $P = Q \cap R \in \mathrm{Ass}(M_R)$.

Proof. We show a sketch of the proof presented by Annin. He begins by proving that for $\mathbf{p} \in \mathrm{Ass}(M_R)$ and $N_R \subseteq M_R$ prime module with $\mathbf{p} = \mathrm{Ann}(N_R)$, we have that

$$\mathbf{p}[x] = \mathrm{Ann}(N[x]_S), \tag{1.4}$$

and

$$N[x]_S \text{ is prime.} \tag{1.5}$$

We begin by cheking (1.4). Let $n(x) = n_0 + n_1x + \cdots + n_kx^k \in N[x]$ and $g(x) = a_0 + a_1x + \cdots + a_lx^l \in S$. We assume that $g(x) \in \mathbf{p}[x]$, so that $n_ja_d = 0$, for all j and d. Thus, by Proposition 1.7 part (1), $n(x)a_d = 0$, for each $0 \le d \le l$, so $n(x)g(x) = 0$. On the other hand, if some $a_d \notin \mathbf{p}$, then there exists $n \in N$ with $na_d \neq 0$. Thus, $ng(x) \neq 0$, so $g(x) \notin \mathrm{Ann}(N[x]_S)$. With the aim of proving (1.5), he shows that for every nonzero element $n(x) \in N[x]_S$, we have that $\mathrm{Ann}(n(x)S_S) = \mathbf{p}[x]$. The proof of $\mathrm{Ann}(n(x)S_S) \supseteq \mathbf{p}[x]$ is obtained from (1.4) and to show the opposite implication he supposes an element $g(x) \notin \mathbf{p}[x]$ such that $n(x)S_Sg(x) = 0$, and choosing $g(x)$ of smallest possible degree. Under these conditions a contradiction is reached, since if $g(x) \notin \mathbf{p}[x]$ then $n(x)S_Sg(x) \neq 0$, concluding this part of the proof.

Next, Annin prove the other implication. He considers an ideal $I \in \mathrm{Ann}(M[x]_S)$ and a prime module $P_S \subseteq M[x]_S$ such that $I = \mathrm{Ann}(P_S)$. We can pick any $m(x) \neq 0$ in P, and by Proposition 1.8, we may assume that $m(x)$ is annihilator-compliant with $m_k \neq 0$ be leading coefficient of $m(x)$. Now, let $Q_R := m(x)R$ and since $Q \cdot S_S \subseteq P_S$, so we get $\mathrm{Ann}(Q \cdot S_S) = I$ and by Proposition 1.7 parts (2) and (5), we have that Q_R is prime. By Proposition 1.7 part (4) we conclude that $\mathrm{Ann}(Q_R) = I \cap R$. Therefore, he ends the proof by showing that $I = (I \cap R)[x]$. □

Next, we show some examples that illustrate Proposition 1.12.

EXAMPLE 1.5.

1. **([Ann04], Example 4.1).** Let $R := R_0[t]$ with R_0 any domain of characteristic zero. Take $M := R_0$, with the action of t on M given by $M \cdot t := 0$. Now, we consider the endomorphism $\sigma : R \to R$ as $\sigma(r) = r$, for every $r \in R$, and a σ-derivation $\delta : R \to R$ as the usual differentiation. The δ-compability fails since for all $m \in M$, $mt = 0$ but $m\delta(t) = m$. Note that $\mathrm{Ass}(M_R) = \{(t)\}$. We must show that $(t)[x] \notin \mathrm{Ass}(M[x]_S)$. For this, we assume that there exists a prime submodule $Q_S \subseteq M[x]_S$ with $(t)[x] = \mathrm{Ann}(Q_S)$. Let $m(x) = m_0 + m_1x + \cdots + m_kx^k \neq 0 \in Q_S$, thus $m(x)tx = 0$. However, $m(x)tx = \sum_{j=0}^{k} m_j(j+1)x^j$, and it would follows from this that each $m_j = 0$, which is a contradiction by the assumption on R_0.

2. **([Ann04], Example 4.3).** Let R_0 be a domain and $R := R_0[s,t]$, and a σ-derivation $\delta : R \to R$ the usual differentiation. Now set $M_R := R_0[s]$ with $M \cdot t := 0$. For $m \in M$ and $f \in R$, if $mf = 0$, then $f = tg$, for some $g \in R$. Thus, $m\delta(f) = m\delta(tg) = m(tg_s + t_sg) = 0$. Thus,

M_R is δ-compatible. Note that $\text{Ass}(M_R) = \{(t)\}$ and by Proposition 1.12 we have that $\text{Ass}(M[x]_S) = \{(t)[x]\}$.

3. **([Ann02a], Example 64).** Let $(R, \mathbf{m})$ be a commutative local ring, and $\sigma : R \to R$ any automorphism. Set $M := R/\mathbf{m}$. We verify that M_R is σ-compatible. Let $\overline{s}$ be any nonzero element of M_R, where $s \in R$. Necessarily, $s \notin \mathbf{m}$. Now for any $r \in R$, we have $\overline{s}r = \overline{0} \Longleftrightarrow sr \in \mathbf{m} \Longleftrightarrow r \in \mathbf{m} \Longleftrightarrow \sigma(r) \in \mathbf{m}$ (since nonunits are invariant under an automorphism of a local ring) $\Longleftrightarrow s\sigma(r) \in \mathbf{m} \Longleftrightarrow \overline{s}\sigma(r) = \overline{0}$. Hence, M_R is σ-compatible. Since M_R is simple, it is prime, so $\text{Ass}(M_R) = \text{Ann}(M_R) = \{\mathbf{m}\}$. By Proposition 1.12, $\text{Ass}((R/\mathbf{m})[x]_S) = \{\mathbf{m}[x]\}$.

4. **([Ann02a], Example 67).** Let R_0 be any ring and set $R := R_0[t]$. Let $\sigma(t) = 0$ and $\sigma|_{R_0} = I_d$ on R and define $\delta : R \to R$ by $\delta(f(t)) := f'(t)h(t)$. Consider $h(t)$ be any element in the center of R, and let $c \in R_0$ be a root of the polynomial $h(t)$, e.g., $h(t) = t^k$, for some $k \in \mathbb{N}$ and $c = 0$. Now, let $M = R_0$, which becomes a right R-module with the R-action as for $m \in M$ and $f(t) \in R$, we have $m \cdot f(t) := mf(c)$. Hence, we have that M_R is (σ, δ)-compatible and therefore, Proposition 1.12 applies.

1.3.3 σ-RIGID AND WEAK σ-RIGID RINGS

Now, we focus on the following algebraic structure in which Bhat in [Bha10a] developed his work.

For a ring R and an endomorphism $\sigma : R \to R$, Krempa in [Kre96] defined σ to be a *rigid endomorphism*, if $a\sigma(a) = 0$ implies $a = 0$, for $a \in R$. A ring R is said to be *σ-rigid*, if there exists a rigid endomorphism σ of R. We have that any rigid endomorphism of a ring is a monomorphism. Also, we have that R is *reduced*, if $r^2 = 0$ implies $r = 0$, for any $r \in R$.

Example 1.6 ([Bha10b], Page 697). Let $R = \mathbb{C}$, and $\sigma : \mathbb{C} \to \mathbb{C}$ be the aplication defined by $\sigma(a + ib) = a - ib$, with $a, b \in \mathbb{R}$. Then, we have that σ is a rigid endomorphism of R.

We recall that an element $r \in R$ (with R be a ring) is *nilpotent*, if $r^n = 0$, for some $n > 0$. The set of nilpotent elements of R is denoted by $N(R)$. Next, we show some properties of rigid rings.

Remark 6 ([HKK00], Page 218). Note that σ-rigid rings are reduced rings. In fact, if R is a σ-rigid ring and $a^2 = 0$ for $a \in R$, then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a^2)\sigma^2(a) = 0$. Thus, $a\sigma(a) = 0$ and so $a = 0$. Therefore, R is reduced.

It is important to highlight that there exists an endomorphism of a reduced ring which is not a rigid endomorphism, (see [HKK00], Example 9).

Proposition 1.13 ([HKK00], Lemma 4). *Let R be a σ-rigid ring and $a, b \in R$. Then we have the following:*

(i) If $ab = 0$ then $a\sigma^n(b) = \sigma^n(a)b = 0$, for any positive integer n.

(ii) If $ab = 0$ then $a\delta^m(b) = \delta^m(a)b = 0$, for any positive integer m.

(iii) If $a\sigma^k(b) = 0 = \sigma^k(a)b$, for some positive integer k, then $ab = 0$.

Proposition 1.14 ([HKK00], Proposition 5). *A ring R is σ-rigid if, and only if, the Ore extension $R[x;\sigma,\delta]$ is a reduced ring and σ is a monomorphism of R. In this case, $\sigma(e)=e$, $\delta(e)=0$, for every $e=e^2\in R$.*

Proposition 1.15 ([MR01], Corollary 2.7). *The following conditions on a ring R are equivalent: (i) R has no nonzero nilpotent right ideal; (ii) R has no nonzero nilpotent ideal; (iii) $N(R)=0$. These properties characterize semiprime rings.*

Recall that a ring R is *2-primal*, if the set of nilpotent elements of R equals the prime radical of R and R is *completely semiprime* if $a^2\in R$ implies that $a\in R$ (see [BHL93]). In this work, the notions of semiprime and 2-primal ring are very important. Marks in [Mar97] established necessary and sufficient conditions for a ring to be 2-primal.

Remark 7 ([Mar97], page 243). Let R be a ring. R is 2-primal and semiprime if and only if R is reduced.

Example 1.7.

1. **([Kre96], Theorem 3.1).** Let R be a reduced ring and let δ be a derivation of R. Then the skew polynomial ring $R[x;\delta]$ is reduced.

2. **([HKK03], Example 2).** Let $R=\mathbb{Z}_2\oplus\mathbb{Z}_2$ where $\mathbb{Z}_2$ is the ring of integers modulo 2. Then R is a commutative reduced ring. Let $\sigma:R\to R$ be an endomorphism defined by $\sigma((a,b))=(b,a)$. Then for $p=(1,0)+(1,0)x$, $q=(0,1)+(1,0)x$ in $R[x;\sigma]$, $pq=0$ but $(1,0)=\sigma((0,1))\neq 0$. Thus $R[x;\sigma]$ is not reduced by Proposition 1.14.

Next, we present a more general case than σ-rigid rings.

Definition 1.14 ([Ouy08]). Let R be a ring and σ be an endomorphism of R. R is said to be a *weak σ-rigid* ring, if $a\sigma(a)\in N(R)$ if and only if $a\in N(R)$, for $a\in R$.

The previous definition is an extension of the σ-rigid rings since every σ-rigid ring is also a weak σ-rigid ring, but every weak σ-rigid ring is not always a σ-rigid ring. The next example illustrates this situation.

Example 1.8 ([Ouy08], Example 2.1). Let σ be an endomorphism of a ring R which is a σ-rigid ring. Consider the ring

$$R_3:=\left\{\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix} \mid a,b,c\in R\right\}.$$

If we extend the endomorphism σ of R to the endomorphism $\overline{\sigma}:R_3\to R_3$ defined by $\overline{\sigma}(a_{ij})=(\sigma(a_{ij}))$, then R_3 is a weak $\overline{\sigma}$-rigid ring but R_3 is not $\overline{\sigma}$-rigid. In fact,

- R_3 is weak $\overline{\sigma}$-rigid. Let $A\in R_3$ with $A\overline{\sigma}(A)\in \mathrm{Nil}(R_3)$, we must see that $A\in\mathrm{Nil}(R_3)$. Since $A\overline{\sigma}(A)\in\mathrm{Nil}(R_3)$, then there exists $n\in\mathbb{N}$ such that $(A\overline{\sigma}(A))^n=0$, thus

$$(A\overline{\sigma}(A))^n=\left(\begin{pmatrix} a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{pmatrix}\begin{pmatrix} \sigma(a) & \sigma(b) & \sigma(c)\\ \sigma(0) & \sigma(a) & \sigma(d)\\ \sigma(0) & \sigma(0) & \sigma(a)\end{pmatrix}\right)^n=0$$

which implies that

$$\begin{pmatrix} a^n\sigma(a)^n & \alpha & \beta \\ 0 & a^n\sigma(a)^n & \gamma \\ 0 & 0 & a^n\sigma(a)^n \end{pmatrix} = \begin{pmatrix} (a\sigma(a))^n & \alpha & \beta \\ 0 & (a\sigma(a))^n & \gamma \\ 0 & 0 & (a\sigma(a))^n \end{pmatrix} = 0$$

where α, β and γ are products of powers between the elements of the A matrix and the $\overline{\sigma}(A)$ matrix. Then, $(a\sigma(a))^n = 0$, but this implies that $a\sigma(a) = 0$ because R is also reduced, thus $a = 0$. This way we get that $A := \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$. Hence, $A^3 = 0$ and so $A \in \text{Nil}(R_3)$.

- R_3 is not $\overline{\sigma}$-rigid. For this, consider the matrix $A := \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$, with $c, d \neq 0$, hence $A \neq 0$, but we have that

$$A\overline{\sigma}(A) = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma(0) & \sigma(0) & \sigma(c) \\ \sigma(0) & \sigma(0) & \sigma(d) \\ \sigma(0) & \sigma(0) & \sigma(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \sigma(c) \\ 0 & 0 & \sigma(d) \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

  Therefore, $A\overline{\sigma}(A) = 0$ but $A \neq 0$.

1.3.4 Associated prime ideals of weak σ-rigid rings

Bhat [Bha08] considered associated prime ideals of skew polynomial rings over a Noetherian ring R and an automorphism σ of R. Also, Bhat [Bha10a] studied the associated prime ideals of weak σ-rigid rings and their extensions.

Following Bhat, we denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$, $R[x;\sigma]$ by $S(R)$, $R[x;\delta]$ by $D(R)$ and we denote

$$O(U) := \left\{ m(x) \in O(R) \mid m(x) = \sum_{i=0}^{n} u_i x^i;\ u_i \in U \text{ for every } 0 \leq i \leq n \right\},$$

where $U \subseteq R$. Similary, we denote $S(U)$ and $D(U)$, for $U \subseteq R$.

Bhat presented the next theorems that characterize the associated prime ideals over weak σ-rigid rings.

Proposition 1.16 ([Bha10a], Theorem 2.6). *Let R be a semiprime right Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ be a σ-derivation of R. Then $P \in \text{Ass}(O(R)_{O(R)})$ if and only if there exist $U \in \text{Ass}(R_R)$ such that $P \cap R = U$ and $O(P \cap R) = P$.*

Proposition 1.17 ([Bha10a], Theorem 2.10). *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$, σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and $\sigma(U) = U$, for all $U \in \mathbb{A}(R_R)$. Then $P \in \mathrm{Ass}(O(R)_{O(R)})$ if and only if there exists $U \in \mathrm{Ass}(R_R)$ such that $P \cap R = U$ and $O(P \cap R) = P$.*

With the aim of proving these theorems, Bhat established the following propositions that are necessary for the main proof. Recall that the set of minimal prime ideals of R is denoted by $\mathrm{MinSpec}(R)$ and for any right R-module J, the assassinator of J is denoted by $\mathrm{Assas}(J)$.

Proposition 1.18 ([Bha07], Lemmas 2.1 and 2.2). *Let R be a ring. Let σ be an automorphism of R and δ be a derivation of R. Then:*

1. *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*

2. *If Q is a prime ideal of R such that $\sigma(Q) = Q$, then $S(Q)$ is a prime ideal of $S(R)$ and $S(Q) \cap R = Q$.*

3. *If P is a prime ideal of $D(R)$, then $P \cap R$ is a prime ideal of R and $\delta(P \cap R) \subseteq P \cap R$.*

4. *If U is a prime ideal of R such that $\delta(U) \subseteq U$, then $D(U)$ is a prime ideal of $D(R)$ and $D(U) \cap R = U$.*

Proposition 1.19 ([Bha10a], Theorem 2.1). *Let R be a Noetherian ring. Let σ be a automorphism of R. Then R is a weak σ-rigid ring if and only if $N(R)$ is completely semiprime.*

Proof. We follow the proof presented by Bhat [Bha10a]. First of all, he shows that $\sigma(N(R)) = N(R)$. For this, we have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So $I = \sigma^{-1}(N(R)) = \{a \in R \text{ such that } \sigma(a) = n \in N(R)\}$ is an ideal of R. Since I is nilpotent, $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$. Now let R be a weak σ-rigid ring. We show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R)$. Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$, so $N(R)$ is completely semiprime.
Conversely, let $N(R)$ be completely semiprime. We show that R is a weak σ-rigid ring. Let $a \in R$ be such that $a\sigma(a) \in N(R)$. Now $a\sigma(a)\sigma^{-1}(a\sigma(a)) \in N(R)$ implies that $a^2 \in N(R)$, and so $a \in N(R)$. Hence R is a weak σ-rigid ring. □

Proposition 1.20 ([Bha10a], Proposition 2.2). *Let R be a 2-primal right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R. Then $\sigma(U) = U$ and $\delta(U) \subseteq U$ for all $U \in \mathrm{MinSpec}(R)$.*

Proof. Following the proof that Bhat does in his paper, he firts shows that $P(R)$ is completely semiprime. For this, he uses the fact that $\sigma(P(R)) = P(R)$, and concludes that if $a\sigma(a) \in P(R)$, then $a \in P(R)$. Next, he shows that $\sigma(U) = U$, for all $U \in \mathrm{MinSpec}(R)$ by contradiction. He suppose that there exist other minimal primes $U_2, \ldots, U_n$ of R asuming that $U = U_1$. He, using that $P(R)$ is completely semiprime, concludes that $U_i \subseteq U_n$, for some $i \neq n$, which is impossible. Next, he shows that $\delta(U) \subseteq U$, for $U \in \mathrm{MinSpec}(R)$. For this, let $V := \{a \in U \mid \text{such that } \delta^k(a) \in$

U for all integers $k \geq 1\}$. First, he proof that V is a δ-invariant ideal of R, $V \in \mathrm{Spec}(R)$ and $V \subseteq U$, so $V = U$. Hence $\delta(U) \subseteq U$. □

PROPOSITION 1.21 ([BHA10A], LEMMA 2.4). *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a weak σ-rigid ring and δ a σ-derivation of R. Then*

1. *If U is a minimal prime ideal of R, then $O(U)$ is a minimal prime ideal of $O(R)$ and $O(U) \cap R = U$.*
2. *If P is a minimal prime ideal of $O(R)$, then $P \cap R$ is a minimal prime ideal of R.*

Proof. We follow the proofs presented by Bhat [Bha10a].

1. Let U be a minimal prime ideal of R. Then by Proposition 1.20, $\sigma(U) = U$ and $\delta(U) \subseteq U$. Now, using [GW04], Theorem 2.22, we have $O(U) \in \mathrm{Spec}(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R, a contradiction. Therefore $O(U) \in \mathrm{MinSpec}(O(R))$. Now, we see that $O(U) \cap R = U$.

2. We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Now the proof follows of Gooderl and Warfierld [GW04], Theorem 2.22 and using Propositions 1.18 and 1.20 he concludes the proof.

□

REMARK 8. In the reconstruction of the proofs of the Proposition 1.21 we have found that Theorem 2.22 of Gooderl and Warfierld does not exist in the reference [GW04]. However, below we present an alternate proof that to solve this problem. For this we need the next results:

PROPOSITION 1.22 ([GW04], EXERCISE 2ZA). *Let $S = R[x;\sigma,\delta]$ be a skew polynomial ring and I an ideal of R such that $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$. Let $\hat{\sigma}$ and $\hat{\delta}$ denote the ring endomorphism and skew derivation on R/I induced by σ and δ. Show that IS is two-sided ideal of S such that $IS \cap R = I$, and that $IS = SI$ in case σ is an automorphism and $\delta(I) = I$. Then show that $S/IS \cong (R/I)[\hat{x};\hat{\sigma},\hat{\delta}]$.*

PROPOSITION 1.23 ([GW04], EXERCISE 2O). *Let $R[x;\sigma,\delta]$ be a skew polynomial ring. Show that if $r \in R$ and $n \in \mathbb{N}$, then $x^n r = \sigma^n(r)x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + \delta^n(r)$, for some $a_{n-1},\ldots,a_1 \in R$. Hence, if $r \neq 0$ and σ is injective, $x^n r$ has degree n and leading coefficient $\sigma^n(r)$. Now show that if R is a domain and σ is injective, then $\deg(pq) = \deg(p) + \deg(q)$ for all $p, q \in R[x;\sigma,\delta]$, and consequently $R[x;\sigma,\delta]$ is a domain.*

Now, we present the proof of the Proposition 1.21.

Proof. 1. Let U be a minimal prime ideal of R. Then by Proposition 1.20, $\sigma(U) = U$ and $\delta(U) \subseteq U$. We have that U is a (σ,δ)-ideal of R, thus let $u \in U$, whence $xu = \sigma(u)x + \delta(u)$, since $\sigma(u) \in U$ and $\delta(u) \in U$, then we have that $xu \in O(U)$. Thus $O(U)$ is an ideal of $O(R)$. Now, we must see that $O(U)$ is a prime ideal of $O(R)$, for this from Proposition 1.22 we have that $O(R)/O(U) \cong (R/U)[\hat{x};\hat{\sigma},\hat{\delta}]$, where $\hat{\sigma}$ and $\hat{\delta}$ denote the ring endomorphism and skew derivation on R/I induced by σ and δ. Since R/U is a domain, then $O(R)/O(U)$ is a domain by Proposition 1.23 and hence $O(U)$ is a prime ideal of $O(R)$. Thus, we have $O(U) \in \text{Spec}(O(R))$. Suppose $L \subset O(U)$ be a minimal prime ideal of $O(R)$. Then $L \cap R \subset U$ is a prime ideal of R, a contradiction. Therefore $O(U) \in \text{MinSpec}(O(R))$. Now, we see that $O(U) \cap R = U$.

2. We note that $x \notin P$ for any prime ideal P of $O(R)$ as it is not a zero divisor. Since P is a prime ideal of $O(R)$ then $P \cap R$ is a prime ideal of R by Proposition 1.18 part (1) and (3). Now, since P is a minimal prime ideal, we have there is no other ideal of $O(R)$ such that P is contained. Thus, because $P \cap R$ is prime ideal of R, then there can not be an ideal of R that contains this ideal, so $P \cap R$ is a minimal prime ideal of R.

□

Bhat [Bha10a] presents the following examples that, he asserts, illustrates the Proposition 1.16.

Example 1.9 ([Bha10a], Example 2.7). Let τ be the application that sends an element in $\mathbb{C}$ to its conjugate. Thus, τ is a rigid endomorphism of $\mathbb{C}$ from Example 1.6. Now, consider the ring

$$R := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

If we extend the endomorphism τ to the endomorphism $\sigma : R \to R$ defined by $\sigma((a_{ij})) = (\tau(a_{ij}))$, then R is a weak σ-rigid ring. In fact, let $A \in R$ with $A\sigma(A) \in \text{Nil}(R)$, we must see that $A \in \text{Nil}(R)$. Since $A\sigma(A) \in \text{Nil}(R)$, then there exists $n \in \mathbb{N}$ such that $(A\sigma(A)) = 0$, thus

$$(A\sigma(A))^n := \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} \tau(a) & \tau(b) \\ 0 & \tau(a) \end{pmatrix} \right)^n = 0,$$

which implies that

$$(A\sigma(A))^n := \begin{pmatrix} a^n\tau(a)^n & \beta \\ 0 & a^n\tau(a)^n \end{pmatrix} = \begin{pmatrix} (a\tau(a))^n & \beta \\ 0 & (a\tau(a))^n \end{pmatrix} = 0$$

where β is products of powers between the element of the A matrix and the $\sigma(A)$ matrix. Then, $(a\tau(a)) = 0$, but this implies that $a\tau(a) = 0$ because $\mathbb{C}$ is also reduced, thus $a = 0$. This way we get that

$$A := \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

Hence, $A^2 = 0$ and so $A \in \text{Nil}(R)$. Now for any $D \in R$, define $\delta_D : R \to R$ by $\delta_D(A) = AD - D\sigma(A)$, for $A \in R$. Then δ_D is a σ-derivation of R. Since, let $A, B \in R$ then,

$$\begin{aligned}
\delta_D(A+B) &= (A+B)D - D\sigma(A+B) \\
&= AD + BD - D(\sigma(A) + \sigma(B)) \\
&= AD + BD - D\sigma(A) - D\sigma(B) \\
&= AD - D\sigma(A) + BD - D\sigma(B) \\
&= \delta_D(A) + \delta_D(B).
\end{aligned}$$

Also, we must see that $\delta_D(AB) = \sigma(A)\delta(B) + \delta(A)B$. Note that for any elements $A, B \in R$ we have that $AB = BA$. Using this fact, we have

$$\begin{aligned}
\sigma(A)\delta_D(B) + \delta_D(A)B &= \sigma(A)[BD - D\sigma(B)] + [AD - D\sigma(A)]B \\
&= \sigma(A)BD - \sigma(A)D\sigma(B) + ADB - D\sigma(A)B \\
&= -\sigma(A)D\sigma(B) + ADB \\
&= ABD - \sigma(A)\sigma(B)D \\
&= ABD - \sigma(AB)D \\
&= \delta_D(AB).
\end{aligned}$$

Let

$$U := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b, \in \mathbb{C} \right\} \in \operatorname{Ass}(R_R),$$

and let I be a uniform R-module defined as,

$$I := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{C} \right\}.$$

In fact, $U = \operatorname{Ann}(I)$ and $\operatorname{Ann}(I) = \operatorname{Assas}(I)$, because I is uniform.

Now, Bhat [Bha10a] says that $\delta_D(I) \subseteq I$, but this is not true. Since, the σ-derivation is defined as $\delta_D(A) = AD - D\sigma(A)$, for any element $A \in I$, suppose that

$$A := \begin{pmatrix} 0 & 0 \\ 0 & a+bi \end{pmatrix}$$

and we have that

$$\sigma(A) := \begin{pmatrix} 0 & 0 \\ 0 & a-bi \end{pmatrix}.$$

Therefore, for any $D \in R$ and calculated $\delta_D(A)$ we get a matrix of the form

$$\begin{aligned}\delta_D(A) &= \begin{pmatrix} 0 & 0 \\ 0 & a+bi \end{pmatrix}\begin{pmatrix} c+di & e+fi \\ 0 & c+di \end{pmatrix} - \begin{pmatrix} c+di & e+fi \\ 0 & c+di \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a-bi \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & ac-bd+(ad+bc)i \end{pmatrix} - \begin{pmatrix} 0 & ae+bf+(af-be)i \\ 0 & ac+bd+(ad-bc)i \end{pmatrix} \\ &= \begin{pmatrix} 0 & -ae-bf+(be-af)i \\ 0 & -2bd+2bci \end{pmatrix}\end{aligned}$$

thus $\delta_D(I) \nsubseteq I$. For this reason, we consider that the Proposition 1.16 can not be applied as the author affirms, since it fails in one of the hypotheses.

EXAMPLE 1.10 ([BHA10A], EXAMPLE 2.11). Let

$$R := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\},$$

and let

$$U := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \in \mathrm{Ass}(R_R).$$

Now, we consider

$$I := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in \mathbb{R} \right\},$$

where I is a right ideal of R. Also, we have that $U = \mathrm{Ann}(I) = \mathrm{Assas}(I)$.

Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then σ is an endomorphism of R and $\sigma(U) \subseteq U$. For any $s \in R$, define $\delta_s : R \to R$ by $\delta_s(a) = as - s\sigma(a)$, for $a \in R$. Then δ_s is a σ-derivation of R, as we show in the previous example.

Now, we have that $\sigma(\delta_s(u)) = \delta_s(\sigma(u))$ for all $u \in R$. Since, let $u = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ and $s = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix}$. Then

$$\sigma(\delta_s(u)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \delta_s(\sigma(u)).$$

Now, we note that $\sigma(I) = I$, $\delta_s(I) \subseteq I$ and $\delta_s(U) \subseteq U$. Also $O(U) \in \mathrm{Ass}(O(R)_{O(R)})$. In fact, $O(U) = \mathrm{Ann}(O(I)) = \mathrm{Assas}(O(I))$.

CHAPTER 2

Associated prime ideals over SPBW and SPBW over weak Σ-rigid rings

In this chapter we establish the main results of this work. We characterize the associated prime ideals in two specific structures, which extend the results that we shown in Chapter 1. In Section 2.1 we focus on skew PBW extensions and we show some properties as the (Σ, Δ)-compatibility and the notion of annihilator-compliant over these extensions. On this structure we develop the first result of this work, Theorem 2.1.4. This consists in to characterize the associated prime ideals over skew PBW extensions and it is important to remark that this characterization is an extension of the work done by Annin [Ann04]. In Section 2.2, we study the skew PBW extensions over weak Σ-rigid rings. First we show its definition and some properties together the second result of our work, Theorem 2.2.2; this extends the characterization presented by Bhat [Bha10a]. Finally, it is important to mention that the results presented in Section 2.1 and Section 2.2 have been submitted for publication.

2.1 Associated prime ideals over skew PBW extensions

The skew Poincaré-Birkhoff-Witt extensions (SPBW) are a generalization of Poincaré-Birkhoff-Witt extensions (PBW) defined by Bell and Goodearl in [BG88]. These extensions were introduced by Gallego and Lezama in [GL11] and include some important classes of rings as commutative polynomials rings, Weyl algebras, enveloping algebras of Lie algebras, q-Heisenberg algebras, 3-dimensional skew polynomial algebras, the algebra for multidimensional descrete linear systems, among others, see [LR14], [LAR15], [RS17b] and [RS17c] for a detailed list of algebras.

2.1.1 Skew Poincaré-Birkhoff-Witt extensions

We present the definition of skew PBW extension and establish some useful properties about these objects. For more information of these structures and to know some ring-theorical and homological properties of them, see [SLR15], [LG16], [LV17], [SLR17], [LL17] and [RS19a].

Definition 2.1 ([GL11], Definition 1). Let R be an associative ring and A be a ring. We say that A is a *skew PBW extension* (also known as *σ-PBW extension*) of R, which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same identity element;

(ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} | \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, and $x_1^0 \cdots x_n^0 = 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R\backslash\{0\}$, there exists an element $c_{i,r} \in R\backslash\{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements $1 \leq i, j \leq n$, there exists $c_{i,j} \in R\backslash\{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Remark 9 ([GL11], Theorem 7). (i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1 are unique.

(ii) If $r = 0$, it follows that $c_{i,0} = 0$. In fact, from $0 = x_i 0 = c_{i,0} x_i + s_i$, with $s_j \in R$, we obtain $c_{i,0} = 0 = s_j$, for all j.

(iii) In Definition 2.1 (iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i} x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.

(iv) Let $i < j$. By (i), there exist elements $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + Rx_1 + \cdots + Rx_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$, and hence $1 = c_{j,i} c_{i,j}$, that is, for each $1 \leq i < j \leq n$ has a left inverse and $c_{j,i}$ has a right inverse.

(v) Each element $f \in A\backslash\{0\}$ has a unique representation as $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R\backslash\{0\}$ and $X_i \in \mathrm{Mon}(A)$ for $1 \leq i \leq t$.

Proposition 2.1 ([GL11], Proposition 3). *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r) x_i + \delta_i(r)$, for each $r \in R$. We write $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and $\Delta := \{\delta_1, \dots, \delta_n\}$.*

Proof. For each $1 \leq i \leq n$ and every $r \in R$, we have elements $c_{i,r}, r_i \in R$ with $x_i r = c_{i,r} x_i + r_i$. Since $\mathrm{Mon}(A)$ is a left R-basis of A we have that $c_{i,r}$ and r_i are unique for r. Hence, we define $\sigma_i, \delta_i : R \to R$, by $\sigma_i(r) := c_{i,r}$ and $\delta_i(r) := r_i$, so we can see that σ_i is an endomorphism ans δ_i is a σ_i-derivation of R, this is $\delta_i(r + r') = \delta_i(r) + \delta_i(r')$ and $\delta_i(rr') = \sigma_i(r)\delta_i(r') + \delta_i(r)r'$, for each elements $r, r' \in R$. By Definition 2.1 part (iii), $c_{i,r} \neq 0$ for $r \neq 0$, which shows that σ_i is inyective for all $1 \leq i \leq n$. □

Definition 2.2 ([GL11], Definition 4; [LAR15], Definition 2.3 (ii)). Let A be a skew PBW extension of a ring R.

(a) A is *quasi-commutative*, if the conditions $(iii) - (iv)$ in Definition (2.1) are replaced by:

(iii') For every $1 \leq i \leq n$ and $r \in R\backslash\{0\}$, there exists $c_{i,j} \in R\backslash\{0\}$ such that $x_i r = c_{i,r} x_i$.

(iv') For every $1 \leq i, j \leq n$, there exists $c_{i,j} \in R\backslash\{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) A is *bijective*, if σ_i is bijective, for every $1 \leq i \leq n$, and $c_{i,j}$ is invertible, for any $1 \leq i < j \leq n$.

(c) A is called of *endomorphism type*, if $\delta_i = 0$, for every i. In addition, if every σ_i is bijective, A is a skew PBW extension of automorphism type.

Proposition 2.2 ([GL11], Theorem 7). *If A is a polynomial ring with coefficients in R with respect to the set of indeterminates $\{x_1, \dots, x_n\}$, then A is a skew PBW extension of R if and only if the following conditions hold:*

(i) for each $x^\alpha \in \mathrm{Mon}(A)$ and every $0 \neq r \in R$, there exist unique elements $r_\alpha := \sigma^\alpha(r) \in R\setminus\{0\}$, $p_{\alpha,r} \in A$, such that $x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r}$, where $p_{\alpha,r} = 0$, $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_α.

(ii) For each $x^\alpha, x^\beta \in \mathrm{Mon}(A)$, there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that $x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta}$, where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$, or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

Proposition 2.3 ([Rey15], Proposition 2.9). *If $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $r \in R$, then*

$$
\begin{aligned}
x^\alpha r &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} r = x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} \left(\sum_{j=1}^{\alpha_n} x_n^{\alpha_n - j} \delta_n(\sigma_n^{j-1}(r)) x_n^{j-1} \right) \\
&+ x_1^{\alpha_1} \cdots x_{n-2}^{\alpha_{n-2}} \left(\sum_{j=1}^{\alpha_{n-1}} x_{n-1}^{\alpha_{n-1}-j} \delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r))) x_{n_2}^{j-1} \right) x_n^{\alpha_n} \\
&+ x_1^{\alpha_1} \cdots x_{n-3}^{\alpha_{n-3}} \left(\sum_{j=1}^{\alpha_{n-2}} x_{n-2}^{\alpha_{n-2}-j} \delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\alpha_n^{\alpha_n}(r)))) x_{n-2}^{j-1} \right) x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \cdots + x_1^{\alpha_1} \left(\sum_{j=1}^{\alpha_2} x_2^{\alpha_2 - j} \delta_2 \left(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r))))) \right) x_2^{j-1} \right) x_3^{\alpha_3} x_4^{\alpha_4} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \\
&+ \sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))) x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad \sigma_j^0 := \mathrm{id}_R, \ \ \textit{for} \ \ 1 \leq j \leq n.
\end{aligned}
$$

Proposition 2.4 ([Rey15], Remark 2.10, (iv)). *Using Proposition 2.3, it follows that for the product $a_i X_i b_j Y_j$, if $X_i := x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}$ and $Y_j := x_1^{\beta_{j1}} \cdots x_n^{\beta_{jn}}$, then when we compute every summand of $a_i X_i b_j Y_j$ we obtain products of the coefficient a_i with several evaluations of b_j in σ's and δ's depending of the coordinates of α_i. This assertion follows from the expression:*

$$
\begin{aligned}
a_i X_i b_j Y_j = a_i \sigma^{\alpha_i}(b_j) x^{\alpha_i} x^{\beta_j} &+ a_i p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_2^{\alpha_{i2}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} p_{\alpha_{i1}, \sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))} x_3^{\alpha_{i3}} \cdots x_n^{\alpha_{in}} x^{\beta_j} \\
&+ \cdots + a_i x_1^{\alpha_{i1}} x_2^{\alpha_{i2}} \cdots x_{i(n-2)}^{\alpha_{i(n-2)}} p_{\alpha_{i(n-1)}, \sigma_{in}^{\alpha_{in}}} x_n^{\alpha_{in}} x^{\beta_j} \\
&+ a_i x_1^{\alpha_{i1}} \cdots x_{i(n-1)}^{\alpha_{i(n-1}} p_{\alpha_{in}, b_j} x^{\beta_j},
\end{aligned}
$$

where the polynomials $p_{\alpha_1, \sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_2, \sigma_3^{\alpha_3}(\cdots(\sigma_n^{\alpha_n}(r)))}$, $p_{\alpha_3, \sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))}, \dots, p_{\alpha_{n-1}, \sigma_n^{\alpha_n}(r)}$, and $p_{\alpha_n, r}$, involve elements of R obtained evaluating σ_j and δ_j in the element r of R.

Next, we have a description of polynomial modules over skew PBW extensions.

REMARK 10 ([REY19], PAGE 7). If A is a skew PBW extension of a ring R, then A is a left free R- module. We can consider the polynomial module $M\langle X\rangle_A$ over A. More precisely, as a set, the elements of $M\langle X\rangle_A$ are of the form $m_0 + m_1X_1 + \cdots + m_tX_t$, $m_i \in M_R$ and $X_i \in \mathrm{Mon}(A)$, for every i. If $\alpha := (\alpha_1,...,\alpha_n) \in \mathbb{N}^n$ and $r \in R$, the action of A on these elements follows the rule established in Proposition 2.4. This fact is precisely because it suffices to define the action of monomials of A on monomial in $M\langle X\rangle_A$. In other words, if $m_i x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}$ and $b_j x_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}}$ are elements of $M\langle X\rangle_A$ and A, respectively, then we multiply these both elements following the rule

$$\begin{aligned} m_1x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}b_jx_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}} &= m_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + m_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\ &+ m_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_{i3}^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\ &+ m_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j}\\ &+\cdots+ m_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b)}x_n^{\alpha_{in}}x^{\beta_j}\\ &+ m_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j}. \end{aligned} \tag{2.1}$$

This guarantees that $M\langle X\rangle_A$ is an A-module. In this way, when we compute every summand of $m_1x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}b_jx_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}}$, we obtain products of the coefficient m_i with several evaluation of b_j in σ's and δ's depending of the coordinates of α_i.

2.1.1.1 MONOMIAL ORDERS IN SKEW PBW EXTENSIONS

In this section we present some results about monomial orders in skew PBW extensions. These results concern properties that characterize monomial orders for noncommutative rings. In the literature we can find several works concerning above these orders. Some of them were realized by Becker and V. Weispfenning [BW93] and Bueso, Gómez-Torrecillas and Verschoren [BGTV03].

Next, we formulate some results presented in [GL11], where we mention some properties over monomial orders in skew PBW extensions.

DEFINITION 2.3 ([GL11], DEFINITION 6). Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \le i \le n$. Then:

(i) for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, $\sigma_i^{\alpha_i} := \sigma_i(\sigma_i(\sigma_i\cdots(\sigma_i(r))))$ composition α_i-times of σ_i and $r \in R$, $\sigma^\alpha := \sigma_1^{\alpha_1}\cdots\sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1+\cdots+\alpha_n$. If $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$, then $\alpha+\beta := (\alpha_1+\beta_1,\dots,\alpha_n+\beta_n)$.

(ii) For $X = x^\alpha \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) Let $0 \neq f \in A$. $t(f)$ is the finite set of terms that conform f, i.e., if $f = c_1X_1+\cdots+c_tX_t$, with $X_i \in \mathrm{Mon}(A)$ and $c_i \in R\setminus\{0\}$, then $t(f) := \{c_1X_1,\dots,c_tX_t\}$.

(iv) Let f be as in (iii). Then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

(v) Each element $f \in A$ can be represented in a unique way as $f = c_1X_1+\cdots+c_tX_t$, with $c_i \in R\setminus\{0\}$, $1 \le i \le t$, and $X_t \succ \cdots \succ X_1$ are the monomials of f, we define $\mathrm{lm}(f) := X_t$ is the leading monomial of f. $\text{lc}(f) := c_t$ is the leading coefficient of f and $\text{lt}(f) := c_t X_t$ is the leading term of f. If $f = 0$, we define $\text{lm}(0) := 0$, $\text{lc}(0) := 0$, $\text{lt}(0) := 0$ and we set $X \succ 0$ for any $X \in \text{Mon}(A)$. Thus, we extend $\succeq$ to $\text{Mon}(A) \cup \{0\}$.

DEFINITION 2.4 ([GL11], DEFINITION 11). Let $\succeq$ be a total order on $\text{Mon}(A)$. We say that $\succeq$ is a *monomial order* on $\text{Mon}(A)$, if the following conditions hold:

(i) for every $x^{\beta}, x^{\alpha}, x^{\gamma}, x^{\lambda} \in \text{Mon}(A)$, $x^{\beta} \succ x^{\alpha} \Rightarrow \text{lm}(x^{\gamma}x^{\beta}x^{\lambda}) \succeq \text{lm}(x^{\gamma}x^{\alpha}x^{\lambda})$;

(ii) $x^{\alpha} \succeq 1$, for every $x^{\alpha} \in \text{Mon}(A)$;

(iii) $\succeq$ is degree compatible, i.e, $|\beta| \geq |\alpha| \Rightarrow x^{\beta} \succeq x^{\alpha}$.

Monomial orders are also called *admissible orders*. The condition (iii) of the previous definition is needed in the proof of Proposition 2.5.

PROPOSITION 2.5 ([GL11], PROPOSITION 12). *Every monomial order on* $\text{Mon}(A)$ *is a well order. Thus, there are not infinite decreasing chains in* $\text{Mon}(A)$.

Proof. We follow the proof presented by Gallego and Lezama [GL11]. Suppose we have a monomial order $\succeq$ on $\text{Mon}(A)$ that is not a well order. This means that we have an infinite sequence of monomials

$$X_1 \succ X_2 \succ X_3 \succ \cdots$$

and since $\succeq$ is degree compatible, then we have an infinite subsequence

$$\deg(X_{i_1}) > \deg(X_{i_2}) > \deg(X_{i_3}) > \cdots,$$

but this is impossible since $\deg(X_{i_1})$ is finite. □

The following are examples of total orders which are compatible. These were adapted from [BGTV03], Examples 2.1.16 and 2.1.17.

EXAMPLE 2.1. 1. The degree lexicographical order $\succeq_{deglex}$ on $\text{Mon}(A)$ is defined by letting

$$x^{\alpha} \succeq_{deglex} x^{\beta} = \begin{cases} x^{\alpha} = x^{\beta} \ \text{ or} \\ x^{\alpha} \neq x^{\beta} \ \text{ but } \ |\alpha| > |\beta|, \text{or} \\ x^{\alpha} \neq x^{\beta}, |\alpha| = |\beta| \ \text{ but } \ \exists\, i \ \text{ with } \ \alpha_1 = \beta_1, \ldots, \\ \qquad\qquad \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases}$$

For example. If $\alpha = (2,4,5,0,\ldots,0)$ and $\beta = (4,3,0,0,\ldots,0)$ then, $x^{\alpha} \succeq_{deglex} x^{\beta}$, since $|\alpha| = |(2,4,5,0,\ldots,0)| = 11 > |\beta| = |(4,3,0,0,\ldots,0)| = 6$.

2. The degree reverse lexicographical order $\succeq_{degrevlex}$ on $\text{Mon}(A)$ is defined by letting

$$x^{\alpha} \succeq_{degrevlex} x^{\beta} = \begin{cases} x^{\alpha} = x^{\beta} \ \text{ or} \\ x^{\alpha} \neq x^{\beta} \ \text{ but } \ |\alpha| > |\beta|, \text{or} \\ x^{\alpha} \neq x^{\beta}, |\alpha| = |\beta| \ \text{ but } \ \exists\, i \ \text{ with } \ \alpha_1 = \beta_1, \ldots, \\ \qquad\qquad \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i. \end{cases}$$

For example. If $\alpha = (2,6,3,0,\dots,0)$ and $\beta = (5,2,4,0,\dots,0)$ then, $x^{\alpha} \succeq_{degrevlex} x^{\beta}$, since $|\alpha| = |(2,6,3,0,\dots,0)| = 11 = |\beta| = |(5,2,4,0,\dots,0)| = 11$, but $\alpha_1 = 2 < \beta_1 = 5$.

Remark 11. We must note that the proof of Proposition 2.5 could be unfold with more elaborated argument, based upon Dickson's Lemma.

2.1.2 (Σ, Δ)-compatibility over skew PBW extensions

The notion of compatibility for skew PBW extensions was introduced independently by Hashemi et al. [HKA17] and Reyes and Suárez in [RS18a]. This extends the Definition 1.12 introduced by Annin for modules.

Definition 2.5 ([HKA17], Definition 3.1; [RS18a], Definition 3.2). Consider a ring R with a finite family of endomorphisms Σ and a finite family of Σ-derivations Δ. Following the notation established in Definition 2.3 part (i), we have that R is said to be Σ-*compatible*, if for each $a, b \in R$, $a\sigma^{\alpha}(b) = 0$ if and only if $ab = 0$, for every $\alpha \in \mathbb{N}^n$; R is said to be Δ-*compatible*, if for each $a, b \in R$, $ab = 0$ implies $a\delta^{\beta}(b) = 0$, for every $\beta \in \mathbb{N}^n$. If R is both Σ-compatible and Δ-compatible, R is called (Σ, Δ)-*compatible*. From now on, we consider finite families of endomorphisms and derivations, so we say *family* to mean *finite family*.

Proposition 2.6 ([HKA17], Lemma 3.3; [RS18a], Proposition 3.8). *Let R be a (Σ, Δ)-compatible ring. For every $a, b \in R$, we have:*

(1) *If $ab = 0$, then $a\sigma^{\theta}(b) = \sigma^{\theta}(a)b = 0$, for each $\theta \in \mathbb{N}^n$.*

(2) *If $\sigma^{\beta}(a)b = 0$ for some $\beta \in \mathbb{N}^n$, then $ab = 0$.*

(3) *If $ab = 0$, then $\sigma^{\theta}(a)\delta^{\beta}(b) = \delta^{\beta}(a)\sigma^{\theta}(b) = 0$, for every $\theta, \beta \in \mathbb{N}^n$.*

Remark 12. The notion of compatibility over skew PBW extensions has been very useful in the characterization of several ring and homological properties of these extensions (e.g., [HKA17], [JR18], [Rey19], [RR19], [RS16], [RS18a], [RS19b] and [RS19a]).

In this section, we extend the results presented by Annin in [Ann04] for Ore extensions to the context of skew PBW extensions.

Definition 2.6 ([Rey19], Proposition 3.6). Consider a ring R with a family of endomorphisms Σ and a family of Σ-derivations Δ and M_R be a right R-module. We have that M_R is said to be Σ-*compatible*, if for each $m \in M_R$ and $r \in R$, $m\sigma^{\alpha}(r) = 0$ if and only if $mr = 0$, for every $\alpha \in \mathbb{N}^n$. Additionally, M_R is said to be Δ-*compatible*, if for each $m \in M_R$ and $r \in R$, $mr = 0$ implies $m\delta^{\beta}(r) = 0$, for every $\beta \in \mathbb{N}^n$. If M_R is both Σ-compatible and Δ-compatible, M_R is called (Σ, Δ)-compatible.

Proposition 2.7 ([Rey19], p. 8). *Consider a ring R with a family of endomorphisms Σ and a family of Σ-derivations Δ. If M_R is a (Σ, Δ)-compatible module, $m \in M$ and $a, b \in R$, then we have the following assertions:*

(1) *if $ma = 0$, then $m\sigma^{\theta}(a) = 0 = m\delta^{\theta}(a)$, for any element $\theta \in \mathbb{N}^n$;*

(2) *If* $mab = 0$*, then* $m\sigma_i(\delta^{\theta}(a))\delta_i(b) = m\sigma^{\beta}(\delta_i(a))\delta^{\theta}(b)$*, and so,* $ma\delta^{\theta}(b) = 0 = m\delta^{\theta}(a)b$*, for any elements* $\beta, \theta \in \mathbb{N}^n$*, and* $i = 1, \dots, n$*.*

(3) $\mathrm{Ann}_R(\{ma\}) = \mathrm{Ann}_R(\{m\sigma_i(a)\}) = \mathrm{Ann}_R(\{m\delta_i(a)\})$*, for every* $i = 1, \dots, n$*.*

The following result is the analogue to the established in [Ann04], Lemma 2.1.

Proposition 2.8 ([Rey19], Proposition 3.7). *Let* $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ *and* M_R *a* (Σ, Δ)*-compatible right* R*-module. If* $m = m_0 + m_1X_1 + \cdots + m_kX_k$ *is an element of* $M\langle X\rangle_A$ *and* $r \in R$*, then* $mr = 0$ *if, and only if,* $m_ir = 0$*, for every* $0 \le i \le k$*.*

Proof. Suppose that $m_ir = 0$, for every $0 \le i \le k$. Since

$$
\begin{aligned}
mr &= (m_0 + m_1X_1 + \cdots + m_kX_k)r \\
&= m_0r + m_1X_1r + \cdots + m_kX_kr \\
&= m_0r + m_1(\sigma^{\alpha_1}(r)X_1 + p_{\alpha_1,r}) + \cdots + m_k(\sigma^{\alpha_k}(r)X_k + p_{\alpha_k,r}) \\
&= m_0r + m_1\sigma^{\alpha_1}(r)X_1 + m_1p_{\alpha_1,r} + \cdots + m_k\sigma^{\alpha_k}(r)X_k + m_kp_{\alpha_k,r},
\end{aligned} \tag{2.2}
$$

where $\alpha_i = \exp(X_i)$, $p_{\alpha_i,r} = 0$, or, $\deg(p_{\alpha_i,r}) < |\alpha_i|$ if $p_{\alpha_i,r} \neq 0$, for every i. Using the equality $m_ir = 0$ with the expression

$$
\begin{aligned}
m_1x_1^{\alpha_{i1}}\cdots x_n^{\alpha_{in}}b_jx_1^{\beta_{j1}}\cdots x_n^{\beta_{jn}} &= m_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} + m_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\
&+ m_ix_1^{\alpha_{i1}}p_{\alpha_{i2},\sigma_{i3}^{\alpha_{i3}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\
&+ m_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}p_{\alpha_{i3},\sigma_{i4}^{\alpha_{i4}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_4^{\alpha_{i4}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\
&+ \cdots + m_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}(b_j)}x_n^{\alpha_{in}}x^{\beta_j} \\
&+ m_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-1)}^{\alpha_{i(n-1)}}p_{\alpha_{in},b_j}x^{\beta_j}.
\end{aligned} \tag{2.3}
$$

and the (Σ, Δ)-compatibility of M_R (Proposition **??**), we conclude that $mr = 0$.

Now, suppose that $mr = 0$. From expression (2.2) we can see that $\mathrm{lc}(mr) = m_k\sigma^{\alpha_k}(r)$, so, by the Σ-compatibility of M_R, we obtain $m_kr = 0$. Hence, expression (2.3) and (Σ, Δ)-compatibility of M_R imply that $p_{\alpha_k,r} = 0$, so mr reduces to

$$mr = m_0r + m_1\sigma^{\alpha_1}X_1 + m_1p_{\alpha_1,r} + \cdots + m_{k-1}\sigma^{\alpha_{k-1}}X_{k-1} + m_{k-1}p_{\alpha_{k-1},r}.$$

Again, since $\mathrm{lc}(mr) = m_{k-1}\sigma^{\alpha_{k-1}}(r) = 0$, from Σ-compatibility of M_R we can assert that $m_{k-1}r = 0$. In this way, expression (2.3) and (Σ, Δ)-compatibility of M_R imply that $p_{\alpha_{k-1},r} = 0$, so mr takes the form

$$mr = m_0r + m_1\sigma^{\alpha_1}(r)X_1 + m_1p_{\alpha_1,r} + \cdots + m_{k-2}\sigma^{\alpha_{k-2}}X_{k-2} + m_{k-2}p_{\alpha_{k-2},r}.$$

Continuing in this way we can show that $m_kr = m_{k-1}r = \cdots = m_1r = m_0r = 0$, which concludes the proof. □

Next proposition extends [Ann04], Lemma 2.2.

PROPOSITION 2.9. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ be a skew PBW extension. If M_R is a (Σ,Δ)-compatible right R-module, then $M\langle X\rangle_R$ is (Σ,Δ)-compatible.*

Proof. Consider the elements $r \in R$ and $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$. We have

$$\begin{aligned} mr &= (m_0 + m_1X_1 + \cdots + m_kX_k)r \\ &= m_0r + m_1X_1r + \cdots + m_kX_kr \\ &= m_0r + m_1(\sigma^{\alpha_1}(r)X_1 + p_{\alpha_1,r}) + \cdots + m_k(\sigma^{\alpha_k}(r)X_k + p_{\alpha_k,r}) \\ &= m_0r + m_1\sigma^{\alpha_1}(r)X_1 + m_1p_{\alpha_1,r} + \cdots + m_k\sigma^{\alpha_k}(r)X_k + m_kp_{\alpha_k,r}, \end{aligned} \tag{2.4}$$

where $\alpha_i = \exp(X_i)$, $p_{\alpha_i,r} = 0$, or, $\deg(p_{\alpha_i,r}) < |\alpha_i|$ if $p_{\alpha_i,r} \neq 0$, for every i. First we assume that $mr = 0$. Proposition 2.8 guarantees that for every $0 \le i \le k$, $m_ir = 0$. Using that M_R is (Σ,Δ)-compatible, we obtain that $m_i\sigma^{\alpha_i}(r) = 0$ and $p_{\alpha_i,r} = 0$, for every $0 \le i \le k$. Therefore, using a similar reasoning as above with the scalar $\sigma^\theta(r)$ or $\delta^\theta(r)$ instead of r shows that we can see $m\sigma^\theta(r) = m\delta^\theta(r) = 0$, for every $\theta \in \mathbb{N}^n$.

Now, we prove that if $m\sigma^\theta(r) = 0$, then $mr = 0$, for $\theta \in \mathbb{N}^n$. For this, we assume that $m\sigma^\theta(r) = 0$. Proposition 2.8 implies that $m_j\sigma^\theta(r) = 0$, for each $0 \le j \le k$. So, by Σ-compatibility of M_R, we obtain $m_jr = 0$. Again, using a similar reasoning, we have $mr = 0$. Thus we conclude that $M\langle X\rangle_A$ is (Σ,Δ)-compatible. □

The next proposition extends [Ann04], Lemma 2.3, making use of the (Σ,Δ)-compatibility.

PROPOSITION 2.10. *Consider a ring R with a family of endomorphisms Σ and a family of Σ-derivations Δ. If M_R is a (Σ,Δ)-compatible right R-module, then for all $m \in M$, $r \in R$, and $\theta \in \mathbb{N}^n$, we have*

$$\mathrm{Ann}(mr) = \mathrm{Ann}_R(m\sigma^\theta(r)) \subseteq \mathrm{Ann}_R(m\delta^\theta(r)).$$

Proof. Let $a \in R$ such that $m\sigma^\theta(r)a = 0$ with $\theta \in \mathbb{N}^n$. By Σ-compatibility of M_R we have $m\sigma^\theta(r)\sigma^\beta(a) = 0$, for every $\beta \in \mathbb{N}^n$. In particular, if we take $\theta = \beta$, then $m\sigma^\theta(r)\sigma^\theta(a) = m\sigma^\theta(ra) = 0$, and hence $mra = 0$ because M_R is Σ-compatible. Conversely, if $a \in \mathrm{Ann}(mr)$, then $mra = 0$, and since M_R is Σ-compatible, $m\sigma^\theta(ra) = 0$ for every $\theta \in \mathbb{N}^n$, and so, $m\sigma^\theta(r)\sigma^\theta(a) = 0$, whence $m\sigma^\theta(r)a = 0$, by the Σ-compatibility of M_R.

With the aim of showing $\mathrm{Ann}(mr) \subseteq \mathrm{Ann}_R(m\delta^\theta(r))$, we take $a \in \mathrm{Ann}(mr)$. Then $mra = 0$, and using the Proposition **??** (2) we have that $m\delta^\theta(r)a = 0$, for $\theta \in \mathbb{N}^n$. Thus $a \in \mathrm{Ann}_R(m\delta^\theta(r))$. □

Propositions 2.11 and 2.12 extend [Ann04], Lemma 2.4 and 2.5, respectively.

PROPOSITION 2.11. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ and P_A an A-module. If M_R is a R-submodule of P_A*

and P_R is a (Σ,Δ)-compatible R-module, then

$$R\cap \text{Ann}(M_R\cdot A_A)=\text{Ann}(M_R).$$

Proof. First we prove that $R\cap\text{Ann}(M_R\cdot A_A)\supseteq\text{Ann}(M_R)$. For this, suppose $r\in R$ such that $M_Rr=0$, $q\in M_R$ and $m=bx_1^{\alpha_1}\cdots x_n^{\alpha_n}=bx^{\alpha}\in A_A$, with $\alpha=(\alpha_1,\dots,\alpha_n)$. With the aim of proving that $qmr=0$, due to Proposition (2.3) we have

$$\begin{aligned}
qmr &= q(bx^{\alpha}r)\\
&= qb[x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}\left(\sum_{j=1}^{\alpha_n}x_n^{\alpha_n-j}\delta_n(\sigma_n^{j-1}(r))x_n^{j-1}\right)\\
&+x_1^{\alpha_1}\cdots x_{n-2}^{\alpha_{n-2}}\left(\sum_{j=1}^{\alpha_{n-1}}x_{n-1}^{\alpha_{n-1}-j}\delta_{n-1}(\sigma_{n-1}^{j-1}(\sigma_n^{\alpha_n}(r)))x_{n_2}^{j-1}\right)x_n^{\alpha_n}\\
&+x_1^{\alpha_1}\cdots x_{n-3}^{\alpha_{n-3}}\left(\sum_{j=1}^{\alpha_{n-2}}x_{n-2}^{\alpha_{n-2}-j}\delta_{n-2}(\sigma_{n-2}^{j-1}(\sigma_{n-1}^{\alpha_{n-1}}(\alpha_n^{\alpha_n}(r))))x_{n-2}^{j-1}\right)x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}\\
&+\cdots+x_1^{\alpha_1}\left(\sum_{j=1}^{\alpha_2}x_2^{\alpha_2-j}\delta_2\left(\sigma_2^{j-1}(\sigma_3^{\alpha_3}(\sigma_4^{\alpha_4}(\cdots(\sigma_n^{\alpha_n}(r)))))\right)x_2^{j-1}\right)x_3^{\alpha_3}x_4^{\alpha_4}\cdots x_{n-1}^{\alpha_{n-1}}x_n^{\alpha_n}\\
&+\sigma_1^{\alpha_1}(\sigma_2^{\alpha_2}(\cdots(\sigma_n^{\alpha_n}(r))))x_1^{\alpha_1}\cdots x_n^{\alpha_n}],\qquad \sigma_j^0:=\text{id}_\text{R},\ \text{ for }\ 1\le j\le n.
\end{aligned}$$

Since M_R is a submodule P_R, by the (Σ,Δ)-compatibility of P_R, we have that M_R is (Σ,Δ)-compatible. Thus, $q\sigma^{\theta}(r)=0$ and $q\delta^{\theta}(r)=0$, with $\theta\in\mathbb{N}$. So we have $qmr=0$.

Conversely, $R\cap\text{Ann}(M_R\cdot A_A)\subseteq\text{Ann}(M_R)$. For this, let $r\in R$, $q\in M_R$ and $m\in A_A$, such that $qmr=0$. Since we have $qm\in M_R$, then $r\in\text{Ann}(M_R)$, so the proof concludes. □

Proposition 2.12. *Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ and P_A an A-module. If P_A is an A-module prime and P_R is Σ-compatible, then P_R is prime.*

Proof. Let M_R be a submodule of P_R with $M_R\neq 0$. The idea is to prove that $\text{Ann}(M_R)=\text{Ann}(P_R)$. We assume that $M_Rr=0$, for some $r\in R$. By the previous proposition $(M_R\cdot A_A)r=0$, so r annihilates a nonzero A-submodule of P_A. Since P_A is prime we have $P_Ar=0$ and given that $P_R\subseteq P_A$, we obtain $P_Rr=0$. Conversely, we have $M_R\subseteq P_R$ which implies $\text{Ann}(P_R)\subseteq\text{Ann}(M_R)$. □

2.1.3 Annihilator-compliant over skew PBW extensions

Our purpose below is to study the notion of annihilator-compliant over skew PBW extensions, under the hypothesis that M_R is (Σ,Δ)-compatible. Therefore, we extend this notion introduced by Annin [Ann04] for Ore extensions. The total order introduced in Remark 2.3 is used in what follows.

Definition 2.7. Let $A=\sigma(R)\langle x_1,\dots,x_n\rangle$ and M_R a (Σ,Δ)-compatible right R-module. If we let $m=m_1X_1+\cdots+m_kX_k$ an element of $M\langle X\rangle_A$ of leading monomial $X_k=x^{\alpha_k}$, with $X_k\succ X_{k-1}\succ$

$\cdots \succ X_1$ and leading coefficient $m_k \neq 0$, then m is called *annihilator-compliant*, if for each $i \leq k$, we have $\text{Ann}(m_k) \subseteq \text{Ann}(m_i)$.

Next, we extend Lemmas 3.1, 3.2 and 3.3 presented by Annin [Ann04] to the context of modules over skew PBW extensions. Suppose that the elements $c_{i,j}$ in Definition 2.1 are invertible.

Proposition 2.13. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ and M_R a Σ-compatible right R-module. If $m = m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$ with $m \neq 0$ then, there exists $r \in R$ such that mr is annihilator-compliant.*

Proof. Let us assume that the result is false. Let $m = m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$ of leading monomial $X_k = x^{\alpha_k}$, with $X_k \succ X_{k-1} \succ \cdots \succ X_1$. Given that every monomial orden on $\text{Mon}(A)$ is a well order by Proposition 2.5, consider m as a counterexample the statement with minimal leading monomial X_k. Since $1 \in R$ then $m1 = m$ is not annihilator-compliant. Hence, there exists $1 \leq i < k$ such that $\text{Ann}(m_k) \nsubseteq \text{Ann}(m_i)$. Thus, there is $r \in R$ such that $m_kr = 0$ but $m_ir \neq 0$. Since,

$$\begin{aligned}
mr &= (m_1X_1 + \cdots + m_kX_k)r \\
&= m_1X_1r + \cdots + m_kX_kr \\
&= m_1(\sigma^{\alpha_1}(r)X_1 + p_{\alpha_1,r}) + \cdots + m_k(\sigma^{\alpha_k}(r)X_k + p_{\alpha_k,r}) \\
&= m_1\sigma^{\alpha_1}(r)X_1 + m_1p_{\alpha_1,r} + \cdots + m_k\sigma^{\alpha_k}(r)X_k + m_kp_{\alpha_k,r},
\end{aligned}$$

we have $\text{lt}(mr) = m_k\sigma^{\alpha_k}(r)X_k$. Given that $m_kr = 0$, due to the Σ-compatibility of M_R it follows that $m_k\sigma^{\alpha_k}(r) = 0$. Therefore, $\text{lm}(mr) = 0$, thus $\text{lm}(mr) \prec X_k$ with $mr \in M\langle X\rangle_A$. On the other hand, since $m_i\sigma^{\alpha_i}(r) \neq 0$, by the Σ-compatibility of M_R, we obtain $m_ir \neq 0$, whence $mr \neq 0$. Since, we assume that m is a polynomial with minimal leading monomial and mr has leading monomial smaller than m, then there exists $a \in R$ such that mra is annihilator-compliant, whence $ra \in R$, but this is a contradiction, since we consider m was a counterexample. □

Next proposition shows us that any scalar multiple of the annihilator-compliant is also a annihilator-compliant.

Proposition 2.14. *Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ and M_R a (Σ,Δ)-compatible right R-module. Let $m = m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$ annihilator-compliant. If $mr \neq 0$, then for all $r \in R$, it is still annihilator-compliant.*

Proof. Let $m = m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$ with leading monomial $X_k = x^{\alpha_k}$ such that $X_k \succ X_{k-1} \succ \cdots \succ X_1$ and assume that for $r \in R$, $mr \neq 0$. We have that

$$\begin{aligned}
mr &= (m_1X_1 + \cdots + m_kX_k)r \\
&= m_1X_1r + \cdots + m_kX_kr \\
&= m_1(\sigma^{\alpha_1}(r)X_1 + p_{\alpha_1,r}) + \cdots + m_k(\sigma^{\alpha_k}(r)X_k + p_{\alpha_k,r}) \\
&= m_1\sigma^{\alpha_1}(r)X_1 + m_1p_{\alpha_1,r} + \cdots + m_k\sigma^{\alpha_k}(r)X_k + m_kp_{\alpha_k,r},
\end{aligned} \tag{2.5}$$

where $\alpha_i = \exp(X_i)$, $p_{\alpha_i,r} = 0$, or, $\deg(p_{\alpha_i,r}) < |\alpha_i|$ if $p_{\alpha_i,r} \neq 0$, for every i. We have $\text{lt}(mr) = m_k\sigma^{\alpha_k}(r)X_k$ and $\text{lc}(mr) = m_k\sigma^{\alpha_k}(r)$ which must be nonzero. Since $m_k\sigma^{\alpha_k}(r) = 0$, by the Σ-

compatibility of M_R we have $m_k r = 0$, and since that m is annihilator-compliant, we obtain $m_i r = 0$, for all i, and the Proposition 2.8 implies that $mr = 0$. So $\mathrm{lc}(mr) = m_k\sigma^{\alpha_k}(r)$.

From expression (2.5) we know that for each $1 \le i < k$, the coefficients of mr are composed for $m_i\sigma^{\alpha_i}(r)$ and $m_i p_{\alpha_i,r}$. The idea is to show that for $a \in R$ with $m_k\sigma^{\alpha_k}(r)a = 0$ then a kills these coefficients. Let us assume that $m_k\sigma^{\alpha_k}(r)a = 0$. By the Σ-compatibility of M_R and Proposition 2.11, we can assert that $m_k ra = 0$. Using that m is annihilator-compliant for $1 \le i \le k$, we have $m_i ra = 0$ and by the Σ-compatibility of M_R and Proposition 2.11, $m_i\sigma^{\alpha_i}(r)a = 0$. In this way, from the expresion (2.5), by (Σ, Δ)-compatibility of M_R and Proposition 2.11, $m_i p_{\alpha_i,r} = 0$. Again, the Proposition 2.11, for each i, concludes the proof. □

Proposition 2.15. *Let $A = \sigma(R)\langle x_1,\dots,x_n\rangle$ and M_R a Σ-compatible right R-module. Let $m = m_0 + m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$ be an annihilator-compliant polynomial and $\mathrm{lc}(m) = m_k \neq 0$. Consider the set $\mathscr{B} := \mathrm{Ann}(m_kR_R)$. We have*

$$\mathrm{Ann}(mA_A) = \mathscr{B}\langle X\rangle_A.$$

Proof. First, we prove that $\mathscr{B}\langle X\rangle \subseteq \mathrm{Ann}(mA_A)$. For this, let $g \in \mathscr{B}\langle X\rangle$ with $g = a_0 + a_1Y_1 + \cdots + a_tY_t$, $Y_t \succ Y_{t-1} \succ \cdots \succ Y_1$ and $\mathrm{lc}(g) \neq 0$. Then $m_kRa_j = 0$ for all $1 \le j < t$, and by Σ-compatibility of M_R, we have $m_k\sigma^{\alpha_k}(Ra_j) = 0$. Hence, m is a annihilator-compliant polynomial, $m_i\sigma^{\alpha_i}(Ra_j) = 0$, for all $1 \le j < t$ and $1 \le i < k$, and therefore $m_iRa_j = 0$, for every i, j. Now, let $h = h_0 + h_1X_1 + \cdots + h_lX_l \in A_A$, with $h \neq 0$ such that $mh \in mA_A$. We consider a term of mhg like $(m_iX_i)(h_dX_d)(a_jY_j)$. With this aim, consider the product

$$\begin{aligned}(m_iX_i)(h_dX_d)(a_jY_j) &= m_i\sigma^{\alpha_i}(h_d)\left(\sigma^{\alpha_i}(\sigma^{\alpha_d}(a_j))X_i + p_{\alpha_k,\sigma^{\alpha_d}(a_j)}\right)X_dY_j\\ &\quad + m_i\sigma^{\alpha_k}(h_d)X_ip_{\alpha_d,a_j}Y_j + m_ip_{\alpha_i,h_d}\sigma^{\alpha_d}(a_j)X_dY_j\\ &\quad + m_ip_{\alpha_i,h_d}p_{\alpha_d,a_j}Y_j.\end{aligned}$$

The terms that arise consist of products of the m_iR_R with several evaluarion of a_j in σ's and δ's depending of the coordinates of α_i. Then, these terms must be zero since a_j kills m_iR_R. Thus, $g \in \mathrm{Ann}(mA_A)$.

Conversely, $\mathrm{Ann}(mA_A) \subseteq \mathscr{B}\langle X\rangle$, suppose that the inclusion fails. Let $g \in \mathrm{Ann}(mA_A)\setminus\mathscr{B}\langle X\rangle$ writing as before, whence $mAg = 0$. Following Definition 2.3, we can fix a monomial order on $\mathrm{Mon}(A)$, with $X_t \succ X_{t-1} \succ \cdots \succ X_1$ and given that every monomial orden on $\mathrm{Mon}(A)$ is a well order, suppose that g is an element of $\mathrm{Ann}(mA_A)$ of minimal leading monomial $Y_t = x^{\alpha_t}$. Consider the product

$$(m_0 + m_1X_1 + \cdots + m_kX_k)A(a_0 + a_1Y_1 + \cdots + a_tY_t) = 0,$$

in particular, if we only take coefficients in R, we have consider the product $(m_0 + m_1X_1 + \cdots + m_kX_k)R(a_0 + a_1Y_1 + \cdots + a_tY_t) = 0$. Since for any $r \in R$ using the Proposition 2.2, then

$$
\begin{aligned}
0 &= (m_0 + m_1X_1 + \cdots + m_kX_k)r(a_0 + a_1Y_1 + \cdots + a_tY_t) \\
&= m_0ra_0 + m_0ra_1Y_1 + \cdots + m_0ra_tY_t + m_1X_1ra_0 + m_1X_1ra_1Y_1 + \cdots + m_1X_1ra_tY_t \\
&+ \cdots + m_kX_kra_0 + m_kX_kra_1Y_1 + \cdots + m_kX_kra_yY_t \\
&= m_0ra_0 + m_0ra_1Y_1 + \cdots + m_0ra_tY_t + m_1\sigma^{\alpha_1}(ra_0)X_1 + m_1p_{\alpha_1,ra_0} + m_1\sigma^{\alpha_1}(ra_1)X_1Y_1 \\
&+ m_1p_{\alpha_1,ra_1}Y_1 + \cdots + m_1\sigma^{\alpha_1}(ra_t)X_1Y_t + \cdots + m_1p_{\alpha_1,ra_t}Y_t + \cdots + m_k\sigma^{\alpha_k}(ra_0)X_k \\
&+ m_kp_{\alpha_k,ra_0} + m_k\sigma^{\alpha_k}(ra_1)X_kY_1 + m_kp_{\alpha_k,ra_1}Y_1 + \cdots + m_k\sigma^{\alpha_k}(ra_t)X_kY_t + m_kp_{\alpha_k,ra_t}Y_t \\
&= m_0ra_0 + m_0ra_1Y_1 + \cdots + m_0ra_tY_t + m_1\sigma^{\alpha_1}(ra_0)X_1 + m_1p_{\alpha_1,ra_0} \\
&+ m_1\sigma^{\alpha_1}(ra_1)c_{\alpha_1,\beta_1}x^{\alpha_1+\beta_1} + m_1\sigma^{\alpha_1}(ra_1)p_{\alpha_1,\beta_1} + m_1p_{\alpha_1,ra_1}Y_1 \\
&+ \cdots + m_1\sigma^{\alpha_1}(ra_t)d_{\alpha_1,\beta_1}x^{\alpha_1+\beta_1} + m_1\sigma^{\alpha_1}(ra_t)p_{\alpha_1,\beta_1} + \cdots + m_1p_{\alpha_1,ra_t}Y_t \\
&+ \cdots + m_k\sigma^{\alpha_k}(ra_0)X_k + m_kp_{\alpha_k,ra_0} + m_k\sigma^{\alpha_k}(ra_1)d_{\alpha_k,\beta_1}x^{\alpha_k+\beta_1} + m_k\sigma^{\alpha_k}(ra_1)p_{\alpha_k,\beta_1} \\
&+ m_kp_{\alpha_k,ra_1}Y_1 + \cdots + m_k\sigma^{\alpha_k}(ra_p)d_{\alpha_k,\beta_t}x^{\alpha_k+\beta_t} + m_k\sigma^{\alpha_k}(ra_p)p_{\alpha_k,\beta_t} + m_kp_{\alpha_k,ra_t}Y_t.
\end{aligned}
$$

We can see that the leading coefficient of this product is $m_k\sigma^{\alpha_k}(ra_t)d_{\alpha_k,\beta_t} = 0$, whence we obtain $m_k\sigma^{\alpha_k}(ra_t) = 0$, and so $m_kra_t = 0$, for all $r \in R$, because M_R is Σ-compatible. Now, since m is annihilator-compliant, whence $m_ira_t = 0$, for all $r \in R$ and each $1 \leq i < k$. Again, using Proposition 2.11 we have to $mAa_t = 0$. In addition, consider $f = a_1Y_1 + \cdots + a_{t-1}Y_{t-1}$ with $\mathrm{lm}(f) \prec Y_t$, then

$$(m_0 + m_1X_1 + \cdots + m_kX_k)A(a_1Y_1 + \cdots + a_{t-1}Y_{t-1}) = mAf = 0,$$

so that, $f \in \mathrm{Ann}(mA_A)$. But since $m_kra_t = 0$ for all $r \in R$, $a_t \in \mathscr{B}$. Thus, $f \notin \mathscr{B}\langle X\rangle$, which is a contradiction since g has a leading minimial monomial. □

The following result presents a characterization of the annihilator-compliant that extends Corollary 3.1 presented by Annin in [Ann04].

Corollary 2.16. *Let $A = \sigma(R)\langle x_1, \ldots, x_n\rangle$ and M_R a Σ-compatible right R-module. Let $m = m_1X_1 + \cdots + m_kX_k \in M\langle X\rangle_A$ with $X_k \succ X_{k-1} \succ \cdots \succ X_1$ and leading monomial $X_k = x^{\alpha_k}$. Then m is annihilator-compliant polynomial if, and only if, for all $f \in mA_A \subseteq M\langle X\rangle_A$, we have that* $\mathrm{lm}(f) \succeq X_k$.

Proof. First assume that m is annihilator-compliant. Due to Proposition 2.14, for all $r \in R$, mr is annihilator-compliant with $\mathrm{lm}(mr) = X_k$. Let $g = r_1Y_1 + \cdots + r_pY_p \in A_A$ with $Y_p \prec Y_{p-1} \prec \cdots \prec Y_1$ and $mg \neq 0$. Then

$$mg = (m_1X_1 + \cdots + m_kX_k)(r_1Y_1 + \cdots + r_pY_p),$$

which implies that $\mathrm{lt}(mg) = m_kX_kr_pY_p$, whence using Proposition 2.2 we have,

$$\mathrm{lt}(mg) = m_k\sigma^{\alpha_p}(r_p)X_kY_p = m_k\sigma^{\alpha_p}(r_p)d_{\alpha_k,\beta_p}x^{\alpha_k+\beta_p} + p_{\alpha_k,\beta_p},$$

thus, $\mathrm{lm}(mg) \prec X_k$.

For the converse, consider m a polinomial in $M\langle X\rangle_A$ that is not annihilator-compliant. Then, $\mathrm{Ann}(m_k) \nsubseteq \mathrm{Ann}(m_i)$, i.e., there exists $r \in R$ such that $m_k r = 0 \neq m_i r$. From this, we have $mr \neq 0$ with $\mathrm{lt}(mr) \prec X_k$. □

2.1.4 Main result

At this point, we establish the main result of this section making use of the previous results. This result generalizes [Ann04], Theorem 2.1.

Theorem 2.17. *Let A be a skew PBW extension of R and M_R a right R-module. Let Σ be a family of endomorphisms and Δ a family of Σ-derivations. If M_R is (Σ, Δ)-compatible, then*

$$\mathrm{Ass}(M\langle X\rangle_A) = \{P\langle X\rangle : P \in \mathrm{Ass}(M_R)\}$$

Proof. With the aim of establishing the desired equality, we proof the two implications.

- We first prove the opposite implication, that is, $\mathrm{Ass}(M\langle X\rangle_A) \supseteq \{P\langle X\rangle : P \in \mathrm{Ass}(M_R)\}$. For this, suppose that $P \in \mathrm{Ass}(M_R)$, and find a prime submodule $N_R \subseteq M_R$ with $P = \mathrm{Ann}(N_R)$. We must proof that for a submodule prime $N\langle X\rangle_A \subseteq M\langle X\rangle_A$, then

$$P\langle X\rangle = \mathrm{Ann}(N\langle X\rangle_A). \tag{2.6}$$

  We start proving (2.6).

  Consider the elements $m = m_1X_1 + \cdots + m_tX_t \in N\langle X\rangle_A$ and $g = b_1Y_1 + \cdots + b_lY_l \in A$. First assume that $g \in P\langle X\rangle$, so that $m_jb_i = 0$, for all j, i. Thus, by Proposition 2.8, $mb_i = 0$ for every $1 \leq i \leq l$, whence we have $mg = 0$, and thus $g \in \mathrm{Ann}(N\langle X\rangle_A)$. On the other hand, assume that $g \notin P\langle X\rangle$, so, for some $1 \leq j \leq l$, we have $b_j \notin P$. Then there exists $u = a_1X_1 + \cdots + a_tX_t \in N_R$ with $ub_j \neq 0$. Given that,

$$\begin{aligned} ug &= (a_1X_1 + \cdots + a_tX_t)(b_1Y_1 + \cdots + b_lY_l) \\ &= \sum_{k=1}^{t+l} \left(\sum_{i+j=k} a_iX_ib_jY_j \right) \end{aligned}$$

  where each

$$\begin{aligned} a_iX_ib_jY_j = a_i\sigma^{\alpha_i}(b_j)x^{\alpha_i}x^{\beta_j} &+ a_ip_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_2^{\alpha_{i2}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\ &+ a_ix_1^{\alpha_{i1}}p_{\alpha_{i1},\sigma_{i2}^{\alpha_{i2}}(\cdots(\sigma_{in}^{\alpha_{in}}(b_j)))}x_3^{\alpha_{i3}}\cdots x_n^{\alpha_{in}}x^{\beta_j} \\ &+ \cdots + a_ix_1^{\alpha_{i1}}x_2^{\alpha_{i2}}\cdots x_{i(n-2)}^{\alpha_{i(n-2)}}p_{\alpha_{i(n-1)},\sigma_{in}^{\alpha_{in}}}x_n^{\alpha_{in}}x^{\beta_j} \\ &+ a_ix_1^{\alpha_{i1}}\cdots x_{i(n-1)}^{\alpha_{i(n-1}}p_{\alpha_{in},b_j}x^{\beta_j}. \end{aligned}$$

  We have that $ug \neq 0$, so $g \notin \mathrm{Ann}(N\langle X\rangle_A)$, as desired. Therefore, $P\langle X\rangle = \mathrm{Ann}(N\langle X\rangle_A)$.

Now we have to prove that $N\langle X\rangle_A$ is prime. From definition of module prime, if $N\langle X\rangle_A$ is prime then $\text{Ann}(N\langle X\rangle_A)$ is constant across all nonzero submodule of $N\langle X\rangle_A$. We must see that $\text{Ann}(m\cdot A_A) = \text{Ann}(N\langle X\rangle_A) = P\langle X\rangle$.

The inclusion opposite follows because $P\langle X\rangle = \text{Ann}(N\langle X\rangle_A)$ and $m\cdot A_A$ is a submodule of $N\langle X\rangle_A$, whence $\text{Ann}(N\langle X\rangle_A) = P\langle X\rangle \subseteq \text{Ann}(m\cdot A_A)$.

Thinking in the opposite inclusion, suppose this fails. Thus, there exists $f \in \text{Ann}(m\cdot A_A)$ with $0 \neq f = a_1X_1 + \cdots + a_mX_m$ such that $f \notin P\langle X\rangle$. Since $f \in \text{Ann}(m\cdot A_A)$ we have that $mA_Af = 0$. Following Remark 2.3 part (ii), we have fixing a monomial order on $\text{Mon}(A)$, with $X_m \succ X_{m-1} \succ \cdots \succ X_1$ and given that every monomial orden on $\text{Mon}(A)$ is a well order, suppose that f is an element of $\text{Ann}(m\cdot A_A)$ of minimal leading monomial $X_m = x^{\alpha_m}$

Now, suppose that $a_m \in P$. Thus, given that $f' = a_0 + a_1X_1 + \cdots + a_{m-1}X_{m-1} \notin P\langle X\rangle$, and since $a_mX_m \in P\langle X\rangle \subseteq \text{Ann}(m\cdot A_A)$, we have $mA_Af' = 0$. We get $\text{lm}(f') \prec X_m$, but this is a contradiction since f has a leading minimal monomial.

Continuing with the same argument and given this contradiction, one must assume then $a_m \notin P$. Following Remark 2.3, let $\text{lc}(m) = m_t$ with $m_t \neq 0$. Since, by hypothesis N_R is module prime, then $\text{Ann}(m_tR_R) = P$. Hence, there exists $r \in R$ such that $m_tra_m \neq 0$. By the Σ-compatibility of M_R and since $N_R \subseteq M_R$, then $m_t\sigma^{\theta}(ra_m) \neq 0$, for $\theta \in \mathbb{N}^n$. From Proposition 2.2 we have

$$mr = m_0r + m_1X_1r + \cdots + m_t(\sigma^{\alpha_t}(r)x^{\alpha_t} + p_{\alpha_t,r}),$$

where $p_{\alpha_t,r} = 0$ or $\deg(p_{\alpha_t,r}) < |\alpha_t|$ if $p_{\alpha_t,r} \neq 0$. So that,

$$\begin{aligned} mrf &= (m_0r + m_1X_1r + \cdots + m_t(\sigma^{\alpha_t}(r)x^{\alpha_t} + p_{\alpha_t,r}))(a_0 + a_1X_1 + \cdots + a_mX_m) \\ &= m_0ra_0 + \cdots + m_1X_1ra_1X_1 + \cdots + m_t\sigma^{\alpha_t}(ra_m)X_tX_m + m_tp_{\alpha_t,ra_m}, \end{aligned}$$

where $m_tp_{\alpha_t,ra_m} = 0$ or $\deg(p_{\alpha_t,r}) < |\alpha_t + \alpha_m|$ if $m_tp_{\alpha_t,ra_m} \neq 0$, which implies that $m_t\sigma^{\alpha_t}(ra_m)X_tX_m \neq 0$, and hence we have $\text{lt}(mrf) \neq 0$, which contradicts the statement that $mA_Af = 0$. This completes the proof.

- Let us see the other inclusion. Let $I \in \text{Ass}(M\langle X\rangle_A)$, then for a A-module prime $P_A \subseteq M\langle X\rangle_A$ we have $I = \text{Ann}(P_A)$. Now, we choose $m' \in P_A$ with $m' \neq 0$. By Lemma 2.13, we have that $m'r$ for $r \in R$ is annihilator-compliant. Consider the element $m'r = m = m_1X_1 + \cdots + m_tX_t \in P_A$. By Definition 2.3, let $\text{lc}(m) = m_t$ with $m_t \neq 0$.

Let $Q_R := mR_R$ and we have that $QA_A \subseteq P_A$. Since, P_A is submodule prime then $I = \text{Ann}(P_A) = \text{Ann}(QA_A)$. Now, since M_R is Σ-compatible then $M\langle X\rangle_R$ is Σ-compatible (for Proposition 2.9), thus P_R and Q_R they are also Σ-compatible since they are submodules of $M\langle X\rangle_R$. By Proposition 2.12, we have that P_R and Q_R are primes.

From Proposition 2.11, we have that $R \cap \text{Ann}(Q_RA_A) = \text{Ann}(Q_R)$, and since $I = \text{Ann}(QA_A)$ then $R \cap I = \text{Ann}(Q_R)$. Now, we have that $I = \text{Ann}(QA_A) = \text{Ann}((mR_R)A_A) = \text{Ann}(mA_A)$, by Proposition 2.15 $\text{Ann}(mA_A) = \mathscr{B}\langle X\rangle_A = \text{Ann}(m_kR_R)\langle X\rangle_A$.

Since that $I \in \text{Ass}(M\langle X\rangle_A)$, we need to see that $I = (I \cap R)\langle X\rangle$. For this, we know that $I = \text{Ann}(mA_A) = \text{Ann}(m_kR_R)\langle X\rangle_A$, and we must see that $\text{Ann}(m_kR_R) = \text{Ann}(mA_A) \cap R$. This has been given that for $r \in (\text{Ann}(mA_A) \cap R)$. Thus by Proposition 2.11, in particular $r \in \text{Ann}(m)$, i.e., $mr = 0$ by Σ-compatibility of M_R and Proposition 2.8 then $m_ir = 0$, for all $1 \leq i \leq k$, and so $r \in \text{Ann}(m_kR_R)$. Now, given that m is annihilator-compliant then $\text{Ann}(m_k) \subseteq \text{Ann}(m_i)$, thus for $r \in \text{Ann}(m_kR_R)$ then $r \in \text{Ann}(m_iR_R)$ which implies that $r \in \text{Ann}(mA_A) \cap R$.

With the aim of finishing the proof, given that $R \cap I = \text{Ann}(Q_R)$ implies that $R \cap I = \text{Ass}(Q_R)$ (whence, Q_R is prime), consider the product mr for $r \in R$, we can defined an embedding $Q_R \hookrightarrow M_R$ with $mr \hookrightarrow m_kr$. We deduce that $I \cap R \subseteq \text{Ass}(Q_R) \subseteq \text{Ass}(M_R)$ and since $I = (I \cap R)\langle X\rangle$, we get the desired result.

□

2.1.5 Examples

We present remarkable examples of skew PBW extensions over (Σ, Δ)-compatible rings that illustrate the Theorem 2.17. A detailed description of each one of these rings as a skew PBW extension, can be found in [LR14], [RS17a] and [RS17c]. Throughout this part, the letter k denotes a field.

1. **Weyl algebra**. The first Weyl algebra $A_1(k)$ over k is defined to be the k-algebra generated by the indeterminates x, y subject to the relation $yx = xy + 1$. The nth Weyl algebra $A_n(k)$ over k is the k-algebra generated by the $2n$ indeterminates $x_1, \dots, x_n, y_1, \dots, y_n$ where

$$x_jx_i = x_ix_j, \quad y_jy_i = y_iy_j, \qquad 1 \leq i, j \leq n,$$
$$y_jx_i = x_iy_j + \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker delta}, \quad 1 \leq i, j \leq n.$$

This algebra is also known as the first quantum algebra. We can identify that

$$R := k[x_1, \dots, x_n]; \ \ X := \{y_1, \dots, y_n\};$$

$$\Sigma := \{\sigma_i : R \to R \mid \sigma_i(x_j) := x_j, \text{ and } 1 \leq i \leq n\},$$

and

$$\Delta := \{\delta_i : R \to R \mid \ 1 \leq i \leq n\},$$

such that

$$\delta_i(x_j) := \begin{cases} 1, & \text{if } \ i = j \\ 0, & \text{if } \ i \neq j \end{cases}$$

Firts, we have that $A_n(k) \cong \sigma(k[x_1, \dots, x_n])\langle y_1, \dots, y_n\rangle$ (see [LR14], Section 3.1). Also, we take $M_R := R$, thus, M_R is (Σ, Δ)-compatible because for $m, q \in M_R$ we have that $m\sigma(q) =$

$mq = 0 \Longleftrightarrow mq = 0$ and if $mq = 0$ then $m\delta(q) = 0$.

Now, using the Proposition 1.4 for M_R we have that $\mathrm{Ass}(M_R) = \{0\}$ and therefore applying Theorem 2.17 we conclude that $\mathrm{Ass}(M\langle X\rangle_A) = \{0\}$.

2. **[Gal15], Example 6.5.4** Let A be a skew PBW extension with $A \cong \sigma(\mathbb{Q})\langle x, y\rangle$, where $yx = xy + x$. Gallego in her work obtained a module over A for calculation with Gröbner basis. She defined $s_1 := (0, -y+1, x, 0)$, $s_2 := (-xy, 1, 0, y-1)$ and $s_3 := (xy^2 + 2xy, -y-1, 0, 1-y^2)$. Thus, she obtained the following module $A^4/\langle s_1, s_2, s_3\rangle$. For our case, we consider

$$R := \mathbb{Q};\ \ X := \{x_1 := x, x_2 := y\};\ \ M_R := A^4/\langle s_1, s_2, s_3\rangle;$$

   $\Sigma := \{\sigma_i : R \to R \mid \sigma_i(q) := q;\ q \in R$ and $1 \le i \le 2\}$ and $\Delta := \{\delta_i : R \to R \mid \delta_i(q) := 0;\ q \in R$ and $1 \le i \le 2\}$. Thus, M_R is (Σ, Δ)-compatible because σ is the identity application and δ is the zero application. Therefore, Proposition 2.17 applies.

3. **Dispin algebra** $\mathscr{U}(osp(1,2))$. This k-algebra is generated by the variables x, y, z subjected to the relations $yz - zy = z,\ zx + xz = y,\ xy - yx = x$. It is a 3-dimensional skew polynomial algebra (see [RS17c], Example 5.1). This example cannot be applied to Annin's main result, as it is not an Ore extension. The aforementioned algebra is a skew PBW extension over the field k generated by three indeterminates x, y and z. If we consider $R := k$; $X := \{x, y, z\}$; $M_R := R$; $\Sigma := \{\sigma_j : R \to R \mid \sigma_j := i_k, 1 \le j \le 3\}$ and $\Delta := \{\delta_j : R \to R \mid \delta_j := 0, 1 \le j \le 3\}$, then M_R is (Σ, Δ)-compatible. Therefore, Proposition 2.17 implies that the set of associated prime ideals consists of the zero prime ideal.

2.2 Associated prime ideals of skew PBW extensions over weak Σ-rigid rings

In Section 1.3.3 we consider some properties of weak σ-rigid rings. Now we present an extension of these rings which have been characterized by several works (c.f. [RS16], [RS19b], [NR17] and [RS17b]).

2.2.1 Σ-rigid rings and weak Σ-rigid rings

We start by presenting the definition of an extension of σ-rigid and weak σ-rigid rings.

Definition 2.8 ([Rey15], Definition 3.2). Let B be a ring and Σ a finite family of endomorphisms of B. Σ is called a *rigid endomorphisms family*, if $r\sigma^{\alpha}(r) = 0$ implies $r = 0$, where $r \in B$ and $\alpha \in \mathbb{N}^n$. A ring B is said to be Σ-*rigid*, if there exists a rigid endomorphisms finite family Σ of B.

It should be noted that if Σ is a finite family of endomorphisms rigid, then every element $\sigma_i \in \Sigma$ is a monomorphism. In fact, Σ-rigid rings are reduced rings: if B is a Σ-rigid ring and $r^2 = 0$ for $r \in B$, then $0 = r\sigma^{\alpha}(r^2)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r)\sigma^{\alpha}(\sigma^{\alpha}(r)) = r\sigma^{\alpha}(r)\sigma^{\alpha}(r\sigma^{\alpha}(r))$, i.e., $r\sigma^{\alpha}(r) = 0$ and so $r = 0$, that is, B is reduced.

From [RS18a], Proposition 3.4 and Example 3.6, we can see that Σ-rigid rings are strictly contained in (Σ, Δ)-compatible rings. Nevertheless, the next proposition shows the importance of reduced rings in the equivalence of both families of rings.

Proposition 2.18 ([HKA17], Lemma 3.5; [RS18a], Theorem 3.9). *If $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a skew PBW extension, then the following statements are equivalent: (1) R is reduced and (Σ, Δ)-compatible. (2) R is Σ-rigid. (3) A is reduced.*

Now, we present the following definition which extends Σ-rigid rings.

Definition 2.9 ([RS18a], Definition 3.2). Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$ be a family of endomorphisms and Σ-derivations of a ring R, respectively. R is called a *weak Σ-rigid ring*, if $a\sigma^{\theta}(a) \in N(R) \Leftrightarrow a \in N(R)$, for each element $a \in R$ and every $\theta \in \mathbb{N}^n$.

Remark 13. Σ-rigid ring and weak Σ-rigid rings have been studied in several works by obtaining important results in algebra research. We can found these results in [RS19a], [RS16], [RS17b], [RS18c], [RR19] and [NR17].

In Krempa [Kre96], Theorem 3.3, Krempa proved that if σ is a monomorphism of B, then the Ore extension $B[x;\sigma,\delta]$ is reduced if and only if B is reduced and σ-rigid. Even more, Reyes [Rey15], proved the following generalization of this fact which be useful in this work.

Proposition 2.19 ([Rey15], Proposition 3.5). *Let A be a bijective skew PBW extension over a ring R. Then R is Σ-rigid ring if, and only if, A is a reduced ring. In this case, $ex^{\alpha} = x^{\alpha}e$, for every $\alpha \in \mathbb{N}$ and $e = e^2 \in R$.*

The next theorem gives conditions for an equivalence between the notions of Σ-rigid rings and weak Σ-rigid rings.

Proposition 2.20 ([RS18b], Theorem 3.4). *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ and $\Delta = \{\delta_1, \dots, \delta_n\}$ be a family of endomorphisms and Σ-derivations of R, respectively. R is Σ-rigid if and only if R is weak Σ-rigid and reduced.*

Proof. We follow the proof presented by Reyes and Suarez [RS18b]. Suppose that R is Σ-rigid. As we saw above, R is reduced. Let us see that R is weak Σ-rigid. If $a \in \text{Nil}(R)$, then $a = 0$, since R is reduced, whence $a\sigma^{\theta}(a) = 0 \in \text{Nil}(R)$, for all $\theta \in \mathbb{N}^n$ and $1 \leq i \leq n$. Now, if $a\sigma^{\theta}(a) \in \text{Nil}(R)$, for $a \in R$ and every $\theta \in \mathbb{N}^n$, then $a\sigma^{\theta}(a) = 0$, for all $\theta \in \mathbb{N}^n$, since R is reduced, and hence $a = 0$ because R is Σ-rigid. Then R is weak Σ-rigid and reduced. Conversely, suppose that R is weak Σ-rigid and reduced, and let $a\sigma^{\theta}(a) = 0$, for $a \in R$ and $\theta \in \mathbb{N}^n$. Then $a \in \text{Nil}(R)$, since R is weak Σ-rigid, and so $a = 0$ because R is reduced. Therefore R is Σ-rigid. □

With the example taken from Ouyang [Ouy08], Example 1.8, we can see that every weak Σ-rigid ring is not always a Σ-rigid ring. The following rings are examples of Σ-rigid rings:

Example 2.2.
1. Every PBW extension A of a ring R such that the coefficients commute with the variables.
2. The algebra of partial differential operators and the algebra of linear differential operators (see [LR14]).

3. Multiplicative analogue of Weyl algebra: Let k be a field. It is the algebra over k, denoted $\mathcal{O}_n(\lambda_{ji})$ which is generated by $x_1,\dots,x_n$ with the relations $x_jx_i = \lambda_{ji}x_ix_j, 1 \leq i < j \leq n$, with $\lambda_{ji} \in k \setminus \{0\}$.

4. The Woronowicz algebra $\mathcal{W}_v(\mathfrak{sl}_3(\mathbb{C}))$; the q-Heisenberg algebra $\mathbf{H}_n(q)$; the Hayashi algebra $W_q(J)$ (see [LR14]).

5. Lie-deformed Heisenberg: This $\mathbb{C}$-algebra is defined by the commutation relation

$$q_j(1+i\lambda_{jk})p_k - p_k(1-i\lambda_{jk})q_j = ih\delta_{jk}$$
$$[q_j,q_k] = [p_j,p_k] = 0, \quad j,k = 1,2,3,$$

where q_j, p_j are the position and momentum operators, and $\lambda_{jk} = \lambda_k\delta_{jk}$, with λ_k real parameters. If $\lambda_{jk} = 0$ one recovers the usual Heisenberg algebra (see [RS17a]).

2.2.2 MAIN RESULT

In this section we generalize the results presented by Bhat ([Bha09] and [Bha10a]), from Ore extensions $R[x;\sigma,\delta]$ with σ be an endomorphism to skew PBW extensions. The results below were made out together with Arturo Niño, D.Sc. student. He contributed to the development of the proofs that characterize the associated prime ideals.

Throughout this section the notation presented in the Section 1.3.4 is employed. We use the same line of reasoning that the one used in Theorem 2.1 of Bhat in [Bha10a] to formulate the following proposition, which gives neccesary conditions for a Noetherian ring R to be a weak Σ-rigid ring.

REMARK 14. In the following propositions is necessary that R be an algebra over $\mathbb{Q}$, because the results previous to the main theorem require the construction of an automorphism as a power series, see [Bha09], Lemma 2.1.

PROPOSITION 2.21. *Let R be a 2-primal right Noetherian ring which is also an algebra over $\mathbb{Q}$. If $\Sigma = \{\sigma_1,\sigma_2,\dots,\sigma_n\}$ is a family of automorphisms of R such that R is a weak Σ-rigid ring and $\Delta = \{\delta_1,\dots,\delta_n\}$ is a family of Σ-derivations of R, then $\sigma^\alpha(U) = U$ and $\delta^\alpha(U) \subseteq U$, for all $\alpha \in \mathbb{N}^n$ and $U \in \mathrm{MinSpec}(R)$.*

Proof. This assertion is a direct consequence of Proposition 1.20. By hypothesis, R is weak Σ-rigid, but this implies that R is weak σ_i-rigid, for each $1 \leq i \leq n$. By Bhat [Bha10a], Proposition 2.2, $\sigma_i(U) = U$ and $\delta_i(U) \subseteq U$ for all $1 \leq i \leq n$ and $U \in \mathrm{MinSpec}(R)$. An iterative argument gives us that $\sigma^\alpha(U) = U$ and $\delta^\alpha(U) \subseteq U$, for all $\alpha \in \mathbb{N}^n$ and $U \in \mathrm{MinSpec}(R)$. □

The next proposition is a consequence of Bhat in [Bha09], Lemma 2.6.

PROPOSITION 2.22. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$, $\Sigma = \{\sigma_1,\dots,\sigma_n\}$ be a family of automorphisms of R and $\Delta = \{\delta_1,\dots,\delta_n\}$ be a family of Σ-derivations of R such that $\sigma_i(\delta_j(a)) = \delta_j(\sigma_i(a))$, for all $a \in R$ and $1 \leq i,j \leq n$. If $U \in \mathrm{MinSpec}(R)$ with $\sigma^\alpha(U) = U$, for all $\alpha \in \mathbb{N}^n$, then $\delta^\alpha(U) \subseteq U$.*

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$. Since $\sigma^{\theta}(U) = U$, for all $\theta \in \mathbb{N}$ and $U \in \mathrm{MinSpec}(R)$, we have that $\sigma_i(U) = U$ for all $1 \le i \le n$. Using Lemma 2.6 of Bhat [Bha09], we have that $\delta_i(U) \subset U$ for all $1 \le i \le n$. Again, using an iterative argument we can conclude that $\delta^{\alpha_i}(U) \subseteq U$ for each $1 \le i \le n$, even more, the same iterative argument proves that $\delta^{\alpha}(U) \subseteq U$, which is what we wanted to prove. □

In what follows, we denote

$$UA := \left\{ \sum_{i=0}^{n} u_i X_i \mid u_i \in U \text{ for every } 0 \le i \le n \right\},$$

where A is a skew PBW extension of a ring R and $U \subseteq R$.

Now, we present some results which extend the theorems formulated in [Bha10a]. The total order introduced in Definition 2.4 is used in what follows.

We start with the following theorem which generalizes [Bha10a], Theorem 2.6.

THEOREM 2.23. *Let R be a reduced right Noetherian ring which is also an algebra over $\mathbb{Q}$. If $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ is a family of automorphisms such that R is a weak Σ-rigid ring and $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ is a bijective skew PBW extension over R, then $P \in \mathrm{Ass}(A_A)$ if and only if there exists $U \in \mathrm{Ass}(R_R)$ such that $(P \cap R)A = P$ and $P \cap R = U$.*

Proof. From [LR14], Corollary 2.4, we know that A is right Noetherian. Let $P \in \mathrm{Ass}(A_A)$. If A is a right module over a right Noetherian ring, then $\mathrm{Ass}(A_A)$ equals the set of assassinators of uniform submodules of A ([GW04],(5Y)), i.e., $\mathrm{Ass}(A_A) = \mathbb{A}(A_A)$. In this way, there exists an ideal I of A_A such that I is uniform as a right A-module and $P = \mathrm{Ann}(I) = \mathrm{Assas}(I)$. Fixing a monomial order on $\mathrm{Mon}(A)$, consider $0 \neq f = m_1 X_1 + \cdots + m_t X_t$ an element of I of minimal leading monomial $X_t = x^{\alpha_t}$, with $X_t \succ X_{t-1} \succ \cdots \succ X_1$ and $\mathrm{lc}(f) = m_t$. Let $U = \mathrm{Ann}(m_t R) = \mathrm{Assas}(m_t R)$. Since R is right Noetherian, we have $\mathrm{Ass}(R_R) = \mathbb{A}(R_R)$. Since R is a semiprime ring, we know that $U \in \mathrm{MinSpec}(R)$ ([MR01], Proposition (2.2.14)). In consequence, Proposition 2.21 guarantees that $\sigma^{\alpha}(U) = U$ and $\delta^{\alpha}(U) \subseteq U$, for all $\alpha \in \mathbb{N}^n$, whence we conclude that UA is an ideal of A. The idea is to show that $fU = 0$. For this, let $r \in U$. By Proposition 2.3 we have

$$fr = m_0 r + m_1 X_1 r + \cdots + m_t(\sigma^{\alpha_t}(r) x^{\alpha_t} + p_{\alpha_t, r}),$$

where $p_{\alpha_t,r} = 0$ or $\deg(p_{\alpha_t,r}) < |\alpha_t|$ if $p_{\alpha_t,r} \neq 0$. Since $\sigma^{\alpha_t}(r) \in U$, we have $m_t \sigma^{\alpha_t}(r) = 0$, thus $\mathrm{lm}(fr) \prec X_t$, but this implies that $fr = 0$ given that $fr \in I$ and f has minimal leading monomial. Therefore $(fA)U \subseteq (fU)A = 0$, so $U \subseteq \mathrm{Ann}(fU) = \mathrm{Assas}(I) = \mathrm{Ass}(I) = P$, whereby $U \subseteq P \cap R$. On the other hand, it is clear that $P \cap R \subseteq U$. Therefore $U = P \cap R$.

Conversely, let us take $U = \mathrm{Ann}(cR) = \mathrm{Assas}(cR)$ with $c \in R$. Again, $\mathrm{Ass}(R_R) = \mathbb{A}(R_R)$ given that R is Noetherian and semiprime, we have that $U \in \mathrm{MinSpec}(R)$. Proposition 2.21 implies that $\sigma^{\alpha}(U) = U$ and $\delta^{\alpha}(U) \subseteq U$ since R is a weak Σ-rigid ring, so UA is an ideal of A. Our purpose is to prove that $UA = \mathrm{Ann}(chA) = \mathrm{Assas}(chA)$ for all $h \in A$, this would imply that $UA = \mathrm{Assas}(cA)$, i.e., UA is an associated ideal of A. For this, we are going to verify that if $f \in UA$, then $fc = 0$, which implies that $f(chA) = (fc)(hA) = 0$ for all $h \in A$. Let $f = a_0 + a_1 X_1 + \cdots + a_t X_t \in UA$, then

$$\begin{aligned} fc &= (a_0 + a_1 X_1 + \cdots + a_t X_t)c \\ &= a_0 c + a_1 X_1 c + \cdots + a_t X_t c \\ &= a_0 c + (a_1 \sigma^{\alpha_1}(c) x^{\alpha_1} + p_{\alpha_t,c}) + \cdots + (a_t \sigma^{\alpha_t}(c) x^{\alpha_t} + p_{\alpha_t,c}) \end{aligned}$$

Since $\sigma^{\alpha_i}(U) = U$, for all $1 \le i \le t$, there exists $u_i \in U$ such that $\sigma^{\alpha_i}(u_i) = a_i$, for all $1 \le i \le t$, so $a_i \sigma^{\alpha_i}(c) = \sigma^{\alpha_i}(u_i)\sigma^{\alpha_i}(c) = \sigma^{\alpha_i}(u_i c) = 0$, for all $1 \le i \le t$ given that $U = \text{Ann}(cR)$. Analogously, the same argument can be used to conclude that $p_{\alpha_i,c} = 0$, for all $1 \le i \le t$, since $\delta^{\alpha}(U) \subseteq U = \sigma^{\alpha}(U)$, for all $\alpha \in \mathbb{N}^n$. Then, $fc = 0$ and therefore $UA \subseteq \text{Assas}(cA)$. Now, since R is a reduced weak Σ-rigid ring, then R is a Σ-rigid ring (Proposition 2.20), but A is a bijective skew PBW extension of R, Proposition 2.19 allows us to say that A is a reduced ring, thus A is a semiprime ring and by [MR01], Proposition (2.2.14), $\text{Ann}(cA)$ is a minimal prime ideal. This last fact implies that $\text{Assas}(cA) = \text{Ann}(cA) \subseteq UA$. Therefore we conclude that $UA = \text{Assas}(cA)$, i.e., UA is an associated prime ideal of A. □

The following theorem generalizes [Bha10a], Theorem 2.10.

Theorem 2.24. *Let R be a right Noetherian ring which is also an algebra over $\mathbb{Q}$, $\Sigma = \{\sigma_i, \dots, \sigma_n\}$ be a set of automorphisms and $\Delta = \{\delta_1, \dots, \delta_n\}$ be a set of Σ-derivations such that $\sigma_i(\delta_j(a)) = \delta_j(\sigma_i(a))$, for all $a \in R$ and $1 \le i, j \le n$ and $\sigma_i(U) = U$, for all $U \in \mathbb{A}(R_R)$ and every $1 \le i \le n$. Then $P \in \text{Ass}(A_A)$ if and only if there exists $U \in \text{Ass}(R_R)$ such that $P = UA$ and $P \cap R = U$.*

Proof. We know that A is a right Noetherian ring by [LR14], Corollary 2.4. Let $J \in \text{Ass}(A_A)$, if A is a right module over a right Noetherian ring, then $\text{Ass}(A_A)$ equals the set of assassinators of uniform submodules of A ([GW04],(5Y)), i.e., $\text{Ass}(A_A) = \mathbb{A}(A_A)$. Let $P = \text{Ann}(I) = \text{Assas}(I)$ for some ideal I of A such that I is uniform as a right A-module. Again, we can choose $f \in I$ an element of minimal leading monomial. Following the same line of reasoning used in the proof of the Theorem 2, let us take $U = \text{Ann}(a_n R) = \text{Assas}(a_n R)$. By hipothesis, $\sigma^{\alpha}(U) = U$, so by Proposition 2.3, $\delta^{\alpha}(U) \subseteq U$, which implies that UA is an ideal of U. Again, it can be seen that $fAU = 0$ and therefore $P \cap R = U$.

Conversely let $U = \text{Ann}(cR) = \text{Assas}(cR)$, $c \in R$. Again, since R is a Noetherian ring, we know that $\text{Ass}(\mathrm{R_R}) = \mathbb{A}(R_R)$. Our hypothesis is that $\sigma_i(U) = U$, for all $1 \le i \le n$, so by Proposition 2.3 we have that $\delta^{\alpha}(U) \subseteq U$, for each $\alpha \in \mathbb{N}^n$. It can be seen that $UA = \text{Ann}(chA)$, for all $h \in A$. Thus $UA = \text{Assas}(cA)$, i.e., UA is an associated ideal of A. □

Remark 15. Theorem 2.24 can be llustrated with remarkable examples of PBW extensions defined by Bell and Goodearl which satisfy the conditions of this theorem.

Conclusions and future work

In carrying out this work, we have found that the results in commutative algebra on the associated prime ideals have no examples in the literature. We did a search for this and, surprisingly, neither in the main works as those of [BH74] and [Fai00] nor in the works that use their results do we find illustrative examples of this theory. It is a pending task to find examples in the commutative and noncommutative case that illustrate the results obtained in this work since the characterization of the associated prime ideals of the polynomial module is not a simple task.

Now, as a possible future work concerning associated primes over skew PBW extensions, we have in mind the work developed by Nordstrom in [Nor05] and [Nor12] where he characterized the associated primes ideals over skew Laurent polynomial extension $R[x;x^{-1};\sigma]$ and the family of noncommutative rings of generalized Weyl algebras. Also, having in mind the work realized by Leroy and Matczuk in [LM04] where they consider the Goldie dimension of Ore extensions with the aim of studying associated primes over these extensions, and since in [Rey14] Reyes computed this dimension for skew PBW extensions, we think that a natural task is to generalize the results established in [LM04], [Nor05] and [Nor12] to the context of skew PBW extensions.

Last, but not least, the weak compatible rings studied recently in [RS20] as a direct generalization the of compatible rings, are an object of interest in the study of different ring and homological properties of noncommutative rings of polynomial type.

Bibliography

[AM69] M. F. Atiyah and I. G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Company, 1969. III

[Ann02a] S. Annin. *Associated and Attached Primes Over Noncommutative Rings*. PhD thesis, University of California, Berkeley, 2002. III, 6, 8, 11, 13

[Ann02b] S. Annin. Associated primes over skew polynomial rings. *Comm. Algebra*, 30(5):2511–2528, 2002. III

[Ann04] S. Annin. Associated primes over Ore extension rings. *J. Algebra Appl.*, 3(2):193–205, 2004. 0, III, IV, 1, 8, 9, 10, 11, 12, 21, 26, 27, 28, 29, 30, 32, 33

[BG88] A. Bell and K. Goodearl. Uniform rank over differential operator rings and Poincaré-Birkhoff-Witt extensions. *Pacific J. Math.*, 131(1):13–37, 1988. III, 21

[BGTV03] J. Bueso, J. Gómez-Torrecillas, and A. Verschoren. *Algorithmic Methods in non-commutative Algebra: Applications to Quantum Groups*. Dordrecht, Kluwer, 2003. 24, 25

[BH74] J. W. Brewer and W. J. Heinzer. Associated primes of principal ideals. *Duke Math. J.*, 41(1):1–7, 1974. 0, III, 1, 3, 4, 41

[Bha07] V. K. Bhat. Polynomial rings over pseudovaluation rings. *Int. J. Math. Math. Sci.*, 2007. 16

[Bha08] V. K. Bhat. Associated prime ideals of skew polynomial rings. *Beitr. Algebra Geom.*, 49(1):277–283, 2008. III, 15

[Bha09] V. K. Bhat. Transparent rings and their extensions. *New York J. Math.*, 15(291-299), 2009. 38, 39

[Bha10a] V. K. Bhat. Associated prime ideals of weak σ-rigid rings and their extensions. *Algebra Discrete Math.*, 10(1):8–17, 2010. 0, III, IV, 1, 13, 15, 16, 17, 18, 19, 20, 21, 38, 39, 40

[Bha10b] V. K. Bhat. Ore extensions over weak σ-rigid rings and $\sigma(*)$-rings. *Eur. J. Pure Appl. Math.*, 3(4):695–703, 2010. 13

[BHL93] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee. Completely prime ideals and associated radicals. *Ring Theory, eds. S. K. Jain and S. T. Rizvi, World Scientific,Singapore*, pages 102–129, 1993. 14

[Bou72] N. Bourbaki. *Elements of Mathematics Commutative Algebra.* Springer, 1972. 1

[BW93] T. Becker and V. Weispfenning. *Gröbner Bases. A Computational Approach to Commutative Algebra.* 141. Graduate Texts in Mathematics, Springer-Verlag, 1993. 24

[Fai00] C. Faith. Associated primes in commutative polynomial rings. *Comm. Algebra*, 28(8):3983–3986, 2000. III, 1, 3, 4, 6, 41

[Gal15] C. Gallego. *Matrix methods for projective modules over σ-PBW extensions.* PhD thesis, Universidad Nacional de Colombia, Bogotá, 2015. 36

[GL11] C. Gallego and O. Lezama. Gröbner bases for ideals of σ-PBW extensions. *Comm. Algebra*, 39(1):50–75, 2011. 0, III, 21, 22, 23, 24, 25

[GP08] G.M. Greuel and G. Pfister. *A Singular Introduction to Commutative Algebra.* Springer-Verlag Berlin Heidelberg, Second edition, 2008. 1

[GW04] K. R. Goodearl and R. B. Warfield. *An Introduction to Noncommutative Noetherian Rings.* Cambridge University Press. London, 2004. III, 1, 2, 5, 17, 39, 40

[HKA17] E. Hashemi, K. Khalilnezhad, and A. Alhevaz. (Σ,Δ)-compatible skew PBW extension ring. *Kyungpook Math. J.*, 57(3):401–417, 2017. 0, IV, 26, 37

[HKK00] C. Y. Hong, N. K. Kim, and T. K. Kwak. Ore extensions of Baer and p.p.-rings. *J. Pure Appl. Algebra*, 151(3):215–226, 2000. 13, 14

[HKK03] C. Y. Hong, N. K. Kim, and T. K. Kwak. On skew Armendariz rings. *Comm. Algebra*, 31(1):103–122, 2003. 14

[JR18] J. Jaramillo and A. Reyes. Symmetry and reversibility properties for quantum algebras and skew Poincaré-Birkhoff-Witt extensions. *Ingeniería y Ciencia*, 14(27):29–52, 2018. 26

[Kre96] J. Krempa. Some examples of reduced rings. *Algebra Colloq.*, 3(4):289–300, 1996. 13, 14, 37

[Lam98] T. Y. Lam. *Lectures on Modules and Rings, Graduate Texts in Mathematics Vol. 189.* Springer-Verlag, Berlin, 1998. 1, 2

[Lam01] T. Y. Lam. *A First Course in Noncommutative Rings. Graduate Texts in Mathematics Vol. 131.* Springer, New York, NY, 2001. 8

[LAR15] O. Lezama, J. P. Acosta, and A. Reyes. Prime ideals of skew PBW extensions. *Rev. Un. Mat. Argentina*, 56(2):39–55, 2015. III, 21, 22

[Laz69] D. Lazard. Autour de la platitude. *Bull. Soc. Math. France*, 97:81–128, 1969. 3

[Lez19] O. Lezama. *Cuadernos de Álgebra, No. 9: Álgebra no conmutativa.* SAC^2, Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia, 2019. https://sites.google.com/a/unal.edu.co/sac2/. 5

[LG16] O. Lezama and C. Gallego. d-Hermite rings and skew PBW extensions. *São Paulo J. Math. Sci.*, 10(1):60–72, 2016. 21

[LL17] O. Lezama and E. Latorre. Non-commutative algebraic geometry of semi-graded rings. *Internat. J. Algebra Comput.*, 27(4):361–389, 2017. 21

[LM04] A. Leroy and J. Matczuk. On induced modules over Ore extensions. *Comm. Algebra*, 32(7):2743–2766, 2004. 8, 41

[LR14] O. Lezama and A. Reyes. Some homological properties of skew PBW extensions. *Comm. Algebra*, 42(3):1200–1230, 2014. 5, 6, 21, 35, 37, 38, 39, 40

[LV17] O. Lezama and H. Venegas. Some homological properties of skew PBW extensions arising in non-commutative algebraic geometry. *Discuss. Math. Gen. Algebra Appl.*, 37(1):45–57, 2017. 21

[Mar97] G. Marks. Direct product and power series formations over 2-primal rings. *Advances in Ring Theory*, pages 239–245, 1997. 14

[MR01] J. McConnell and J. Robson. *Noncommutative Noetherian Rings*, volume 30 of *Graduate Studies in Mathematics.* American Mathematical Society, Second edition, 2001. 4, 5, 14, 39, 40

[Nor05] H. Nordstrom. Associated primes over Ore extensions. *J. Algebra*, 286(1):69–75, 2005. 41

[Nor12] H. Nordstrom. Simple modules over generalized Weyl algebras and their associated primes. *Comm. Algebra*, 40(9):3224–3235, 2012. III, 41

[NR17] A. Niño and A. Reyes. Some ring theoretical properties of skew Poincaré-Birkhoff-Witt extensions. *Bol. Mat.*, 24(2):131–148, 2017. 36, 37

[NR20] A. Niño and A. Reyes. Some remarks about minimal prime ideals of skew Poincaré-Birkhoff-Witt extensions. *Algebra Discrete Math.*, 2020. To appear. III

[Ore33] O. Ore. Theory of non-commutative polynomials. *Ann. of Math. Second Series*, 34(3):480–508, 1933. III, 4

[Ouy08] L. Ouyang. Extensions of generalized α-rigid rings. *Int. Electron. J. Algebra*, 3:103–116, 2008. III, 14, 37

[Rey14] A. Reyes. Uniform dimension over skew PBW extensions. *Rev. Colombiana Mat.*, 48(1):79–96, 2014. 41

[Rey15] A. Reyes. Skew PBW extensions of Baer, quasi-Baer, p.p. and p.q.-rings. *Rev. Integr. Temas Mat.*, 33(2):173–189, 2015. IV, 23, 36, 37

[Rey19] A. Reyes. Armendariz modules over skew PBW extensions. *Comm. Algebra*, 47(3):1248–1270, 2019. 24, 26, 27

[Ros95] A. Rosenberg. *Non-commutative Algebraic Geometry and Representations of Quantized Algebras*. Math. Appl. (Soviet Ser.), 330 Kluwer Academic Publishers, 1995. 4

[RR19] A. Reyes and C. Rodríguez. The McCoy condition on skew Poincaré-Birkhoff-Witt extensions. *Commun. Math. Stat.*, 2019. https://doi.org/10.1007/s40304-019-00184-5. 26, 37

[RS16] A. Reyes and H. Suárez. A note on zip and reversible skew PBW extensions. *Bol. Mat.*, 23(1):71–79, 2016. 26, 36, 37

[RS17a] A. Reyes and H. Suárez. Bases for quantum algebras and skew Poincaré-Birkhoff-Witt extensions. *Momento*, 54(1):54–75, 2017. 35, 38

[RS17b] A. Reyes and H. Suárez. Enveloping algebra and skew Calabi-Yau Algebras over skew Poincaré-Birkhoff-Witt extensions. *Far East J. Math. Sci.*, 102(2):373–397, 2017. 21, 36, 37

[RS17c] A. Reyes and H. Suárez. PBW bases for some 3-dimensional skew polynomial algebras. *Far East J. Math. Sci. (FJMS)*, 101(6):1207–1228, 2017. 21, 35, 36

[RS18a] A. Reyes and H. Suárez. A notion of compatibility for Armendariz and Baer properties over skew PBW extensions. *Rev. Un. Mat. Argentina*, 59(1):157–178, 2018. 0, IV, 26, 37

[RS18b] A. Reyes and H. Suárez. Skew Poincaré-Birkhoff-Witt extensions over weak Σ-rigid rings. *Far East J. Math. Sci.*, 106(2):421–440, 2018. 0, III, IV, 37

[RS18c] A. Reyes and Y. Suárez. On the ACCP in skew Poincaré-Birkhoff-Witt extensions. *Beitr. Algebra Geom.*, 59(4):625–643, 2018. III, 37

[RS19a] A. Reyes and H. Suárez. Radicals and Köthe's conjecture for skew PBW extensions. *Commun. Math. Stat.*, 2019. https://doi.org/10.1007/s40304-019-00189-0. III, 21, 26, 37

[RS19b] A. Reyes and H. Suárez. Skew Poincaré-Birkhoff-Witt extensions over weak zip rings. *Beitr. Algebra Geom.*, 60(2):197–216, 2019. 26, 36

[RS20] A. Reyes and H. Suárez. Skew Poincaré-Birkhoff-Witt extensions over weak compatible rings. *J. Algebra Appl.*, 2020. https://doi.org/10.1142/S0219498820502254. 41

[Sch72] R. Schock. Polynomial rings over finite dimensional rings. *Pacific J. Math*, 42(1):251–257, 1972. 4

[SLR15] H. Suárez, O. Lezama, and A. Reyes. Some relations between N-Koszul, Artin-Schelter regular and Calabi-Yau algebras with skew PBW extensions. *Revista Ciencia en Desarrollo*, 6(2):205–213, 2015. 21

[SLR17] H. Suárez, O. Lezama, and A. Reyes. Calabi-Yau property for graded skew PBW extensions. *Rev. Colombiana Mat.*, 51(2):221–238, 2017. 21