Some aspects of the obstacle problem
Algunos aspectos del problema de obstáculo

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Algunos aspectos del problema de obstáculo

Resumen

Esta tesis de maestría estudia algunos aspectos del problema de obstáculo: existencia y unicidad de la solución usando desigualdades variacionales; regularidad de la solución utilizando el método de penalización; un resultado de regularidad de la frontera libre de Kinderlehrer y Nirenberg; y la solución de un ejemplo detallado del problema de obstáculo en una dimensión.

Palabras Clave: Problema de obstáculo, desigualdades variacionales, regularidad, frontera libre.

Some aspects of the obstacle problem

Abstract

The aim of this work is to study some aspects of the obstacle problem: existence and uniqueness of a solution using variational inequalities; regularity of the solution using the penalization method; a regularity result for the free boundary by Kinderlehrer and Nirenberg; and a detailed solution of a one-dimensional example of the obstacle problem.

Keywords: Obstacle problem, variational inequality, regularity, free boundary.

Contents

Introduction

The obstacle problem is, roughly speaking, a mathematical problem motivated in Physics which consists in minimizing an energy functional on a closed convex set in a Hilbert space. In order to more precisely state the problem, let us assume $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $\psi \in C^2(\overline{\Omega})$, and $\psi < 0$ en $\partial\Omega$. Let us consider a functional of the form

$$J(u) = \int_\Omega (|\nabla u|^2 - 2fu)\, dx, \qquad (1)$$

where $f \in L^\infty(\Omega)$ and u lies in the subset of the Sobolev space $W_0^{1,2}(\Omega)$ defined by

$$\mathbb{K} := \{v \in W_0^{1,2}(\Omega) : v(x) \geq \psi(x) \text{ a.e. } x \in \Omega\}. \qquad (2)$$

The obstacle problem, in this setting, consists in minimizing J over $\mathbb{K}$, i.e. finding $u \in \mathbb{K}$ such that $J(u) = \min_{v\in\mathbb{K}} J(v)$. Because of the form $\mathbb{K}$ has, function ψ is refered to as an *obstacle*. From the physical point of view this problem consists of studying the equilibrium position u of an elastic membrane that lies above and touches the obstacle ψ (see figure 1 and see chapter 1 in [19]).

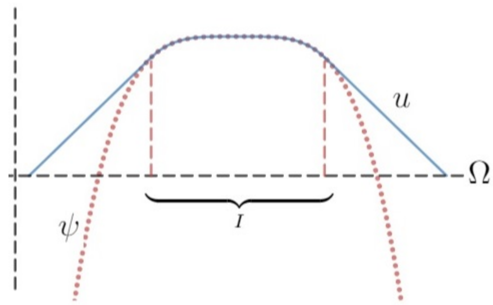

Figure 1: The obstacle problem.

For a continuous function $u \in \mathbb{K}$ such that $J(u) = \min_{v\in\mathbb{K}} J(v)$, the set

$$I_u := \{x \in \Omega : \; u(x) = \psi(x)\}, \qquad (3)$$

is called *coincidence set* or *contact set*. Under some conditions (to be specified below) u satisfies a differential equation in the sub-domain $\{u > \psi\} = \Omega \setminus I$, which is not known a priori. So, u has two regimes or phases in the prescribed domain Ω: one in $\Omega \setminus I$ given by a differential equation, and other in I, given by ψ. In this sense, it is of interest the study of the set $\Gamma_u := \partial I_u$, which represents the interfase between the two regimes. Set Γ is referred to as the *free boundary of the* *obstacle problem.*

In general, those types of problems that involve the solution of a differential equation in a domain that is not known a priori, are called *free boundary problems*, and they appear in situations that include the melting of ice in water, incompressible or compressible flow in porous media, flame propagation, optimal stopping problems, the study of American options, and of course the obstacle problem (see, e.g., chapters VII and VIII in [13]).

The subject of this thesis has been studied by many authors and there are several generalizations of the problem, including Signorini problem (see chapter 9 in [19]). Some of the key known results concerning the obstacle problem, informally summarized, are the following:

1) Existence of solutions: using variational methods it is possible to show that $u \in \mathbb{K}$ actually exists and is unique.

2) Regularity of the solution: Frehse showed in [9] that u is as regular as ψ up to $C^{1,1}$, in particular if ψ is smooth then u has continuous Lipschitz derivatives. This is the best possible regularity that can be expected, since even in the one dimensional case it is possible to construct solutions with discontinuous second derivatives.

3) Regularity of the free boundary: Caffarelli showed in [4] that the free boundary is locally a $C^{1,\alpha}-$manifold of dimension $N-1$ around certain regular points. He also studied in [4] the structure of the singular set.

4) Higher order regularity of the free boundary: Kinderlehrer and Nirenberg showed in [12] that if the free boundary is locally $C^{1,\alpha}$ around a point x then Γ is locally a $C^{\infty}-$manifold of dimension $N-1$ around x.

In this document we focus in aspects 1), 2) and 4). As mentioned above, these are well-known results, but their proofs in the references are mostly sketchy. The main contributions of this document consist, first, in developing a more or less self-contained summary of several results in one piece; and, second, presenting detailed arguments and proofs, as well as including some remarks that could help a beginner to better understand some ideas. Specifically, we include complete details (which are not fully developed in the references) in the proofs of the following results: Proposition 1.2.9, Theorem 2.3.12, Lemma 3.1.4, Theorem 3.1.2, Theorem 4.0.2, details in Example 4.0.3, and Theorem 5.2.3. The main references we used as guides are [13], [19], and [12].

The work is organized as follows: in Chapter 1 we include some preliminary results. We separate them in three sections. The first is devoted to variational inequalities, including some motivation and background from functional analysis, in particular Stampacchia's theorem. In the second, we introduce an order in the Sobolev space $W_0^{1,2}(\Omega)$. The last section in Chapter 1 presents some results from abstract measure theory and regularity theory of elliptic equations. In Chapter 2, we begin with a precise statement of the obstacle problem as well as its variational formulation. Then, existence of a unique weak solution is established using the theory from Chapter 1. Many additional qualitative properties of the solution are also presented here. In Chapter 3 we study regularity of the weak solution found in Chapter 2. The most of Chapter 3 is devoted to explain how the variational formulation of the obstacle problem is related to a “limiting” elliptic problem, which, in turn, is studied through a technique of *penalization*. Then, $C^{1,\alpha}-$regularity of the solution is established in detail. In the final part of this chapter we present a summary (with no proofs) of the results leading to the optimal regularity mentioned above. Chapter 4 is devoted to the study of some properties of the solution in the one-dimensional setting. This chapter includes a explicit example.

Finally, in Chapter 5 we study, up to some detail, the regularity result for the free boundary by Kinderlehrer and Nirenberg mentioned in 4) above. This result is proved essentially in two steps: the first consists, basically, in using a technical device (the Legendre transform) to translate the regularity of the free boundary into the regularity of a solution of a fully nonlinear differential equation. The second step consists in a subtle (non-direct) application of regularity theory for nonlinear elliptic operators.

At the end of the work we have included an appendix mainly regarding some properties of Sobolev Spaces which are used throughout the work.

Chapter 1

Preliminaries

1.1 Variational inequalities

1.1.1 Motivation

The idea of variational inequalities came from the basic problem of minimizing a function over a closed interval. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a smooth function. The problem of minimizing f consists in finding $x_0 \in [a, b]$ such that

$$f(x_0) = \min_{x \in [a,b]} f(x).$$

We know that a possible solution must satisfy one of the following conditions

1. If $x_0 \in (a, b)$, then $f'(x_0) = 0$.
2. If $x_0 = b$ then $f'(x_0) \leq 0$.
3. If $x_0 = a$ then $f'(x_0) \geq 0$.

All of which can be summarized into the inequality

$$f'(x_0)\,(x - x_0) \geq 0, \quad \forall x \in [a, b]. \tag{1.1.1}$$

Solving (1.1.1) would give us a necessary condition for x_0 to be a solution of the minimization problem.

Another example, which will be become more clear after chapter 2, would be to consider a bounded domain $\Omega \subseteq \mathbb{R}^N$ and a function $\psi : \overline{\Omega} \longrightarrow \mathbb{R}$ such that $\max_\Omega \psi \geq 0$ and $\psi \leq 0$ on $\partial\Omega$. If we define $\mathbb{K} := \{v \in C^1\left(\overline{\Omega}\right) : \ v \geq \psi \text{ in } \Omega \text{ and } v|_{\partial\Omega} = 0\}$, the problem of minimizing the functional

$$J : C^1(\overline{\Omega}) \longrightarrow \mathbb{R}, \quad J(v) = \int_\Omega |\nabla v|^2 \, dx \tag{1.1.2}$$

over $\mathbb{K}$, can be translated into a variational inequality as follows: If a solution u of the minimizing problem exists and $v \in \mathbb{K}$, by setting $\phi(t) = \int_\Omega |\nabla\left(u + t(v - u)\right)|^2 \, dx$, for $0 \leq t \leq 1$, we get that

ϕ attains a minimum at $t = 0$ which implies that $\phi'(0) \geq 0$ by the previous argument. All of the previous leave us with

$$u \in \mathbb{K} \text{ and } \phi'(0) = \int_\Omega \nabla u \cdot \nabla (v - u)\, dx \geq 0 \quad \forall v \in \mathbb{K}. \tag{1.1.3}$$

Problem (1.1.2) is known as the obstacle problem as we will precisely introduce in chapter 2, and inequality (1.1.3) is known as the variational formulation of the obstacle problem.

1.1.2 Variational inequalities in finite dimensional spaces

We will denote by $(\cdot,\cdot)$ the inner product on $\mathbb{R}^N$. We start by recalling some important results that will be helpful through the study of this section (see, e.g., [14] and [17]).

Definition 1.1.1. Let S be a metric space. Let $F : S \longrightarrow S$ be a function. We say that F is a contraction mapping if there exists $\alpha \in [0, 1)$: $\operatorname{dist}(F(x), F(y)) \leq \alpha \operatorname{dist}(x, y)$, for all $x, y \in S$.

Theorem 1.1.2 (Banach Fixed point). *Let S be a complete metric space and assume $F : S \longrightarrow S$ is a contraction mapping. Then there exists a unique point $s \in S$ such that $F(s) = s$.*

Theorem 1.1.3 (Brouwer Fixed Point). *Let $B_R \subseteq \mathbb{R}^N$ be an open ball and let $F : \overline{B_R} \longrightarrow \overline{B_R}$ be a continuous mapping. Then F admits a fixed point.*

This theorem can be extended to any compact and convex set of $\mathbb{R}^N$. But first we need to define the projection onto a convex set. Since this can be done in the general setting of (finite or infinite dimensional) Hilbert spaces, and will be helpful in the next subsection (1.1.3), we proceed in this way.

Lemma 1.1.4. *Let $\mathbb{K} \subseteq H$ be a non-empty closed convex subset of the real Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. Then for each $x \in H$ there is a unique $y \in \mathbb{K}$ such that*

$$\|x - y\|_H = \inf_{z\in\mathbb{K}} \|x - z\|_H .$$

The point y is called the projection of x on $\mathbb{K}$, and it is denoted by $y = P_{\mathbb{K}}x$.

The proof of this lemma can be found in [13]. Lemma 1.1.4 can also be translated in the form of an inequality.

Theorem 1.1.5 (Characterization of the projection). *Let $\mathbb{K} \subseteq H$ be a non-empty closed convex subset of the real Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. Let $x \in H$. Then $y = P_{\mathbb{K}}x$ if and only if*

$$y \in \mathbb{K} \text{ and } \langle y, \eta - y\rangle_H \geq \langle x, \eta - y\rangle_H \quad \forall \eta \in \mathbb{K}. \tag{1.1.4}$$

Proof. Let $x \in H$, and set $y = P_{\mathbb{K}}x$. Given that $\mathbb{K}$ is convex, for any $\eta \in \mathbb{K}$ and $0 \leq t \leq 1$, $y + t(\eta - y) \in \mathbb{K}$. The function $\phi(t) = \|x - y - t(\eta - y)\|_H^2$, defined on $[0, 1]$, attains its minimum at $t = 0$. Therefore $\phi'(0) = -2\langle x-y, \eta - y\rangle_H \geq 0$, and we got (1.1.4). On the other hand, if the inequality (1.1.4) is satisfied then for any $\eta \in \mathbb{K}$,

$$\|x-y\|_H^2 = \langle x-y, x-\eta\rangle_H + \underbrace{\langle x-y, \eta-y\rangle_H}_{\leq 0}$$

$$\leq \langle x-y, x-\eta\rangle_H\,,$$

then by Cauchy-Schwarz inequality $\|x-y\|_H \leq \|x-\eta\|_H$.

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An immediate corollary that will allows us to extend Brouwer fixed point theorem to any compact and convex set from $\mathbb{R}^N$ is that the projection operator is non-expansive, in particular a continuous operator.

Corollary 1.1.5.1. *Let $\mathbb{K} \subseteq H$ be a non-empty closed convex subset of the real Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. Then $P_{\mathbb{K}}$ is non-expansive, i.e. $\|P_{\mathbb{K}}x - P_{\mathbb{K}}z\| \leq \|x-z\|$ for all $x, z \in H$.*

Proof. Let $y = P_{\mathbb{K}}x$ and $w = P_{\mathbb{K}}z$. By the previous characterization

$$\langle x, w-y\rangle_H \leq \langle y, w-y\rangle_H\,,$$

$$\langle z, y-w\rangle_H \leq \langle w, y-w\rangle_H\,.$$

Adding up we obtain $\langle x-z, w-y\rangle_H \leq \langle y-w, w-y\rangle_H$. Rearranging terms and using Cauchy-Scharwz inequality yields $\|y-w\| \leq \|x-z\|$ proving the result.

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Theorem 1.1.6. *Let $\mathbb{K} \subseteq \mathbb{R}^N$ be compact and convex. Let $F : \mathbb{K} \longrightarrow \mathbb{K}$ be a continuous mapping. Then F admits a fixed point.*

Proof. Let $\overline{B}_R(0)$ be a closed ball containing $\mathbb{K}$. The function $F \circ P_{\mathbb{K}} : \overline{B}_R(0) \longrightarrow \mathbb{K} \subseteq \overline{B}_R(0)$ is a continuous mapping from $\overline{B}_R(0)$ into itself. From Brouwer's theorem there exists $x \in \overline{B}_R(0)$ with $x = F \circ P_{\mathbb{K}}(x) = F(x) \in \mathbb{K}$. ☑

Theorem 1.1.7. *Let $\mathbb{K} \subseteq \mathbb{R}^N$ be compact and convex. Let $F : \mathbb{K} \longrightarrow (\mathbb{R}^N)'$ be a continuous mapping. Then there exists an $x \in \mathbb{K}$ such that*

$$\langle F(x), \eta - x\rangle \geq 0 \quad \forall \eta \in \mathbb{K}. \tag{1.1.5}$$

Proof. Let π denote the map given by the Riesz Representation Theorem from $(\mathbb{R}^N)' \longrightarrow \mathbb{R}^N$, so that $\langle F(x), \eta - x\rangle = (\pi F(x), \eta - x)$. Note that

$$(\pi F(x), \eta - x) \geq 0 \quad \forall \eta \in \mathbb{K} \iff$$

$$(x, \eta - x) \geq (x - \pi F(x), \eta - x) \quad \forall \eta \in \mathbb{K}. \qquad (*)$$

The map $P_{\mathbb{K}}(I-\pi F):\mathbb{K}\longrightarrow\mathbb{K}$ is continuous. By Theorem 1.1.6 it admits a fixed point $x\in\mathbb{K}$. Such a point satisfies $P_{\mathbb{K}}(x-\pi F(x))=x$. By the characterization of the projection in $\mathbb{R}^N$ (Theorem 1.1.5), x satisfies $(*)$.

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Finally, let us see one way in which Theorem 1.1.7 can help us to solve an equation of the form $F(x)=0$. This illustrates the usefulness of the variational inequality techniques.

Corollary 1.1.7.1. *Let x be a solution of problem (1.1.5) and suppose further that $x\in\operatorname{int}(\mathbb{K})$. Then $F(x)=0$.*

Proof. Because $x\in\operatorname{int}(\mathbb{K})$, there exists $r>0$ such that $B_r(x)\subseteq\mathbb{K}$. By taking $\eta=x+\alpha e_i$, with $\alpha\in(-r,r)$ and $i\in\{1,\cdots,N\}$, we have that

$$\langle F(x),\eta-x\rangle=\alpha\langle F(x),e_i\rangle\geq 0\quad\forall\alpha\in(-r,r).$$

Therefore $\langle F(x),e_i\rangle=0$ for all $i=1,\cdots,N$.

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1.1.3 Variational inequalities in Hilbert spaces

Many problems from the field of mathematical optimization can be viewed as "optimal" inequality problems. By optimal we mean that such inequalities characterize the optimization problems as in Theorem 1.1.9 below.

Let $(H,\langle\cdot,\cdot\rangle_H)_H$ be a real Hilbert space. Let us recall that a *bilinear form* a on H is a function $a(\cdot,\cdot):H\times H\longrightarrow\mathbb{R}$ which is continuous and componentwise linear, i.e. for each $x\in H$ mappings $a(x,\cdot)$ and $a(\cdot,x)$ are linear. We also say that a is *coercive* on H if there exists a constant $\alpha>0$, such that

$$a(v,v)\geq\alpha\,\|v\|_H^2\quad\forall v\in H.\tag{1.1.6}$$

Notice that if a is a bilinear form on H, the mapping $u\overset{A}{\mapsto}a(u,\cdot)$ determines a continuous linear transformation $A:H\longrightarrow H'$. Equivalently, for any continuous linear transformation $A:H\to H'$, we can obtain a bilinear form $a(u,v)=\langle Au,v\rangle_H$, $u,v\in H$. Proposition 1.1.8 (see below), shows that the coercivity of a as in (1.1.6) is equivalent to that of A as a linear operator, i.e.

$$\lim_{\|u\|_H\to\infty}\frac{\langle Au-Av,u-v\rangle_H}{\|u-v\|_H}=\infty\quad\text{for all } v\in H.\tag{1.1.7}$$

Proposition 1.1.8. *Let $a(\cdot,\cdot):H\times H\longrightarrow\mathbb{R}$ be a continuous bilinear form. Let $A:H\longrightarrow H'$ be defined by $\langle Au,\cdot\rangle_H\equiv a(u,\cdot)$ for all $u\in H$. Then, a is coercive on H if and only if A is coercive as a linear operator.*

Proof.
Let u, $v\in H$. Suppose a is coercive on H. By (1.1.6), we have that $\langle Au-Av,u-v\rangle_H=$

$a(u-v,u-v) \geq \alpha \|u-v\|_H^2$. The latter implies that the limit in (1.1.7) holds, proving the coercivity of A as a linear operator. Now, suppose that A is coercive as a linear operator. By contradiction, assume that for every $\alpha > 0$ there exists $z \in S := \{x \in H : \|x\|_H = 1\}$ such that $a(z,z) < \alpha$. Then, for each $n \in \mathbb{N}$, there exists $z_n \in S$ such that $a(z_n, z_n) < 1/n$. Set $u_n = nz_n$ and $v = 0$. By (1.1.7), we have that

$$\lim_{n\to\infty} \frac{\langle Au_n, u_n\rangle_H}{||u_n||_H} = \lim_{n\to\infty} \frac{a\,(nz_n, nz_n)}{n||z_n||_H} = \lim_{n\to\infty} na\,(z_n, z_n) = \infty.$$

The latter contradicts the fact that $a(z_n, z_n) < 1/n$ for all $n \in \mathbb{N}$. Thus, there must be an $\alpha > 0$ such that (1.1.6) holds true.

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The equivalence given by Proposition 1.1.8 will be very helpful to understand the definition of coercivity for an operator that is not necessarily linear (in the following subsection see Definition 1.1.17).

Finally, let us recall that if a is a bilinear form on H, then a is called symmetric when $a(u,v) = a(v,u)$ for all $u, v \in H$.

Given a non-empty closed and convex subset, $\mathbb{K} \subseteq H$, for $f \in H'$, we aim to solve the following problem: find $u \in \mathbb{K}$ such that

$$a(u, v-u) \geq \langle f, v-u\rangle \quad \forall v \in H. \tag{1.1.8}$$

This kind of problems are usually referred to as a variational inequalities. The existence and uniqueness of a solution for (1.1.8) is guaranteed by the well-known Stampacchia's theorem.

Theorem 1.1.9 (Stampacchia's Theorem). *Let $a(\cdot,\cdot) : H \times H \longrightarrow \mathbb{R}$ be a continuous coercive bilinear form on the real Hilbert space H. Let $f \in H'$ and let $\mathbb{K} \subseteq H$ be a non-empty, closed and convex subset from H. Then, there exists a unique solution $u \in \mathbb{K}$ to problem (1.1.8).*

Additionally,

(i) The mapping $f \mapsto u$, where u is the unique solution to problem (1.1.8) is Lipschitz:

$$\|u_2 - u_1\|_H \leq \frac{1}{\alpha} \|f_2 - f_1\|_{H'}$$

where $f_1, f_2 \in H'$, and $u_1,$, u_2 are the corresponding solutions, and α is the constant from (1.1.6).

(ii) If $a(\cdot,\cdot)$ is symmetric, then the solution to problem (1.1.8) is characterized by the following two properties

$$u \in \mathbb{K} \ \text{ and } \ \frac{1}{2}a(u,u) - \langle f, u\rangle = \min_{v\in\mathbb{K}} \left\{\frac{1}{2}a(v,v) - \langle f, v\rangle\right\}.$$

The proof of this theorem is standard from several functional analysis books, see for example [3].

We will make use of Stampacchia's theorem to find a solution of the variational form of the obstacle problem (1.1.3).

1.1.4 Variational inequalities for monotone operators

Throughout this subsection, X will denote a reflexive real Banach space with topological dual space X', $\mathbb{K} \subset X$ will denote a non-empty closed convex subset of X and $L : \mathbb{K} \longrightarrow X'$ will denote a given mapping (L could be a non-linear mapping).

Definition 1.1.10. We say that $L : \mathbb{K} \longrightarrow X'$ is called *monotone* if

$$\langle Lu - Lv, u - v\rangle \geq 0 \quad \forall u, v \in \mathbb{K},$$

where $\langle f, v\rangle$ (with $f \in X'$, $v \in X$) denotes the evaluation of functionals in X' on elements on X . If, additionally, L satisfies

$$\langle Lu - Lv, u - v\rangle = 0 \text{ implies } u = v,$$

then L is called *strictly monotone*.

In the following example we will make use of the sobolev space $H_0^1(\Omega)$ (see appendix 6).

Example 1.1.11. Let $X = H_0^1(\Omega)$ and let $\mathbb{K}$ be the closed convex subset given by $\mathbb{K} = \{v \in H_0^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega\}$. Then the operator $L : \mathbb{K} \longrightarrow X'$ given by

$$\langle L\phi, \psi\rangle = \int_\Omega \nabla\phi \cdot \nabla\psi \, dx, \;\; \psi \in H_0^1(\Omega),$$

is a well-defined monotone operator as we know show: by Cauchy-Schwarz inequality, for each $\phi \in \mathbb{K}$, $\langle L\phi, \cdot\rangle$ is a continuous linear operator from $H_0^1(\Omega)$ to $\mathbb{R}$. Also, it is monotone since for $\phi, \psi \in \mathbb{K}$ we get

$$\langle L\phi - L\psi, \phi - \psi\rangle = \int_\Omega \nabla(\phi - \psi) \cdot \nabla(\phi - \psi) \, dx = \|\phi - \psi\|^2_{H_0^1(\Omega)} \geq 0.$$

Notice that the choosing of the set $\mathbb{K}$ was not really relevant for this example, however it will be important in next sections.

We also define a notion of weak continuity for this type of mappings as follows.

Definition 1.1.12. We say that $L : \mathbb{K} \to X'$ is *continuous on finite dimensional subspaces*, if for any linear subspace M of X, with $dim(M) < \infty$, the restriction of L to $\mathbb{K} \cap M$ is weakly continuous,[1] i.e.

$$\forall \{u_n\}_n \subseteq \mathbb{K} \cap M : \; u_n \overset{w}{\rightharpoonup} u \implies Lu_n \overset{w}{\rightharpoonup} Lu.$$

Or in simple words, the restriction of L to $\mathbb{K} \cap M$ is weakly continuous, if weak convergence in $\mathbb{K} \cap M$ implies weak convergence in X'.

[1] This definition of weak continuity was taken from [8].

Remark 1.1.13. Two observations regarding the previous definition are relevant: first, since X is a reflexive Banach space, then $f_n \overset{w}{\rightharpoonup} f$ in X' is equivalent to $\langle f_n - f, x\rangle \to 0$ for all $x \in X$. Second, note that if M is a linear subspace of X, by virtue of Hahn-Banach theorem, for each $\{x_n\} \subseteq M$ and each $x \in M$, one has that $x_n \overset{w}{\rightharpoonup} x$ in M if and only if $x_n \overset{w}{\rightharpoonup} x$ in X.

With all this in mind, we aim to prove a version of Theorem 1.1.7 in infinite dimensional spaces.

Theorem 1.1.14. *Let $\mathbb{K}$ be a non-empty closed bounded and convex subset of X and let $L : \mathbb{K} \to X'$ be a monotone operator which is continuous on finite dimensional subspaces. Then there exists an $u \in \mathbb{K}$ such that $\langle Lu, v - u\rangle \geq 0$ for all $v \in \mathbb{K}$. If, in addition, L is strictly monotone such a solution u is unique.*

Before proving the theorem, we will prove the following lemma, which says that we can eliminate one of the dependencies of u in the expression $\langle Lu, v - u\rangle \geq 0$ for all $v \in \mathbb{K}$.

Lemma 1.1.15 (Minty's Lemma). *Let $\mathbb{K}$ be a non-empty closed and convex subset of X and let $L : \mathbb{K} \to X'$ be a monotone operator which is continuous on finite dimensional subspaces. The following two are equivalent:*

(i) $u \in \mathbb{K}$ *such that* $\langle Lu, v - u\rangle \geq 0 \ \forall v \in \mathbb{K}$.

(ii) $u \in \mathbb{K}$ *such that* $\langle Lv, v - u\rangle \geq 0 \ \forall v \in \mathbb{K}$.

Proof. (i) $\implies$ (ii): By monotonicity $\langle Lv - Lu, v - u\rangle = \langle Lv, v - u\rangle - \langle Lu, v - u\rangle \geq 0$.

(ii) $\implies$ (i): Let $w \in \mathbb{K}$. For $0 \leq t \leq 1$ set $v = u + t(w - u) \in \mathbb{K}$. By (ii) for any $t > 0$ we have that $\langle L(u + t(w - u)), t(w - u)\rangle \geq 0$, and so

$$\langle L(u + t(w - u)), (w - u)\rangle \geq 0.$$

On $\mathbb{K} \cap \mathrm{span}\{u, w\}$, L is weakly continuous. Let $\{t_n\} \subseteq \mathbb{R}^+$ be a sequence such that $t_n \to 0$ as $n \to \infty$, then $u + t_n(w - u) \to u$ in $\mathbb{K}$ as $n \to \infty$. Therefore $L(u + t_n(w - u)) \rightharpoonup Lu$ in X' as $n \to \infty$. In particular, $\langle Lu, w - u\rangle \geq 0$.

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We can now proceed with the proof of the theorem.

Proof of Theorem 1.1.14. First we offer a small sketch of the proof. For $v \in \mathbb{K}$, define $S(v) = \{u \in \mathbb{K} : \langle Lv, v - u\rangle \geq 0\}$. For each $v \in \mathbb{K}$, we will show that $S(v)$ is non-empty, weakly closed and bounded. Therefore, since X is a reflexive Banach space, $S(v)$ will be weakly compact. Finally, we will prove that there exists $u \in \cap_{v\in\mathbb{K}} S(v)$. By Minty's Lemma u is the solution we are looking for.

In order to show that $S(v)$ is weakly closed we will show that it is convex and (strongly) closed in X. Clearly it is convex, since for $u, w \in S(v)$ we have that for any $0 \leq t \leq 1$,

$$\langle Lv, v - (tw + (1 - t)u)\rangle = t\,\langle Lv, v - w\rangle + (1 - t)\,\langle Lv, v - u\rangle \geq 0.$$

Now we prove that $S(v)$ is closed in X. Let $\{u_n\}_n \subseteq S(v)$ be such that $u_n \to u$ in X. Since $\{u_n\}_n \subseteq S(v)$, then $\langle Lv, v - u_n \rangle \geq 0$ for all $n \in \mathbb{N}$. By continuity of $Lv \in X'$, we can take the limit on both sides to get $\langle Lv, v - u \rangle \geq 0$. Therefore, $u \in S(v)$ and $S(v)$ is closed in X. We can conclude that $S(v)$ is weakly closed. Since $\mathbb{K}$ is weakly closed and bounded in the reflexive Banach space X, we have that it is weakly compact. Then, since $S(v)$ is weakly closed and it is contained in $\mathbb{K}$, we conclude that $S(v)$ is weakly compact as well.

Since $\mathbb{K} \neq \emptyset$, we take $v_0 \in \mathbb{K}$. The set $S(v_0)$ is compact in the weak topology of X. In order to show that $\cap_{v\in\mathbb{K}} S(v) \neq \emptyset$ first we prove that the family $\{S(v)\}_{v\in\mathbb{K}}$ has the finite intersection property with respect to $S(v_0)$, i.e. any finite sub-collection has non-empty intersection with $S(v_0)$. Beyond the notation we are going to introduce, this property is a consequence of Theorem 1.1.7. Indeed, let $\{v_1, \cdots, v_m\} \subseteq \mathbb{K}$ and consider $M = span\{v_1, \cdots, v_m\}$. Set $\mathbb{K}_M \coloneqq \mathbb{K} \cap M$. Define $j : M \longrightarrow X$ as the injection map and let $j' : X' \longrightarrow M'$ be its dual, i.e. $j'(f) = f|_M$ for $f \in X'$. The pairing between M' and M, $\langle \cdot, \cdot \rangle_{M',M}$ is defined as $\langle j'f, x \rangle_{M',M} = \langle f, jx \rangle$, $x \in M$, $f \in X'$.

If we consider the mapping $j'Lj : \mathbb{K}_M \longrightarrow M'$, the set $\mathbb{K}_M$ is convex and compact subset of M, since it is closed and bounded and M is finite dimensional. By hypothesis the map $j'Lj$ is continuous from K_M to M' since in finite dimensional spaces weak convergence is equivalent to strong convergence. Therefore, by Theorem 1.1.7 there exists $u_M \in \mathbb{K}_M$ such that

$$\langle j'Lju_M, v - u_M \rangle \geq 0 \quad \forall v \in \mathbb{K}_M.$$

In particular, for each $i = 1, \cdots, m$ we have that $u_M \in S(v_i)$.

Finally, we prove that $\cap_{v\in\mathbb{K}} S(v) \neq \emptyset$. The family $\{S(v)\}_{v\in\mathbb{K}}$, of weakly closed subsets of $\mathbb{K}$, satisfes the finite intersection property with respect to $S(v_0)$. The compactness of $S(v_0)$ in the weak topology of X and the finite intersection property of $\{S(v)\}_{v\in\mathbb{K}}$ imply that $\cap_{v\in\mathbb{K}} S(v) \neq \emptyset$: otherwise the complements of the sets $\{S(v) \cap S(v_0)\}_{v\in\mathbb{K}}$ would form an open cover of $S(v_0)$ that has no finite sub-cover (see Theorem 26.9 of [16]).

For uniqueness we assume further that L is strictly monotone. Assume u_1 and u_2 are elements in $\mathbb{K}$ such that $\langle Lu_i, v - u_i \rangle \geq 0$ for all $v \in \mathbb{K}$, $i = 1, 2$. In particular,

$$\langle Lu_1, u_1 - u_2 \rangle \leq 0 \text{ and } \langle Lu_2, u_1 - u_2 \rangle \geq 0.$$

After subtracting we obtain

$$\langle Lu_1 - Lu_2, u_1 - u_2 \rangle = \langle Lu_1, u_1 - u_2 \rangle - \langle Lu_2, u_1 - u_2 \rangle \leq 0$$

Therefore, $\langle Lu_1 - Lu_2, u_1 - u_2 \rangle = 0$ and by strict monotonicity $u_1 = u_2$.

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We aim to avoid the hypothesis of boundedness over $\mathbb{K}$ in Theorem 1.1.14, this will have a cost as we will see. The first step to do this is to prove the following characterization.

Theorem 1.1.16. *Let $\mathbb{K}$ be a non-empty closed and convex subset of X and let $L : \mathbb{K} \longrightarrow X'$ be a monotone operator which is continuous on finite dimensional subspaces. For any $R > 0$ set $\mathbb{K}_R = \mathbb{K} \cap B_R(0)$. The following are equivalent:*

(i) There exists an $u \in \mathbb{K}$ such that $\langle Lu, v - u\rangle \geq 0\ \forall v \in \mathbb{K}$.

(ii) There are $R > 0$ and $u_R \in \mathbb{K}_R$ such that $\langle Lu_R, v - u_R\rangle \geq 0\ \forall v \in \mathbb{K}_R$.

Proof. $(i) \implies (ii)$: Assuming (i) and taking $R = \|u\| + 1$, then assertion (ii) is true for $u_R = u$. $(ii) \implies (i)$: Let $R > 0$ and $u_R \in \mathbb{K}_R$ such that

$$\langle Lu_R, v - u_R\rangle \geq 0\ \forall v \in \mathbb{K}_R. \quad (*)$$

We want to find $u \in \mathbb{K}$ satisfying $\langle Lu, v - u\rangle \geq 0\ \forall v \in \mathbb{K}$. For $w \in \mathbb{K}$ and $0 < \varepsilon < 1$ we set $v_\varepsilon = u_R + \varepsilon(w - u_R) \in \mathbb{K}$. It follows that

$$\langle Lu_R, v_\varepsilon - u_R\rangle = \varepsilon \langle Lu_R, w - u_R\rangle .$$

In view of the above, $\langle Lu_R, w - u_R\rangle \geq 0$ as long as $\langle Lu_R, v_\varepsilon - u_R\rangle \geq 0$. By virtue of $(*)$, the latter inequality is true as long as $v_\varepsilon \in \mathbb{K}_R$. Which is true whenever $0 < \varepsilon < \frac{R - \|u_R\|_X}{\|w - u_R\|_X}$.

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We want to prove a similar result to that of Theorem 1.1.14 without the hypothesis of boundedness on $\mathbb{K}$, the cost is an extra hypothesis on L. Motivated by the equivalence (1.1.6)-(1.1.7) (see subsection 1.1.3) we introduce the following definition.

Definition 1.1.17. We say that L is *coercive* on $\mathbb{K}$ if there exists $\varphi \in \mathbb{K}$ such that

$$\lim_{\substack{\|u\|_X \to \infty \\ u \in \mathbb{K}}} \frac{\langle Lu - L\varphi, u - \varphi\rangle}{\|u - \varphi\|_X} = \infty.$$

The last result of this subsection is the following.

Corollary 1.1.17.1. *Let $\mathbb{K}$ be a non-empty closed and convex subset of X and let $L : \mathbb{K} \to X'$ be a monotone operator which is coercive and continuous on finite dimensional subspaces. Then there exists an $u \in \mathbb{K}$ such that $\langle Lu, v - u\rangle \geq 0\ \forall v \in \mathbb{K}$.*

Proof.

We will establish the existence of u by proving (ii) in Theorem 1.1.16. The coercivity of L implies that there exists an $u_0 \in \mathbb{K}$ such that for any $h > 0$, there exists $R > 0$ such that

$$\langle Lu - Lu_0, u - u_0\rangle \geq h \|u - u_0\|_X , \text{ for all } u \in \mathbb{K} \text{ satisfying } \|u\|_X \geq R.$$

Using the continuity of $Lu_0 \in X'$ and the latter inequality we obtain that

$$\langle Lu, u - u_0\rangle \geq (h - \|Lu_0\|_{X'}) \|u - u_0\|_X , \text{ for all } u \in \mathbb{K} \text{ satisfying } \|u\|_X \geq R.$$

Notice that in the previous inequalities R can be enlarged as much as we want. By choosing $h > \|Lu_0\|_{X'}$ and $R > \|u_0\|_X$, we get that

$$\langle Lu, u-u_0\rangle > 0, \text{ for all } u\in\mathbb{K} \text{ satisfying } \|u\|_X \geq R. \quad (*)$$

Finally, since $\mathbb{K}\cap \overline{B}_R(0)$ is closed, convex and bounded, Theorem 1.1.14 implies that there exists $u_R\in\mathbb{K}\cap\overline{B}_R(0)$ such that

$$\langle Lu_R, v-u_R\rangle \geq 0 \;\forall v\in\mathbb{K}\cap\overline{B}_R(0). \qquad (**)$$

In fact, let us see that $u_R\in\mathbb{K}_R=\mathbb{K}\cap B_R(0)$. By contradiction, if $u_R\notin\mathbb{K}_R$ we would have that $\|u_R\|=R$ and by $(*)$

$$\langle Lu_R, u_0-u_R\rangle < 0.$$

Contradicting $(**)$, which is particularly true for $v=u_0$ since $\|u_0\|_X<R$ and $u_0\in\mathbb{K}_R$. Finally, Theorem 1.1.16 guarantees the existence of the solution we needed.

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1.2 An order in $H^1(\Omega)$

Throughout this section, $\Omega\subseteq\mathbb{R}^N$ will denote a smooth bounded open connected set. Also, we will make use of different functional spaces. For a definition and a summary of their properties see Appendix 6.

In order to introduce the obstacle problem in section 2.2 of chapter 2, in which (as we will see) the obstacle function is in $H^1(\Omega)$ and satisfies $\psi\leq 0$ on $\partial\Omega$, first we need to precise what it means that the function ψ satisfies an inequality on a set whose measure could be zero as $\partial\Omega$.

Roughly speaking, the following definition will say that $u\in H^1(\Omega)$ will be non-negative on a given set $E\subseteq\overline{\Omega}$, if it can be approximated by functions in $H^1(\Omega)$ for which the pointwise inequality makes sense.

Definition 1.2.1. Let $u\in H^1(\Omega)$ and let $E\subseteq\overline{\Omega}$. The function u is said to be *non-negative on E in the sense of* $H^1(\Omega)$, if there exists a sequence $\{u_n\}_n\subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that

$$u_n\to u \text{ in } H^1(\Omega), \text{ and for all } n\in\mathbb{N},\ x\in E:\ u_n(x)\geq 0.$$

We will abbreviate the expression "u is non-negative on E in the sense of $H^1(\Omega)$" by writing that $u\geq 0$ on E in $H^1(\Omega)$. Analogously, when $-u\geq 0$ we say that $u\leq 0$ on E in $H^1(\Omega)$. If both things happen, $u\geq 0$ and $u\leq 0$ on E in $H^1(\Omega)$, we say that $u=0$ on E in $H^1(\Omega)$.

Remark 1.2.2. The fact that $C^{0,1}\left(\overline{\Omega}\right)\subseteq H^1(\Omega)$ is proved, for instance, in chapter II, section 4 of [13].

Remark 1.2.3. Notice that if $u\in H^1_0(\Omega)$, then by definition there exists a sequence of functions $\{u_n\}_n\subseteq C_0^\infty(\Omega)$, such that $u_n\to u$ in $H^1(\Omega)$. In particular, $u=0$ on $\partial\Omega$ in $H^1(\Omega)$.

Observe that, for a given $E\subseteq\Omega$, the set $P=\left\{u\in H^1(\Omega):\ u\geq 0 \text{ on } E \text{ in } H^1(\Omega)\right\}$ is a closed convex cone, i.e. P is closed in the topology induced by the norm of $H^1(\Omega)$ and

(P1) If u_1, $u_2 \in P$, then $u_1 + u_2 \in P$.

(P2) If $\alpha > 0$ and $u \in P$, then $\alpha u \in P$.

(P3) $P \cap (-P) = \{u \in H^1(\Omega) : u = 0 \text{ on } E \text{ in } H^1(\Omega)\}$.

The following lemma allows us to prove that in Definition 1.2.1 we can change the convergence in norm $u_n \to u$ in $H^1(\Omega)$ by $u_n \overset{w}{\to} u$ in $H^1(\Omega)$.

Lemma 1.2.4. *Let X be a Banach space. Let $u \in X$ and let $\{u_n\}_n \subseteq X$ be a sequence such that $u_n \overset{w}{\to} u$ in X. Then there exists a sequence $\{y_n\}_n$ of convex combinations of the elements u_n, $n \in \mathbb{N}$, that strongly converges to u in X.*

For a proof of the above lemma see Corollary 3.8 of [3]. Now we explain why in Definition 1.2.1 we can change the convergence in norm $u_n \to u$ in $H^1(\Omega)$ by weak convergence, $u_n \overset{w}{\to} u$ in $H^1(\Omega)$.

Proposition 1.2.5. *Let $u \in H^1(\Omega)$ and let $E \subseteq \overline{\Omega}$. Then, $u \geq 0$ on E in $H^1(\Omega)$ if and only if there exists a sequence $\{u_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that*

$$u_n \overset{w}{\to} u \text{ in } H^1(\Omega) \text{ and for all } n \in \mathbb{N},\ x \in E:\ u_n(x) \geq 0.$$

Proof. First, if $u \geq 0$ on E in $H^1(\Omega)$, then by Definition 1.2.1 there exists a sequence $\{u_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that $u_n \to u$ in $H^1(\Omega)$. Since strong convergence implies weak convergence, then $u_n \overset{w}{\to} u$ in $H^1(\Omega)$ and for all $n \in \mathbb{N}$, $x \in E$: $u_n(x) \geq 0$.

On the other hand, if there exists $\{u_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that $u_n \overset{w}{\to} u$ in $H^1(\Omega)$, then Lemma 1.2.4 allows us to show the existence of another sequence $\{y_n\}_n$ of convex combinations of the elements u_n, $n \in \mathbb{N}$, such that $y_n \to u$ in $H^1(\Omega)$. Given that $C^{0,1}\left(\overline{\Omega}\right)$ is a convex subset of $H^1(\Omega)$, then $\{y_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$. Finally, since for all $n \in \mathbb{N}$, $x \in E$: $u_n(x) \geq 0$, then clearly $y_n(x) \geq 0$.

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Now, a natural question is how do we compare the usual almost everywhere order in $H^1(\Omega)$ with this new notion of order.

Proposition 1.2.6. *Let $u \in H^1(\Omega)$ and let $E \subseteq \overline{\Omega}$.*

(i) If $u \geq 0$ on E in $H^1(\Omega)$, then $u \geq 0$ a.e. $x \in E$.

(ii) If $u \geq 0$ a.e. $x \in \Omega$, then $u \geq 0$ on Ω in $H^1(\Omega)$.

(iii) If $u \in H^1_0(\Omega)$ and $u \geq 0$ a.e. $x \in \Omega$, then there exists a sequence $\{u_n\}_n \subseteq C^{0,1}_0(\Omega)$ such that $u_n \geq 0$ in Ω and $u_n \to u$ in $H^1_0(\Omega)$.

Part (i) in the above proposition tells us that the order in $H^1(\Omega)$ is finer than the almost everywhere order. Together, (i) and (ii), say that in the whole domain Ω both orders are equivalent. Before we prove the proposition let us enunciate a very important lemma about the functions in $H^1(\Omega)$.

Lemma 1.2.7. *Let $u \in H^1(\Omega)$. Then the functions $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ belong to $H^1(\Omega)$, and satisfy*

$$\nabla u^+ = \begin{cases} \nabla u & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\}. \end{cases} \tag{1.2.1}$$

$$\nabla u^- = \begin{cases} -\nabla u & \text{a.e. on } \{u < 0\} \\ 0 & \text{a.e. on } \{u \geq 0\}. \end{cases} \tag{1.2.2}$$

Also,

$$\nabla u = 0 \text{ a.e. on the set } \{u = 0\}. \tag{1.2.3}$$

Additionally, if $u \in H_0^1(\Omega)$, then the functions u^+ and u^- belong to $H_0^1(\Omega)$.

The proof of this lemma requires several results about the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$, results that are not the subject of this work. For a detailed proof see Theorem 4.4 in [6], section 2.2 of [11], and section 5.10 of [5].

Proof of Proposition 1.2.6.

(i) If $u \geq 0$ on E in $H^1(\Omega)$, then there exists a sequence $\{u_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that $u_n(x) \geq 0$ for all $x \in E$ and $u_n \to u$ in $H^1(\Omega)$. By the definition of the norm in (6.0.4), $u_n \to u$ in $L^2(\Omega)$. Hence, there exists a subsequence $\{u_{n_k}\}_k$ such that $u_{n_k}(x) \to u(x)$ a.e. $x \in \Omega$, which implies that $u(x) \geq 0$ a.e $x \in E$.

(ii) $H^1(\Omega)$ can be seen as $\overline{C^{0,1}\left(\overline{\Omega}\right)}$ with respect to the norm (6.0.4) (see page 29 in [13]). Accordingly, let $\{v_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that $v_n \to u$ in $H^1(\Omega)$. Using the same argument as in the proof of (i), we can ask this sequence to converge a.e. $x \in \Omega$ to u. Notice that by hypothesis, $u(x) = u^+(x)$ a.e. $x \in \Omega$. Now, set $u_n = v_n^+$ for $n \in \mathbb{N}$. Given that the function $\max\{\cdot, 0\}$ is a Lipschitz function

$$\|u_n - u\|_{L^2(\Omega)} = \|\max\{v_n, 0\} - \max\{u, 0\}\|_{L^2(\Omega)} \leq \|v_n - u\|_{L^2(\Omega)}.$$

This implies that $u_n \to u$ in $L^2(\Omega)$. By Lemma 1.2.7, $\|u_n\|^2_{H^1(\Omega)} = \|v_n^+\|^2_{H^1(\Omega)} \leq \|v_n\|^2_{H^1(\Omega)}$, the latter being a convergent sequence in $H^1(\Omega)$. The reflexivity of $H^1(\Omega)$ implies that $\{u_n\}_n$ has a weakly convergent subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \xrightarrow{w} \tilde{u}$ in $H^1(\Omega)$, for some $\tilde{u} \in H^1(\Omega)$. Given that weakly convergence in $H^1(\Omega)$ implies weakly convergence in $L^2(\Omega)$, $u_{n_k} \xrightarrow{w} \tilde{u}$ in $L^2(\Omega)$. Since $u_n \to u$ in $L^2(\Omega)$, the subsequence $\{u_{n_k}\}_k$ also converges weakly to u in $L^2(\Omega)$. Thus, $u = \tilde{u} \in H^1(\Omega)$. By Proposition 1.2.5, $u \geq 0$ on Ω in $H^1(\Omega)$.

(iii) Proof of this part follows the same argument as in part (ii) starting with $\{v_n\}_n \subseteq C_0^{0,1}(\Omega)$.

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We end the section by proving two propositions that will be use in the following sections.

Proposition 1.2.8. *Let $f \in H_0^1(\Omega)$ and let $\psi \in H^1(\Omega)$ be such that $\psi \leq 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$. Then, $\max(f, \psi) \in H_0^1(\Omega)$.*

Proof.

First we rewrite the function $\max(f,\psi)$ as $\max(f,\psi) = f + \max(\psi - f, 0)$. In this way, it is enough to show that $\max(\psi - f, 0) = (\psi - f)^+ \in H_0^1(\Omega)$. Given that $\psi \le 0$ on $\partial\Omega$ in $H^1(\Omega)$, there is a sequence $\{\psi_n\}_n \subseteq C^{0,1}\left(\overline{\Omega}\right)$ such that $\psi_n(x) \le 0$, for all $x \in \partial\Omega$, and $\psi_n \to \psi$ in $H^1(\Omega)$. Additionally, let $\{f_n\}_n \subseteq C_0^\infty(\Omega)$ be a sequence such that $f_n \to f$ in $H_0^1(\Omega)$. Notice that, for every $n \in \mathbb{N}$, $(\psi_n - f_n)^+(x) = 0$ for $x \in \partial\Omega$ (extending f_n to be boundary $\partial\Omega$ as 0), and $(\psi_n - f_n)^+ \in C\left(\overline{\Omega}\right)$. Therefore $(\psi_n - f_n)^+ \in H_0^1(\Omega)$ (see Theorem 2.2.6 in [11]). Arguing as in the proof of part (ii) of Proposition 1.2.6, taking $u_n = \psi_n - f_n$, we can conclude that $(\psi_n - f_n)^+ \stackrel{w}{\to} (\psi - f)^+$ in $H^1(\Omega)$. Given that $\{(\psi_n - f_n)^+\}_n \subseteq H_0^1(\Omega)$, and $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, then $(\psi - f)^+ \in H_0^1(\Omega)$.

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Proposition 1.2.9. *Let $f \in H_0^1(\Omega)$ and let $\{\psi_k\}_k \subseteq H^1(\Omega)$ be such that, for each $k \in \mathbb{N}$, $\psi_k \le 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$. Let $\psi \in H^1(\Omega)$ be such that $\psi_k \longrightarrow \psi$ in $H^1(\Omega)$. Then $\max(f,\psi_k) \longrightarrow \max(f,\psi)$ in $H^1(\Omega)$ and $\max(f,\psi) \in H_0^1(\Omega)$.*

Proof.

By Proposition 1.2.8, $\max(f,\psi_k) \in H_0^1(\Omega)$ for each $k \in \mathbb{N}$. Given that $\max(f,\psi_k) = f + (\psi_k - f)^+ \in H_0^1(\Omega)$, and the latter is a closed subspace of $H^1(\Omega)$, it is enough to prove that $(\psi_k - f)^+ \to (\psi - f)^+$ in $H^1(\Omega)$. To simplify the proof we set $\varphi_k = \psi_k - f$, for each $k \in \mathbb{N}$, and set $\varphi = \psi - f$. So we have to show that $\varphi_k^+ \to \varphi^+$ in $H^1(\Omega)$. The function $\max\{\cdot, 0\}$ is a Lipschitz function, then

$$\begin{aligned}\left\|\varphi_k^+ - \varphi^+\right\|_{L^2(\Omega)} &= \|\max\{\psi_k - f, 0\} - \max\{\psi - f, 0\}\|_{L^2(\Omega)} \\ &\le \|\psi_k - \psi\|_{L^2(\Omega)}.\end{aligned}$$

The fact that $\psi_k \to \psi$ in $H^1(\Omega)$ implies that $\varphi_k^+ \to \varphi^+$ in $L^2(\Omega)$. Now, we will show that $\nabla\varphi_k^+ \to \nabla\varphi^+$ in $L^2(\Omega)$, which concludes the proof. For each $k \in \mathbb{N}$

$$\int_\Omega \left|\nabla\varphi_k^+ - \nabla\varphi^+\right|^2 \, dx = \int_\Omega \left|\nabla\varphi_k - \nabla\varphi^+\right|^2 \chi_{\{\varphi_k > 0\}} \, dx + \int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi_k \le 0\}} \, dx. \tag{1.2.4}$$

Let us see why the above integrals converge to 0. First, we prove that the second integral on the right-hand side of (1.2.4) converges to 0. We will make use of the monotone convergence theorem. For any $0 < \delta \le 1$, we have

$$\int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi_k \le 0\}} \, dx = \int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi_k \le 0\} \cap \{\varphi \le \delta\}} \, dx + \int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi_k \le 0\} \cap \{\varphi > \delta\}} \, dx. \tag{1.2.5}$$

The first integral of the right-hand side of (1.2.5) is bounded by above by

$$\int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi \le \delta\}} \, dx.$$

For a.e. $x \in \Omega$, $\chi_{\{\varphi\leq\delta\}}(x)$ monotonically decreases to $\chi_{\{\varphi\leq 0\}}(x)$ as $\delta \to 0^+$. By the monotone convergence theorem (applied to $\chi_{\{\varphi\leq 1\}} - \chi_{\{\varphi\leq\delta\}}$), as $\delta \to 0^+$

$$\int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi\leq\delta\}}\, dx \to \int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi\leq 0\}}\, dx = 0. \quad (*)$$

On the other hand, let us show that for any $\delta > 0$, the second integral of the right-hand side of (1.2.5) converges to 0 as $k \to \infty$. Let $\delta > 0$. Let $\{\varphi_{k_l}\}_l$ be a subsequence of $\{\varphi_k\}_k$. Since $\varphi_{k_l} \to \varphi$ in $H^1(\Omega)$, the subsequence $\{\varphi_{k_l}\}_l$ has a subsequence $\{\varphi_{k_{l_m}}\}_m$ such that $\varphi_{k_{l_m}} \to \varphi$ a.e. in Ω (see Theorem 4.9 in [3]). For $x \in \Omega$ there exists $m_x \in \mathbb{N}$ such that $\varphi_{k_{l_m}}(x) - \varphi(x) > -\delta/2$, for all $m \geq m_x$. Therefore,

$$\chi_{\{\varphi_{k_{l_m}}\leq 0\}\cap\{\varphi>\delta\}}(x) = 0 \quad \text{for all } m \geq m_x.$$

The latter implies, by dominated convergence theorem,

$$\int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi_{k_{l_m}}\leq 0\}\cap\{\varphi>\delta\}}\, dx \to 0 \quad \text{as } m \to \infty.$$

Which in fact proves that for any $\delta > 0$, the second integral of the right-hand side of (1.2.5) converges to 0 as $k \to \infty$.

We can now prove that the integral on the left-hand side of (1.2.5) converges to 0. Let $\varepsilon > 0$, by $(*)$ there exists $\delta > 0$ such that the first integral on the right-hand side of (1.2.5) is bounded from above by $\varepsilon/2$. For such a $\delta > 0$, there exists $k_0 \in \mathbb{K}$ such that for all $k \geq k_0$, the second integral on the right hand side of (1.2.5) is bounded from above by $\varepsilon/2$. Thus,

$$\int_\Omega \left|\nabla\varphi^+\right|^2 \chi_{\{\varphi_k\leq 0\}}\, dx \leq \varepsilon \quad \text{for all } k \geq k_0,$$

proving our first claim. Similarly, to prove that the first integral on the right-hand side of (1.2.4) converges to 0, we split the integral into

$$\int_\Omega \left|\nabla\varphi_k - \nabla\varphi^+\right|^2 \chi_{\{\varphi_k>0\}}\, dx = \int_\Omega \left|\nabla\left(\varphi_k - \varphi\right)\right|^2 \chi_{\{\varphi_k>0\}\cap\{\varphi>0\}}\, dx + \int_\Omega \left|\nabla\varphi_k\right|^2 \chi_{\{\varphi_k>0\}\cap\{\varphi\leq 0\}}\, dx. \tag{1.2.6}$$

The first integral on the right-hand side of (1.2.6) is bounded from above by $\|\psi_k - \psi\|^2_{H^1(\Omega)}$, which converges to 0 by hypothesis. For the second integral on the right-hand side of (1.2.6), we reason as above: let $\{\varphi_{k_l}\}_l$ be a subsequence of $\{\varphi_k\}_k$. Since $\varphi_{k_l} \to \varphi$ in $H^1(\Omega)$, the subsequence $\{\varphi_{k_l}\}_l$ has a subsequence $\{\varphi_{k_{l_m}}\}_m$ such that $\varphi_{k_{l_m}} \to \varphi$ a.e. in Ω, and $\nabla\varphi_{k_{l_m}} \to \nabla\varphi$ a.e. in Ω. Additionally, (by Theorem 4.9 in [3]) there exists a function $h \in L^2(\Omega)$ such that $\left|\nabla\varphi_{k_{l_m}}\right| \leq h$ a.e. in Ω, for all $m \in \mathbb{N}$. Then, one can show (using the ideas above) that

$$\int_\Omega \left|\nabla\varphi_{k_{l_m}}\right|^2 \chi_{\{\varphi_{k_{l_m}}>0\}\cap\{\varphi\leq 0\}}\, dx \to 0 \quad \text{as } m \to \infty.$$

From which we can conclude that the first integral on the right-hand side of (1.2.4) converges to 0.

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Throughout the rest of this work, whenever we say that a function is greater than another function in a set of measure zero we will be referring to the order in the sense of $H^1(\Omega)$.

1.3 Results from measure theory and regularity theory

In this section we enunciate a very important result from measure theory and integration, namely, Riesz Representation Theorem for positive functionals. We also prove a generalization of this theorem following the ideas in [6]. Later in subsection 1.3.2, we prove some basic facts about harmonic functions. Also, we enunciate some regularity results from partial differential equations. Throughout this section, $\Omega \subseteq \mathbb{R}^N$ will denote a smooth bounded open connected set. We clarify that most of the results presented in this section are still valid in more general domains. However, these are the assumptions that we will need in order to study the obstacle problem in chapter 2.

1.3.1 Results from measure theory

The results presented in this subsection will be very important in chapter 2. In particular, we will need a generalization of the Riesz Representation Theorem for positive functionals. In order to state this result, we first recall some terminology from measure theory. Let $\mathcal{M}$ be a $\sigma-$algebra in Ω containing all the Borel sets in Ω. Let μ be a positive Borel measure defined on $\mathcal{M}$. We say that μ is a ***Radon measure*** on Ω, if it satisfies the following three properties:

(i) For every compact set $K \subseteq \Omega$, $\mu(K) < \infty$.

(ii) For every $E \in \mathcal{M}$, we have

$$\mu(E) = \inf\{\mu(V) : E \subseteq V, V \text{ open in } \Omega\}.$$

(iii) For every open set $E \subseteq \Omega$, and for every $E \in \mathcal{M}$ such that $\mu(E) < \infty$, it holds

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact in } \Omega\}.$$

Let μ be a Radon measure on Ω. The ***support*** of μ is defined as

$$supp(\mu) := \{x \in \Omega | \ \forall \ U_x \text{ neighborhood of } x \text{ in } \Omega : \mu(U_x) > 0\}.$$

It can be proved that the support of μ can be characterized as the complement in Ω of the union of the collection $\{O \subseteq \Omega : O \text{ is open in } \Omega \text{ and } \mu(O) = 0\}$. As a consequence, it can be proved that for any set $E \in \mathcal{M}$ such that $E \subseteq \Omega \setminus supp(\Omega)$ it holds $\mu(E) = 0$.

The proof of the following theorem can be found in chapter 2 of [20].

Theorem 1.3.1 (Riesz Representation Theorem). *Let $\Lambda : C_0(\Omega) \longrightarrow \mathbb{R}$ be a linear functional. Assume that Λ is positive, i.e. for any $f \in C_0(\Omega)$ such that $f \geq 0$ in Ω it holds $\Lambda(f) \geq 0$. Then there exists a $\sigma-$algebra $\mathcal{M}$ in Ω which contains all Borel sets in Ω, and there exists a unique positive Radon measure μ on $\mathcal{M}$ which represents Λ in the sense that*

$$\Lambda(f) = \int_\Omega f \, d\mu \quad \text{for every } f \in C_0(\Omega).$$

Now following the ideas in [6] (see Theorem 1.39), we prove the following generalization of the previous theorem.

Theorem 1.3.2. *Let* $\Lambda : C_0^\infty(\Omega) \longrightarrow \mathbb{R}$ *be a positive linear functional. Then all the conclusions of the Riesz Representation Theorem (Theorem 1.3.1) are still valid.*

Proof.

We want to extend the linear functional Λ to the larger space $C_0(\Omega)$, while preserving the positivity of Λ. Then we will have the conditions to apply Theorem 1.3.1.

In order to make the extension, we prove the following continuity result about Λ on $C_0^\infty(\Omega)$: let $K \subseteq \Omega$ be a compact set, then

$$\exists\, C = C(K) : \ |\Lambda(f)| \le C \|f\|_{L^\infty(\Omega)} \quad \text{for any } f \in C_0^\infty(\Omega) \text{ such that } supp(f) \subseteq K. \tag{1.3.1}$$

Let $\xi \in C_0^\infty(\Omega)$ be such that $0 \le \xi \le 1$ and $\xi \equiv 1$ on K (the existence of such a function is guaranteed for instance by Lemma 9.3 in [3]). Let $f \in C_0^\infty(\Omega)$ be such that $supp(f) \subseteq K$. Then, $g = \|f\|_{L^\infty(\Omega)} \xi - f$ is such that $g \in C_0^\infty(\Omega)$ and $g \ge 0$ in Ω. Thus,

$$\Lambda(g) = \|f\|_{L^\infty(\Omega)} \Lambda(\xi) - \Lambda(f) \ge 0.$$

Taking $C = \Lambda(\xi)$, we obtain that $\Lambda(f) \le C \|f\|_{L^\infty(\Omega)}$. The previous argument is still valid if we replace f by $-f$, from where we obtain (1.3.1). To show that Λ can be extended to $C_0(\Omega)$, we need to be able to suitably approximate any function in $C_0(\Omega)$ by a sequence of functions in $C_0^\infty(\Omega)$. Let $\varphi \in C_0(\Omega)$, and set $K = supp(\varphi)$. Let $\{\rho_n\}_n$ be a sequence of functions such that for each $n \in \mathbb{N}$

$$\rho_n \in C_0^\infty(\mathbb{R}^N), \quad supp(\rho_n) \subseteq \overline{B}\left(0, \frac{1}{n}\right), \quad \int_{\mathbb{R}^N} \rho_n \, dx = 1, \ \rho_n \ge 0 \text{ on } \mathbb{R}^N$$

(see the definition of mollifiers in chapter 4 of [3]). The sequence given by $\{\varphi_n\}_n = \{\rho_n \star \varphi\}_n$ converges uniformly on compact sets of Ω to φ, as $n \to \infty$ (see Proposition 4.21 in [3]). Set $\varepsilon_0 = \text{dist}(K, \partial\Omega)/2 > 0$. Since (by Proposition 4.18 in [3])

$$supp(\rho_n \star \varphi) \subseteq \overline{supp(\rho_n) + supp(\varphi)} \ \text{ for each } n \in \mathbb{N},$$

then taking $N \in \mathbb{N}$ such that $N > 1/\varepsilon_0$ it holds that for all $n \ge N$ $supp(\varphi_n) \subseteq K_0 := \overline{\overline{B}(0, 1/N) + K} \subset \Omega$. Up to a subsequence, we can ask $\{\varphi_n\}_n$ to satisfy the property that $supp(\varphi_n) \subseteq K_0$ for all $n \in \mathbb{N}$. The conclusion is that there exist a sequence of functions $\{\varphi_n\}_n \subseteq C_0^\infty(\Omega)$ and a compact set $K_0 \subseteq \Omega$ such that $\varphi_n \to \varphi$ uniformly on compact sets of Ω, and $supp(\varphi_n) \subseteq K_0$ for all $n \in \mathbb{N}$. By (1.3.1), there exists $C = C(K_0) \ge 0$ such that

$$|\Lambda(\varphi_n) - \Lambda(\varphi_k)| \le C \|\varphi_n - \varphi_k\|_{L^\infty(\Omega)} \qquad \text{for all } n, k \in \mathbb{N}.$$

The latter inequality proves that $\{\Lambda(\varphi_n)\}_n$ is a Cauchy sequence in $\mathbb{R}$. We define $\Lambda(\varphi)$ as the limit of such a sequence. It remains to prove that $\Lambda(\varphi)$ is well-defined. For $i \in \{1, 2\}$, let $\{\varphi_n^i\}_n \subseteq C_0^\infty(\Omega)$, and let $K^i \subseteq \Omega$ be a compact set such that $\varphi_n^i \to \varphi$ uniformly on compact sets of Ω, and $supp(\varphi_n^i) \subseteq K^i$ for all $n \in \mathbb{N}$. The set $K^1 \cup K^2 \subseteq \Omega$ is compact, then there exists $C = C(K^1 \cup K^2) \ge 0$ such that (1.3.1) is true on $K^1 \cup K^2$, which implies

$$\left|\Lambda(\varphi_n^1) - \Lambda(\varphi_n^2)\right| \le C \left\|\varphi_n^1 - \varphi_n^2\right\|_{L^\infty(\Omega)} \quad \text{for all } n \in \mathbb{N}.$$

Since $\left\|\varphi_n^1 - \varphi_n^2\right\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$, the sequences $\{\Lambda(\varphi_n^1)\}_n$ and $\{\Lambda(\varphi_n^2)\}_n$ have the same limit in $\mathbb{R}$. Therefore, $\Lambda(\varphi)$ is well-defined. Now that we have proved that our extension of Λ to $C_0(\Omega)$ is well-defined, we just have to prove that this extension is still linear and positive on $C_0(\Omega)$. The proof the linearity is straightforward, and we now prove the positivity on $C_0(\Omega)$: let $\varphi \in C_0(\Omega)$ be such that $\varphi \ge 0$ in Ω. By the properties of mollifiers $\{\rho_n\}_n$, $\rho_n \ge 0$ on $\mathbb{R}^N$ for all $n \in \mathbb{N}$. By definition of the convolution it follows that $\varphi_n = \rho_n \star \varphi \ge 0$ in Ω for all $n \in \mathbb{N}$. Therefore, since $\Lambda(\varphi_n) \ge 0$ for all $n \in \mathbb{N}$,

$$\Lambda(\varphi) = \lim_{n\to\infty} \Lambda(\varphi_n) \ge 0.$$

The conclusion of the theorem follows after applying Theorem 1.3.1.

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1.3.2 Results from regularity theory

In this subsection we prove some basic facts about harmonic functions that will be helpful in chapters 3 and 5. Also, we enunciate one important regularity results of elliptic partial differential equations.

The following are standard definitions and results concerning harmonic functions. See for instance [10] and [3].

Definition 1.3.3. Let $F \in L^s(\Omega)$, for a given $s \in [1, \infty]$. Consider the problem

$$\begin{cases} -\Delta v = F & \text{in } \Omega \\ v = 0 & \text{in } \partial\Omega. \end{cases} \tag{1.3.2}$$

We say that a function v is a weak solution of (1.3.2), if $v \in H_0^1(\Omega)$, and

$$\int_\Omega \nabla v(x) \cdot \nabla \varphi(x) \; dx = \int_\Omega F(x)\varphi(x) \; dx \quad \forall \varphi \in C_0^1(\Omega). \tag{1.3.3}$$

The above definition can be extended to functions $F \in L^s(\Omega \times \mathbb{R})$, allowing us to define the concept of weak solution when F itself depends on v. In such a case, we just have to replace F by $F(x, v(x))$ in expressions (1.3.2) and (1.3.3). This idea will be very helpful in Chapter 3.

Definition 1.3.4. Let $u \in C(\Omega)$. We say that u satisfies the mean value property in Ω, if for every $x \in \Omega$ and $r > 0$ such that $B_r(x) \subset\subset \Omega$, one of the following is true

$$(i) \quad u(x) = \frac{1}{N\omega_N r^{N-1}} \int_{\partial B_r(x)} u(y) \; dS,$$

$$(ii) \quad u(x) = \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y) \; dy.$$

Where ω_N denotes the volume of the unit sphere in $\mathbb{R}^N$.

Remark 1.3.5. It can be shown (see, e.g., [10] and [5]) that the integral in the right-hand side of (ii) in the above definition can be obtained from (i) by integrating with respect to r. Also, the integral in the right-hand side of (i) can be obtained from (ii) by differentiating with respect to r. Additionally, using a change of variables one may verify that the above formulas (i)-(ii) are equivalent to the following

$$(i') \quad u(x) = \frac{1}{N\omega_N}\int_{\partial B_1(0)} u(x+rw)\ dS_w,$$

$$(ii') \quad u(x) = \frac{1}{\omega_N}\int_{B_1(0)} u(x+rz)\ dz.$$

Theorem 1.3.6. *Let $u \in C(\Omega)$. Then, u is harmonic in Ω if and only if u satisfies the mean value property in Ω.*

See [5] for a proof of the above theorem.

Theorem 1.3.7 (Harnack Inequality in a Ball). *Let u be a non-negative harmonic function in Ω. Let $x^0 \in \Omega$, and let $R > 0$ be such that $B_R(x^0) \subset\subset \Omega$. Then for any $x \in B_R(x^0)$*

$$u(x^0) \geq \left(\frac{R - \|x - x^0\|}{R}\right)^N u(x).$$

The following proof uses ideas from [5].

Proof.

By Theorem 1.3.6, u satisfies the mean value property in any ball compactly contained in Ω, then if $r = R - \|x - x^0\|$

$$u(x^0) = \frac{1}{w_N R^N}\int_{B_R(x^0)} u(y)\ dy \geq \left(\frac{r}{R}\right)^N \frac{1}{w_N r^N}\int_{B_r(x)} u(y)\ dy = \left(\frac{r}{R}\right)^N u(x).$$

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Finally, we prove a weak version of the well-known maximum principle for harmonic functions.

Theorem 1.3.8 (Weak maximum principle). *Let $f \in L^2(\Omega)$ and let $u \in H_0^1(\Omega)$ be such that*

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\ dx = \int_\Omega f(x)v(x)\ dx \quad \forall v \in H_0^1(\Omega). \tag{1.3.4}$$

If $f \geq 0$ a.e. in Ω then $u \geq 0$ a.e. in Ω.

Proof.

Let $u \in H_0^1(\Omega)$ be such that it satisfies (1.3.4). We split u into the positive and negative parts, $u = u^+ - u^-$. By Lemma 1.2.7, u^+ and u^- also belong to $H_0^1(\Omega)$. By the same lemma, we get that

$$\nabla u^- = \begin{cases} 0 & \text{a.e. on } \{u > 0\} \\ -\nabla u & \text{a.e. on } \{u \leq 0\}. \end{cases} \tag{1.3.5}$$

Equality (1.3.4) and the fact that u^- and f are non-negative a.e. in Ω imply that

$$\int_\Omega \nabla u \cdot \nabla u^- \, dx = \int_\Omega f u^- \, dx \geq 0.$$

Thus, by (1.3.5)

$$\int_\Omega |\nabla u^-|^2 \, dx = -\int_\Omega \nabla u \cdot \nabla u^- \, dx = -\int_\Omega f u^- \, dx \leq 0.$$

The latter implies that u^- is constant in $H_0^1(\Omega)$, i.e. $u^-(x) = 0$ for a.e. $x \in \Omega$. It follows that $u = u^+ \geq 0$ a.e. in Ω.

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The following theorem and its proof can be found in chapter 9 of [10].

Theorem 1.3.9 (L^p-estimate for the Poisson equation). *(i) Let $F \in L^s(\Omega)$, for a given $s \in [2, \infty)$. Then, there exists a unique weak solution of problem (1.3.2), $v \in H_0^1(\Omega)$. Also, $v \in H^{2,s}(\Omega)$ and $-\Delta v = F$ a.e. in Ω.*

(ii) Consider the map $F \mapsto v$, where v is the solution given by (i). Such a map is continuous in the following sense: there exists a constant c independent of F such that

$$\|v\|_{H^{2,s}(\Omega)} \leq c \|F\|_{L^s(\Omega)} \qquad \text{for all } F \in L^s(\Omega). \tag{1.3.6}$$

Remark 1.3.10. For a given $F \in L^s(\Omega)$, the latter theorem gives us the existence of a unique weak solution in $H_0^1(\Omega)$ of problem (1.3.2). Also, it gives us some regularity for the solution, which ends up belonging to $H^{2,s}(\Omega)$. In fact, if in addition $F \in C^\infty(\Omega)$, it turns out that the solution given by such a theorem belongs to $C^\infty(\Omega)$ as well (see Corollary 8.11 in [10]).

Chapter 2

Existence of a solution

In this chapter we begin by describing the obstacle problem (sometimes it will be abbreviated as O.P.) in section 2.1. Next, in section 2.2, we show the existence of a unique solution by means of Stampacchia's Theorem (see Theorem 1.1.9). This approach uses the variational inequalities technique that we presented in Chapter 1. There are other ways of finding a solution to the Obstacle Problem. Besides the approach using variational inequalities, there exists two other ways: one through a method known as «penalization» (which we present in Chapter 3), and other through the technique of sub and super solutions (see, e.g., [21]). In section 2.3, we describe some steps of the latter approach. However, we do not fully present this alternative. Also, at the end of section 2.3, we introduce the coincidence set associated with the solution of the O.P.

2.1 Statement of the Obstacle Problem

Before accurately describing the problem, we recall the origins of the obstacle problem can be found in the following minimization problem

$$\min_{v\in\mathbb{K}} \int_{\Omega} |\nabla v(x)|^2 \; dx, \tag{2.1.1}$$

where $\mathbb{K}$ is a closed convex set of suitable functions (see subsection 1.1.1). Typically, $\mathbb{K}$ is described through a function ψ (called the obstacle) and formally defined as $\mathbb{K} = \{v : v \geq \psi\}$.
When the operator $J(v) = |\nabla v|^2$ is replaced by $J(v) = \sqrt{1 + |\nabla v|^2}$, then the minimization problem (2.1.1) can be thought as the problem of finding the element of minimal surface area lying above the obstacle ψ.

We now precisely define the obstacle problem in its more general setting. Throughout this section, $\Omega \subseteq \mathbb{R}^N$ will denote a smooth bounded open connected set.

Definition 2.1.1. For $1 \leq i, j \leq N$, let $a_{ij} \in L^\infty(\Omega)$. Consider the matrix function $A(\cdot) = [a_{ij}(\cdot)]_{i,j=1}^N$. From now on, we will simply call $A(\cdot)$ a matrix. We say that $A(\cdot)$ satisfies the *ellipticity condition*, if there exists $\Lambda > 0$ such that

$$\frac{1}{\Lambda}|\xi|^2 \le \xi^T A(x)\, \xi \le \Lambda\, |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N \text{ and for a.e. } x \in \Omega. \tag{2.1.2}$$

We recall that the constant Λ in the previous definition is independent of x and ξ. For a matrix $A(\cdot)$ that satisfies the above definition, we define a bilinear form $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ as follows: if $u, v \in H^1(\Omega)$, then

$$a(u,v) = \int_\Omega \nabla v(x)^T A(x)\, \nabla u(x)\, dx. \tag{2.1.3}$$

We show that $a(\cdot,\cdot)$ is a well-defined bilinear form in Proposition 2.2.2 below. Note that in particular, $a(\cdot,\cdot)$ can be restricted to the smaller subspace $H_0^1(\Omega) \times H_0^1(\Omega) \subseteq H^1(\Omega) \times H^1(\Omega)$. In what follows, we will precisely state in each case the domain of definition of $a(\cdot,\cdot)$. Also, from now on $H^{-1}(\Omega)$ will denote the normed dual space of $H_0^1(\Omega)$ (see Appendix 6 for a definition). We now precisely state the obstacle problem.

Definition 2.1.2 (The Obstacle Problem)**.** Let $A(\cdot)$ be a given matrix satisfying the ellipticity condition (as in Definition 2.1.1). Let $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ be the bilinear form given by (2.1.3). Let $\psi \in H^1(\Omega)$ be such that $\psi \le 0$ on $\partial\Omega$ in $H^1(\Omega)$, and let us define

$$\mathbb{K} = \mathbb{K}_\psi = \{v \in H_0^1(\Omega) : v \ge \psi \text{ a.e. in } \Omega\}.$$

Given a functional $f \in H^{-1}(\Omega)$ the *obstacle problem* is to find

$$u \in \mathbb{K} \text{ such that } a(u, v-u) \ge \langle f, v-u \rangle \quad \forall v \in \mathbb{K}. \tag{2.1.4}$$

From now on, unless we specify otherwise, $A(\cdot)$, $a(\cdot,\cdot)$ and ψ will be given and understood as in Definition 2.1.2.

2.2 Existence of a unique solution

We begin stating and proving the main theorem of the chapter.

Theorem 2.2.1 (Existence and uniqueness). *Let $f \in H^{-1}(\Omega)$. The Obstacle Problem (as stated in Definition 2.1.2) has a unique solution $u \in \mathbb{K}$. Additionally, if $A(\cdot)$ is a symmetric matrix, i.e. $a_{ij}(x) = a_{ji}(x)$ for all $i \ne j$ and for a.e. $x \in \Omega$, then u is characterized by the following two properties*

$$u \in \mathbb{K} \quad \text{and} \quad \frac{1}{2}a(u,u) - \langle f, u \rangle = \min_{v \in \mathbb{K}} \left\{ \frac{1}{2}a(v,v) - \langle f, v \rangle \right\}. \tag{2.2.1}$$

In order to prove this theorem, we first prove the following proposition.

Proposition 2.2.2. *Let $A(\cdot)$ be a matrix as in Definition 2.1.1. Let $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ be the bilinear form given by (2.1.3).*

(i) $a(\cdot,\cdot)$ is a continuous bilinear form on $H^1(\Omega) \times H^1(\Omega)$.

(ii) If, in addition, $a(\cdot,\cdot)$ is restricted to the subspace $H_0^1(\Omega) \times H_0^1(\Omega) \subseteq H^1(\Omega) \times H^1(\Omega)$, then $a(\cdot,\cdot)$ is a continuous coercive bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$.

(iii) Further, if $A(\cdot)$ is a symmetric matrix, then $a(\cdot,\cdot)$ is a symmetric form.

Proof.

(i) For $u, v \in H^1(\Omega)$, the integrand in (2.1.3) can be rewritten as

$$\nabla v(x)^T A(x)\ \nabla u(x) = \sum_{i,j=1}^{N} a_{ij}(x) D_i u(x) D_j v(x), \tag{2.2.2}$$

where D_i stands for the weak partial derivative in the i-th direction in $H^1(\Omega)$, for $1 \leq i \leq N$ (see (6.0.5) in the appendix). The right-hand side of (2.2.2) shows that the expression for $a(u, v)$ is a well-defined integral, since for $1 \leq i, j \leq N$ (by Cauchy-Schwarz inequality)

$$\left| \int_\Omega a_{ij}(x) D_i u(x) D_j v(x)\ dx \right| \leq \|a_{ij}\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}. \tag{2.2.3}$$

By properties of matrix multiplication and the linearity of the integral, $a(u, \cdot)$ and $a(\cdot, v)$ are linear functionals on $H^1(\Omega)$. Continuity of $a(\cdot,\cdot)$ on $H^1(\Omega) \times H^1(\Omega)$ immediately follows from (2.2.2) and (2.2.3).

(ii) Let us restrict $a(\cdot,\cdot)$ to the subspace $H_0^1(\Omega) \times H_0^1(\Omega) \subseteq H^1(\Omega) \times H^1(\Omega)$. The fact that $a(\cdot,\cdot)$ is a continuous bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ follows exactly as in the proof of (i). Now, the ellipticity condition (2.1.2) implies that $a(\cdot,\cdot)$ is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$, since for any $u \in H_0^1(\Omega)$

$$a(u, u) = \int_\Omega \nabla u(x)^T A(x)\ \nabla u(x)\ dx \geq \int_\Omega \frac{1}{\Lambda} |\nabla u(x)|^2\ dx = \frac{1}{\Lambda} \|u\|_{H_0^1(\Omega)}^2.$$

(iii) Finally, when $A(\cdot)$ is symmetric, it can be easily seen that $a(\cdot,\cdot)$ is also symmetric.

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We now prove the main theorem of this chapter.

***Proof of Theorem** 2.2.1.*

Let $f \in H^{-1}(\Omega)$. We need to verify the hypotheses of Stampacchia's Theorem (Theorem 1.1.9). First, by part (ii) of Proposition 2.2.2, $a(\cdot,\cdot)$ is a continuous coercive bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. Second, $\mathbb{K}$ is a closed convex subset of $H_0^1(\Omega)$. Also, by Proposition 1.2.8, since $\psi \in H^1(\Omega)$ is such that $\psi \leq 0$ on $\partial\Omega$ in $H^1(\Omega)$, then $\max\{\psi, 0\} \in H_0^1(\Omega)$. The latter, combined with the fact that $\max\{\psi, 0\} \geq \psi$ a.e. in Ω, implies that $\mathbb{K} \neq \emptyset$. Theorem 1.1.9 gives us a unique solution $u \in \mathbb{K}$. Finally, if $A(\cdot)$ is a symmetric matrix, by part (iii) of Proposition 2.2.2, $a(\cdot,\cdot)$ is also symmetric. Then, the characterization of u is a consequence of (ii) in Theorem 1.1.9.

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Remark 2.2.3. When $f = 0$ and $A(x) = I_{N \times N}$, solving the Obstacle Problem (as stated in Definition 2.1.2) gives the solution of problem (2.1.1).

2.3 Properties of the solution

There are other ways of finding a solution to the Obstacle Problem as stated in Definition 2.1.2. Besides the approach using variational inequalities, there exists two other ways: one through a method known as «penalization» (which we present in Chapter 3), and other through the technique of sub and super solutions (see, e.g., [21]). We now describe some steps of the latter approach. However, we do not fully present this alternative. Let us begin with the following remark.

Remark 2.3.1. As a consequence of Proposition 2.2.2, we can define an operator $L : H^1(\Omega) \longrightarrow \left(H^1(\Omega)\right)'$ (the latter space denoting the normed dual space of $H^1(\Omega)$) acting through $a(\cdot,\cdot)$ in the following way: for $u \in H^1(\Omega)$

$$\langle Lu, \cdot\rangle \coloneqq a(u,\cdot) : H^1(\Omega) \longrightarrow \mathbb{R}. \tag{2.3.1}$$

Definition 2.3.2. Let $f \in H^{-1}(\Omega)$ and let L be as in Remark 2.3.1. We say that a function $g \in H^1(\Omega)$ is a supersolution of $L(\cdot) - f$, if it satisfies

$$\langle Lg - f, \varphi\rangle \equiv a(g,\varphi) - \langle f, \varphi\rangle \geq 0, \quad \forall\varphi \in H_0^1(\Omega) \text{ such that } \varphi \geq 0 \text{ a.e. in } \Omega.$$

Remark 2.3.3. In particular, the solution u of the Obstacle Problem (as stated in Definition 2.1.2) given by Theorem 1.1.14 is a supersolution of $L(\cdot) - f$, since for any $\varphi \in H_0^1(\Omega)$ such that $\varphi \geq 0$ a.e. in Ω, we have that $w = u + \varphi \in \mathbb{K}$ and

$$0 \leq a(u, w-u) - \langle f, w-u\rangle = a(u,\varphi) - \langle f,\varphi\rangle = \langle Lu - f, \varphi\rangle .$$

Also, by Remark 1.2.3, $u = 0$ on $\partial\Omega$ in $H^1(\Omega)$.

Now we aim to show that any supersolution g of $L(\cdot) - f$, with the property that $g \geq \psi$ a.e. in Ω and $g \geq 0$ on $\partial\Omega$ in $H^1(\Omega)$, remains above the solution u of the O.P., i.e. $g \geq u$ a.e. in Ω. In view of the above comments, the solution of the O.P. will be the smallest supersolution of $L(\cdot) - f$ among the supersolutions of $L(\cdot) - f$ that are above the obstacle ψ and are greater than or equal to 0 on $\partial\Omega$ in $H^1(\Omega)$.

Theorem 2.3.4. *Let $f \in H^{-1}(\Omega)$. Let u be the solution of the Obstacle Problem (as stated in Definition 2.1.2). Let $g \in H^1(\Omega)$ be a supersolution of $L(\cdot) - f$ satisfying $g \geq \psi$ a.e. in Ω and $g \geq 0$ on $\partial\Omega$ in $H^1(\Omega)$. Then*

$$u \leq g \text{ a.e. in } \Omega.$$

Proof. Set $\phi = \min\{u, g\}$. In order to prove the theorem it is enough to show that $\phi = u$ a.e. in Ω. Since $u \in H_0^1(\Omega)$ and $g \geq 0$ on $\partial\Omega$ in $H^1(\Omega)$, by Proposition 1.2.8, we get that $\phi = -\max\{-u, -g\} \in H_0^1(\Omega)$. Also, since by hypothesis $g \geq \psi$ a.e. in Ω, it follows that $\phi \in \mathbb{K}$. Therefore,

$$a(u, \phi - u) \geq \langle f, \phi - u\rangle . \tag{2.3.2}$$

By definition of ϕ, $\phi - u \leq 0$ a.e. in Ω. Given that g is a supersolution of $L(\cdot) - f$, we get that

$$a(g, \phi - u) \leq \langle f, \phi - u \rangle . \tag{2.3.3}$$

Putting together (2.3.2) and (2.3.3)

$$\begin{aligned}
0 &\geq a(g - u, \phi - u) \\
&= \int_\Omega \nabla(g-u)^T A(x) \nabla(\phi - u) dx \\
&= \int_{\{g<u\}} \nabla(g-u)^T A(x) \nabla(\phi - u)\ dx + \int_{\{g\geq u\}} \nabla(g-u)^T A(x) \nabla(\phi - u)\ dx \\
&= \int_{\{g<u\}} \nabla(g-u)^T A(x) \nabla(\phi - u)\ dx + 0 \quad (\text{By Lemma 1.2.7, } \nabla(\phi - u) = 0 \text{ a.e. in } \{g \geq u\}) \\
&= \int_{\{g<u\}} \nabla(\phi-u)^T A(x) \nabla(\phi - u)\ dx \quad (\text{Since } \phi = g \text{ a.e. in } \{g < u\}) \\
&= a(\phi - u, \phi - u) \quad (\text{By Lemma 1.2.7, } \nabla(\phi - u) = 0 \text{ a.e. in } \{g \geq u\}) \\
&\geq \frac{1}{\Lambda} \|\phi - u\|^2_{H^1_0(\Omega)} \quad (\text{By the coercivity of } a(\cdot,\cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \longrightarrow \mathbb{R}).
\end{aligned}$$

In conclusion $\|\phi - u\|_{H^1_0(\Omega)} = 0$, which implies that $\phi = u$ in Ω, i.e. $u \leq g$ in Ω.

☑

The following corollary asserts that in the case $f = 0$, if the obstacle ψ is bounded from above, then so is the solution of the O.P.

Corollary 2.3.4.1. *Let $f = 0$. Let $\psi \in H^1(\Omega)$ be an obstacle and let u be the solution of the Obstacle Problem (as stated in Definition 2.1.2). If there exists $M > 0$ such that $\psi \leq M$ in Ω, then $u \leq M$ in Ω.*

The proof of the above corollary follows by choosing $g = M$ in Theorem 2.3.4, and by recalling that any constant function $M \in H^1(\Omega)$ is a supersolution of L (given that $\langle LM, \cdot \rangle = 0$).

Remark 2.3.5. Fix $f \in H^{-1}(\Omega)$ and $\psi \in H^1(\Omega)$ as in Definition 2.1.2. At this point, notice that we have been solving the O.P. with null Dirichlet datum on $\partial\Omega$, i.e. asking for a solution $u \in H^1_0(\Omega)$. For any non-null datum $h \in H^1(\Omega)$ satisfying the compatibility condition $h \geq \psi$ on $\partial\Omega$, one can try to solve, using Theorem 1.1.9 (Stampacchia's Theorem), the O.P. over the closed convex set

$$\mathbb{K} = \{v \in H^1(\Omega) : \ v \geq \psi \text{ a.e. in } \Omega, \ v - h \in H^1_0(\Omega)\}.$$

However, observe that the bilinear form $a(\cdot,\cdot)$ (as defined in (2.1.3)) is not coercive in $H^1(\Omega) \times H^1(\Omega)$. Instead, we can proceed as follows: set

$$\mathbb{K}_0 = \left\{\eta \in H^1_0(\Omega) : \ \eta \geq \psi - h \text{ a.e. in } \Omega\right\}.$$

Notice that by hypothesis, $\psi - h \in H^1(\Omega)$ and $\psi - h \leq 0$ on $\partial\Omega$, as required in Definition 2.1.2. Set $\langle F, \cdot\rangle \equiv \langle f, \cdot\rangle - a(h, \cdot) \in H^{-1}(\Omega)$. The solution u for the O.P. over the closed convex set $\mathbb{K}_0$ for the functional F, given by Theorem 2.2.1, satisfies

$$u \in \mathbb{K}_0 \text{ and } a(u, v-u) \geq \langle F, v-u\rangle \quad \text{for all } v \in \mathbb{K}_0. \qquad (*)$$

We claim the function $w = u+h \in H^1(\Omega)$ solves the O.P. over $\mathbb{K}$ (a subset of $H^1(\Omega)$) with non-null datum h, and $f \in H^{-1}(\Omega)$, i.e.

$$w \in \mathbb{K} \;\text{ and }\; a(w, z-w) \geq \langle f, z-w\rangle\,,\; \forall z \in \mathbb{K}.$$

Since $u \in \mathbb{K}_0$, by definition $w - h = u \in H_0^1(\Omega)$, and $u \geq \psi - h$. By the latter inequality, $w \geq \psi$ in Ω. Therefore, $w \in \mathbb{K}$. On the other hand, for any $z \in \mathbb{K}$, $z - h \in \mathbb{K}_0$ (by definition of $\mathbb{K}$) and

$$\begin{aligned}
a(w, z-w) &= a(u, (z-h)-u) + a(h, z-w) \\
&\geq \langle F, (z-h)-u\rangle + a(h, z-w) \qquad (\text{Given that } z-h \in \mathbb{K}_0 \text{ and using } (*)) \\
&= \langle f, z-w\rangle - a(h, z-w) + a(h, z-w) \qquad (\text{By definition } \langle F, \cdot\rangle \equiv \langle f, \cdot\rangle - a(h, \cdot)) \\
&= \langle f, z-w\rangle\,.
\end{aligned}$$

In conclusion, $w \in \mathbb{K}$ and satisfies $a(w, z-w) \geq \langle f, z-w\rangle$ for all $z \in \mathbb{K}$. Now, let us see that such a solution w is unique. In fact, let w_1 and w_2 in $\mathbb{K}$ be such that, for all $z \in \mathbb{K}$

(i) $a(w_1, z-w_1) \geq \langle f, z-w_1\rangle$, and

(ii) $a(w_2, z-w_2) \geq \langle f, z-w_2\rangle$.

After taking $z = w_2$ in (i), $z = w_1$ in (ii), and adding up both equations, we obtain

$$a(w_1 - w_2, w_2 - w_1) \geq 0. \qquad (**)$$

Since w_1 and w_2 belong to $\mathbb{K}$, it holds that $w_1 - h$ and $w_2 - h$ belong to $H_0^1(\Omega)$. In particular, $w_1 - w_2 \in H_0^1(\Omega)$. Inequality $(**)$ implies that $a(w_1 - w_2, w_1 - w_2) \leq 0$. The latter combined with the coercivity of $a(\cdot,\cdot)$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ implies that $\|w_1 - w_2\|^2_{H_0^1(\Omega)} = 0$, i.e $w_1 = w_2$ in $H_0^1(\Omega)$.

The above remark will be helpful in proving that the minimum of any two supersolutions of $L(\cdot) - f$ is still a supersolution of $L(\cdot) - f$.

Theorem 2.3.6. *Let $f \in H^{-1}(\Omega)$ and let L be as in Remark 2.3.1. Let $u, v \in H^1(\Omega)$ be two supersolutions of $L(\cdot) - f$. Then $\min\{u, v\}$ is a supersolution of $L(\cdot) - f$.*

Proof. The function $w = \min\{u, v\}$ can be rewritten as $w = u - \max\{u - v, 0\}$. By Lemma 1.2.7, $w \in H^1(\Omega)$. It remains to prove that $\langle Lw - f, \varphi\rangle \geq 0$ for all $\varphi \in H_0^1(\Omega)$ such that $\varphi \geq 0$ a.e. in Ω. In order to show this, we will prove that w is the solution of a very specific O.P. This fact combined with Remark 2.3.3 will prove our claim.

Set $\mathbb{K} = \{\eta \in H^1(\Omega) : \ \eta \geq w \text{ in } \Omega, \text{ and } \eta - w \in H_0^1(\Omega)\}$. For the given $f \in H^{-1}(\Omega)$, by Remark 2.3.5 (taking $\psi = w$ and $h = w$), there exists a unique $\phi \in \mathbb{K}$ such that

$$a(\phi, \eta - \phi) \geq \langle f, \eta - \phi \rangle, \ \forall \eta \in \mathbb{K}. \quad (*)$$

By the same remark, the function $z = \phi - w$ satisfies the following two conditions

$$z \in \mathbb{K}_0 = \{z \in H_0^1(\Omega) : z \geq 0 \text{ in } \Omega\} \quad \text{and} \quad a(z, \eta - z) \geq \langle F, \eta - z \rangle \ \ \forall \eta \in \mathbb{K}_0,$$

where $\langle F, \cdot \rangle = \langle f, \cdot \rangle - a(w, \cdot)$. By Remark 2.3.3, z is a supersolution of $L(\cdot) - F$. Let $\varphi \in H_0^1(\Omega)$ be such that $\varphi \geq 0$ a.e. in Ω. Since u is a supersolution of $L(\cdot) - f$, then

$$0 \leq a(u - w + w, \varphi) - \langle f, \varphi \rangle = a(u - w, \varphi) - (\langle f, \varphi \rangle - a(w, \varphi)).$$

Therefore, $u - w$ is a supersolution of $L(\cdot) - F$. Analogously, $v - w$ is a supersolution of $L(\cdot) - F$. Now we will use Theorem 2.3.4 to show that in Ω, the function $z = \phi - w$ is bounded from above by $u - w$ and $v - w$. In fact, by definition of w, we have that $u - w \geq 0$ and $v - w \geq 0$ a.e. in Ω. By Theorem 2.3.4, we get that $z \leq u - w$ and $z \leq v - w$ a.e. in Ω, impliying that $\phi \leq u$ and $\phi \leq v$ a.e. in Ω. Thus, $\phi \leq w$ a.e. in Ω. Given that $\phi \in \mathbb{K}$, then $\phi \geq w$ a.e. in Ω, which implies that $\phi = w$ a.e. in Ω. Since $z = 0$ is a supersolution of $L(\cdot) - F$, for $\varphi \in H_0^1(\Omega)$ such that $\varphi \geq 0$ a.e. in Ω, we have

$$0 \leq \langle Lz - F, \varphi \rangle = a(w, \varphi) - \langle f, \varphi \rangle.$$

Thus, w is a supersolution of $L(\cdot) - f$. ☑

Remark 2.3.7. Theorem 2.3.6 resembles a first step in a well-known method to find solutions of partial differential equations called Perron's method. Roughly speaking, the idea is that one can construct a solution of a certain type of differential equations by taking the infimum of supersolutions. See for instance section 2.8 of [10].

Now that we have a solution for the obstacle problem, we are interested in describing some of its properties. Particularly, we would like to study some properties of u on the set in which it coincides with the obstacle ψ. In order to do so, we give the following two definitions.

Definition 2.3.8. Let $u \in H^1(\Omega)$ and let $x \in \Omega$. We say that $u(x) > 0$ in the sense of $H^1(\Omega)$, if there exists a neighborhood of x in Ω, $B_\rho(x) \subseteq \Omega$, and a function $\varphi \in C_0^{0,1}(B_\rho(x))$ such that

(a) $\varphi \geq 0$ in $B_\rho(x)$ and $\varphi(x) > 0$, and

(b) $u - \varphi \geq 0$ on $B_\rho(x)$ in $H^1(\Omega)$.

Observe that at this point, we have several notions of order: the one presented in Definition 1.2.1, the one in Definition 2.3.8 (see the above definition), the usual order in Ω, and the almost everywhere order in Ω. In what follows, we will be very precise what order we will be using in each case.

Remark 2.3.9. (i) We want to show that the function φ in Definition 2.3.8 can be asked to be in $C_0^\infty(\Omega)$. Let $u \in H^1(\Omega)$, and let us define the set

$$A_u := \{x \in \Omega : \ u(x) > 0 \ \text{ in the sense of } H^1(\Omega)\}.$$

Let $x \in A_u$. By definition, there exists $\rho > 0$ and $\varphi \in C_0^{0,1}(B_\rho(x))$ satisfying (a) and (b) in the Definition 2.3.8. Continuity of φ implies that there exists a radius $r \in (0, \rho)$ such that $\varphi|_{\overline{B}_r(x)} > 0$. The latter implies that $\alpha = \min_{\overline{B}_r(x)} \varphi(x) > 0$. There exists a function $\widetilde{\varphi} \in C_0^\infty(B_{r/2}(x))$ such that $\widetilde{\varphi} \geq 0$ in $B_{r/2}(x)$ in the usual sense, $\widetilde{\varphi}(x) > 0$, and $\|\widetilde{\varphi}\|_{L^\infty(\Omega)} < \alpha$. It is straightforward to verify that $\widetilde{\varphi}$ satisfies (a) and (b) in Definition 2.3.8 in the ball $B_{r/2}(x)$. Thus, in Definition 2.3.8 we can ask φ to be in $C_0^\infty(\Omega)$.

(ii) Let $u \in H^1(\Omega)$. The set A_u is open: for a given $x \in A_u$ there exists $\rho > 0$ and $\varphi \in C_0^\infty(B_\rho(x))$ satisfying (a) and (b) in the Definition 2.3.8. Also, there exists a radius $r \in (0, \rho)$ such that $\varphi|_{\overline{B}_r(x)} > 0$. Let us see that $B_r(x) \subseteq A_u$. For any $y \in B_r(x)$, we claim there exists a radius $s \in (0, r)$ such that $B_s(y) \subseteq B_r(x)$, and a function $\widetilde{\varphi} \in C_0^\infty(B_s(y))$ such that $\widetilde{\varphi}$ satisfies (a) and (b) in the Definition 2.3.8: let $s > 0$ be such that $B_s(y) \subseteq B_r(x)$. The function $\widetilde{\varphi}$ can be constructed as follows: consider the cut-off function $\widetilde{\varphi}$ with the same values that φ on the ball $B_{s/4}(y)$, vanishing outside $B_{s/2}(y)$, and satisfying $0 \leq \widetilde{\varphi} \leq \varphi$ in $B_s(y)$ (for this type of construction see for instance Section 16 in [18]). In this way, we get that $\widetilde{\varphi}(y) = \varphi(y) > 0$ and that $u \geq \varphi \geq \widetilde{\varphi}$ in $B_s(y)$. Thus, $B_r(x) \subseteq A_u$ and then A_u is an open set.

(iii) Let $u \in H^1(\Omega) \cap C(\Omega)$, and let $x \in \Omega$. Then $u(x) > 0$ in the sense of $H^1(\Omega)$ if and only if $u(x) > 0$ in the usual sense, as we now show. Assume $u(x) > 0$ in the the sense of $H^1(\Omega)$, then there exists $\rho > 0$ and $\varphi \in C_0^\infty(B_\rho(x))$ satisfying (a) and (b) in the Definition 2.3.8. In particular, $u - \varphi \geq 0$ on $B_\rho(x)$ in $H^1(\Omega)$. By Proposition 1.2.6, $u - \varphi \geq 0$ a.e. in $B_\rho(x)$, and by continuity of $u - \varphi$ it holds $u(x) - \varphi(x) \geq 0$. Thus, $u(x) \geq \varphi(x) > 0$ in the usual sense. On the other hand, if $u(x) > 0$ in the usual sense, by continuity there exists $r > 0$ such that $u|_{\overline{B}_r(x)} > 0$. Following the ideas in (i) we can find $\rho \in (0, r)$ and $\varphi \in C_0^\infty(B_\rho(x))$ satisfying (a) and (b) in the Definition 2.3.8.

Definition 2.3.10. Let u be the solution of the O.P. (as stated in Definition 2.1.2). The set $I = I[u] = \Omega \setminus \{x \in \Omega : u(x) - \psi(x) > 0 \ \text{ in the sense of } H^1(\Omega)\}$ is called the coincidence set of the solution u.

Remark 2.3.11. By Remark 2.3.9 (iii), when $u - \psi \in C(\Omega)$, it follows that $\{x \in \Omega : u(x) - \psi(x) > 0 \ \text{ in the sense of } H^1(\Omega)\} = \{x \in \Omega : u(x) - \psi(x) > 0 \text{ in the usual sense}\}$. In this case, $I = \{x \in \Omega : u(x) - \psi(x) = 0\}$, and the set Ω can be decomposed as $\Omega = \{x \in \Omega : u(x) - \psi(x) > 0\} \cup I$. By Remark 2.3.9, I is a closed subset in Ω.

The above decomposition of Ω allows us to prove an important property of the solution u. In what it remains of this section, we will make extensive use of the definitions and theorems from Subsection 1.3.1. Given $f \in H^{-1}(\Omega)$, $u \in H_0^1(\Omega)$, a Radon measure μ on Ω, and an open subset $O \subseteq \Omega$, we say that $Lu = f + \mu$ ***in O in the distributional sense***, if for all $\varphi \in C_0^\infty(O)$

$$\langle Lu, \varphi \rangle = \langle f, \varphi \rangle + \int_O \varphi \, d\mu.$$

Theorem 2.3.12. *Let $f \in H^{-1}(\Omega)$ and let L be as in Remark 2.3.1. Let u be the solution of the O.P. (as stated in Definition 2.1.2). Then there exists a non-negative Radon measure μ on Ω satisfying all of the following*

(a) $Lu = f$ in $\Omega \setminus I$ in the distributional sense,

(b) $Lu = f + \mu$ in Ω in the distributional sense, and

(c) $supp(\mu) \subseteq I$.

As a consequence of (c) in the above theorem, and the properties of $supp(\mu)$, it holds that $\mu(\Omega \setminus I) = 0$.

Proof.

The scheme of the proof is the following: first we will show that (a) holds. The fact that u is a supersolution of $L(\cdot) - f$ allows us to make use of Riesz Representation Theorem for positive functionals (Theorem 2.14 in [20]). Finally, we prove (b) and (c). Let us prove that $Lu = f$ in $\Omega \setminus I$ in the distributional sense, i.e. for any $\eta \in C_0^\infty(\Omega \setminus I)$

$$\begin{aligned} \langle Lu, \eta \rangle &= \langle f, \eta \rangle \quad \text{(or equivalently)} \\ a(u, \eta) &= \langle f, \eta \rangle . \end{aligned} \tag{2.3.4}$$

In order to prove (2.3.4), we first prove a "local version" of it: for every $x_0 \in \Omega \setminus I$ there exists $\rho > 0$ such that

$$a(u, \xi) = \langle f, \xi \rangle , \quad \text{for all } \xi \in C_0^\infty(B_{\rho/2}(x_0)).$$

Then we apply a partition of unity argument to conclude (2.3.4). Let $x_0 \in \Omega \setminus I$. By Remark 2.3.9 applied to $u - \psi$, there exist $\rho > 0$ such that $B_\rho(x_0) \subseteq (\Omega \setminus I)$, and a function $\varphi \in C_0^\infty(B_\rho(x_0))$, satisfying

(i) $\varphi > 0$ on $\overline{B}_{\rho/2}(x_0)$, and

(ii) $u - \psi \geq \varphi$ on $B_\rho(x_0)$ in $H^1(\Omega)$.

Given $\xi \in C_0^\infty(B_{\rho/2}(x_0))$, let us see that there exists $\varepsilon > 0$ such that

$$\varepsilon \xi + \frac{1}{2}\varphi \geq 0 \ \text{ on } B_{\rho/2}(x_0) \text{ in the usual sense.} \qquad (*)$$

If $\xi \geq 0$ on $B_{\rho/2}(x_0)$, then $(*)$ is trivially true for any $\varepsilon > 0$, since $\varphi > 0$ on $\overline{B}_{\rho/2}(x_0)$. On the other hand, if there exists $z \in B_{\rho/2}(x_0)$ such that $\xi(z) < 0$, then $\sup_{\overline{B}_{\rho/2}(x_0)}(-\xi) > 0$. In such a case, any ε satisfying $0 < \varepsilon < \left(\inf_{\overline{B}_{\rho/2}(x_0)} \varphi/2\right) / \sup_{\overline{B}_{\rho/2}(x_0)}(-\xi)$ verifies $(*)$.

In view of (ii) and (∗), we get that

$$u - \psi + \varepsilon\xi + \frac{1}{2}\varphi \geq \varphi \text{ on } B_{\rho/2}(x_0) \text{ in } H^1(\Omega)$$

$$u + \varepsilon\xi \geq \psi + \frac{1}{2}\varphi \text{ on } B_{\rho/2}(x_0) \text{ in } H^1(\Omega).$$

By the above inequality, (i), and part (i) of Proposition 1.2.6, it follows that the function $v = u + \varepsilon\xi \geq \psi$ a.e. in $B_{\rho/2}(x_0)$. Given that $\xi = 0$ in $\Omega \setminus B_{\rho/2}(x_0)$, and $u \geq \psi$ a.e. in Ω, it holds $v \geq \psi$ a.e. in Ω. Also, since u and ξ are in $H_0^1(\Omega)$, $v \in H_0^1(\Omega)$. Thus, $v \in \mathbb{K}$ and, since u is the solution of the O.P.

$$\begin{gathered} a(u, v - u) = a(u, \varepsilon\xi) \geq \langle f, \varepsilon\xi \rangle \\ \therefore \; a(u, \xi) \geq \langle f, \xi \rangle . \end{gathered} \tag{2.3.5}$$

Since this is true for an arbitrary $\xi \in C_0^\infty(B_{\rho/2}(x_0))$, applying (2.3.5) to $\pm\xi$, we conclude

$$a(u, \xi) = \langle f, \xi \rangle , \quad \text{for all } \xi \in C_0^\infty(B_{\rho/2}(x_0)). \qquad (**)$$

We now prove (2.3.4). Let $\eta \in C_0^\infty(\Omega \setminus I)$. Let $\Gamma = supp(\eta)$. For each $x \in \Gamma$, by the previous step, there exists $\rho_x > 0$ such that (∗∗) holds. The family $\{B_{\rho_x/2}(x)\}_{x\in\Gamma}$ is an open covering of Γ. By compactness, there exists a finite collection of open balls $U_i = B_{\rho_{x_i}/2}(x_i)$, for $1 \leq i \leq n$, that covers Γ. Using Lemma 9.3 (Partition of unity) in [3], we know that there exist functions $\{\varphi_i\}_{i=1}^n$ such that

(1) For each $1 \leq i \leq n$, $\varphi_i \in C_0^\infty(U_i)$,

(2) $\Gamma = supp(\eta) \subseteq \bigcup_{i=1}^n U_i$, and

(3) $\sum_{i=1}^n \varphi_i(x) = 1$ for all $x \in \Gamma$.

Then, observe $\eta = \sum_{i=1}^n \eta\varphi_i$ on $\mathbb{R}^N$, and

$$\begin{aligned} a(u, \eta) &= a\left(u, \sum_{i=1}^n \eta\varphi_i\right) \qquad \text{(by (3))} \\ &= \sum_{i=1}^n a(u, \eta\varphi_i) \qquad \text{(by linearity of } a(u, \cdot)) \\ &= \sum_{i=1}^n \langle f, \eta\varphi_i \rangle \qquad \text{(by } (**)) \\ &= \left\langle f, \sum_{i=1}^n \eta\varphi_i \right\rangle = \langle f, \eta \rangle . \end{aligned}$$

Concluding the proof of (2.3.4). Now we prove (b): by Remark 2.3.3, u is a supersolution of $L(\cdot) - f$, then

$$a(u, h) - \langle f, h \rangle \geq 0 \quad \forall h \in H_0^1(\Omega) \text{ such that } h \geq 0 \text{ a.e. in } \Omega.$$

In particular, the operator defined by $\Lambda(\cdot) = a(u,\cdot) - \langle f, \cdot\rangle$ is positive and linear in $C_0^\infty(\Omega)$. By Theorem 1.3.2, there exists a $\sigma-$algebra $\mathcal{M}$ in Ω which contains all Borel sets in Ω, and there exists a unique positive Radon measure μ on $\mathcal{M}$ which represents $a(u,\cdot) - \langle f, \cdot\rangle$ in the sense that

$$a(u,\varphi) - \langle f, \varphi\rangle = \int_\Omega \varphi \; d\mu, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Finally, we prove (c). To show this, we will prove that if $x \in \Omega \setminus I$, then $x \notin supp(\mu)$, i.e. there exists a neighborhood of x, $U_x \subseteq \Omega$, such that $\mu(U_x) = 0$. Let $x \in \Omega \setminus I$, and let $r > 0$ be such that $\overline{B}_r(x) \subseteq \Omega \setminus I$. By (a), we now that for any $\xi \in C_0^\infty(B_r(x))$, it holds $a(u,\xi) - \langle f, \xi\rangle = 0$. By (b), we have that

$$\int_\Omega \xi \; d\mu = 0 \qquad \text{for all } \xi \in C_0^\infty(B_r(x)). \quad (***)$$

By similar arguments to those in Remark 2.3.9, we can construct a function $\xi \in C_0^\infty(B_r(x))$ such that $\xi \geq 0$ on Ω in the usual sense, and $\xi \equiv 1$ on $B_{r/2}(x)$. By $(***)$,

$$0 \leq \mu(B_{r/2}(x)) = \int_{B_{r/2}(x)} 1 \; d\mu \leq \int_\Omega \xi \; d\mu = 0.$$

☑

Chapter 3

$C^{1,\alpha}$-Regularity of the solution

We wish to go deeper into the properties that the solution u of the O.P. satisfies. Specifically, we want to show that under certain conditions over ψ, f and Ω, the solution of the O.P. (as stated in Definition 2.1.2) belongs to $C^{1,\alpha}\left(\overline{\Omega}\right)$, with α depending on N and certain parameter associated to f. For simplicity we will limit ourselves to the case $A(\cdot) = I_{N\times N}$, where $I_{N\times N}$ is the identity matrix. However, the results presented in this chapter are still valid, with some additional technicalities, in the case in which $A(\cdot)$ is a smooth matrix (see Part III of [15]). We will use a method known as «Penalization» (see, e.g., [13]). Also, we want to mention that the $C^{1,\alpha}$-regularity of u is not the optimal one. In fact, it was first proved by Frehse in 1972 (see [9]) that the optimal regularity for the solution is $u \in C^{1,1}_{loc}(\Omega)$ (see section 3.2 at the end of this chapter for additional comments).

3.1 $C^{1,\alpha}$-Regularity of the solution

In this section, we will make use of the theory of variational inequalities for monotone operators from subsection 1.1.4. The set $\Omega \subseteq \mathbb{R}^N$ will denote a smooth bounded open connected set. For a given $p \in (N, \infty)$, the function ψ will denote an obstacle such that

$$\begin{gathered} \psi \in H^{2,p}(\Omega),\ \psi < 0 \text{ on } \partial\Omega \text{ in } H^1(\Omega), \text{ and} \\ \Delta\psi \in L^2(\Omega). \end{gathered} \tag{3.1.1}$$

The function f will be assumed to satisfy

$$f \in L^2(\Omega) \cap L^p(\Omega). \tag{3.1.2}$$

Set

$$\mathbb{K} = \mathbb{K}_\psi := \{v \in H^1_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega\}.$$

The variational form of the obstacle problem consists in

$$\text{finding } u \in \mathbb{K} \text{ such that } \int_\Omega \nabla u \cdot \nabla(v-u)\, dx \geq \int_\Omega f(v-u)\, dx \quad \forall v \in \mathbb{K}. \tag{3.1.3}$$

Remark 3.1.1. Notice that $p > N$, implies that $L^p(\Omega) \subseteq L^2(\Omega)$ when $N \geq 2$ (since Ω is bounded). In this way, the hypotheses $\Delta\psi \in L^2(\Omega)$ and $f \in L^2(\Omega)$ are only relevant in the case $N = 1$ (i.e. they can be disregarded when $N \geq 2$). Also, let us recall the Sobolev's embedding of $H^{2,p}(\Omega)$ into $C^{1,\lambda}(\overline{\Omega})$ for any $\lambda \in (0, 1 - N/p]$ (see, e.g., [1]). Thus, $\psi \in C^{1,\lambda}(\overline{\Omega})$ for any $\lambda \in (0, 1 - N/p]$. In particular by Definition 1.2.1, the hypothesis $\psi < 0$ on $\partial\Omega$ in $H^1(\Omega)$ is equivalent to $\psi < 0$ on $\partial\Omega$ in the usual sense.

Since $f \in L^2(\Omega)$, then f induces an element in $H^{-1}(\Omega)$. Let u be the solution to problem (3.1.3). The goal of this chapter is to prove the following theorem.

Theorem 3.1.2. *Let ψ as in (3.1.1) and let f as in (3.1.2). Then, $u \in H^{2,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, where $\alpha = 1 - \frac{N}{p}$.*

Remark 3.1.3. The idea to prove the above result is the following. Given $\varepsilon > 0$, consider the function

$$\vartheta_\varepsilon(t) = \begin{cases} 1, & t \leq 0 \\ 1 - \frac{t}{\varepsilon}, & 0 \leq t \leq \varepsilon \\ 0, & t \geq \varepsilon. \end{cases} \tag{3.1.4}$$

For any $\varepsilon > 0$, the function ϑ_ε is non-increasing and uniformly Lipschitz. Let us consider the parameterized family of problems

$$\begin{cases} -\Delta u_\varepsilon = \max(-\Delta\psi - f, 0)\vartheta_\varepsilon(u_\varepsilon - \psi) + f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.5}$$

For any $\varepsilon > 0$, we will show that each of the above problems has a solution $u_\varepsilon \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$. Thus, getting a family of solutions $\{u_\varepsilon\}_{\varepsilon>0} \subseteq H_0^1(\Omega) \cap H^{2,p}(\Omega)$. Such a family of solutions (up to a subsequence) will be weakly convergent in $H^{2,p}(\Omega)$, as $\varepsilon \to 0$, with limit $\tilde{u}$. The function $\tilde{u}$ solving (3.1.3). Therefore, by uniqueness in Theorem 2.2.1, $u = \tilde{u}$ a.e. in Ω. Allowing us to conclude that $u \in H^{2,p}(\Omega)$. Finally, we will use Sobolev's embedding (see, e.g., [1] and [3]) to conclude that $u \in C^{1,\alpha}(\overline{\Omega})$, where $\alpha = 1 - \frac{N}{p}$.

Before proving Theorem 3.1.2, let us try to show, **formally**, how the family of penalized problems (3.1.5) arises from the properties given by Theorem 2.3.12.

Recall that by Remark 2.3.11, if u, $\psi \in C(\Omega)$, then the coincidence set I is equal to $I = \{x \in \Omega : u(x) = \psi(x)\}$. By Theorem 2.3.12, we know that, **at least formally**

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus I, \\ -\Delta u = -\Delta\psi & \text{in } I : \end{cases} \tag{3.1.6}$$

the first of the above equalities may sense in the distributional sense (see the proof of Theorem 2.3.12), while the second equality makes sense on $\operatorname{Int}(I)$. Therefore, we would like to obtain u (the solution of the O.P. (3.1.3)) as the solution of the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus I, \\ -\Delta u = -\Delta\psi & \text{in } I, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.7}$$

Additionally, from the same Theorem 2.3.12, we know that there exists a non-negative Radon measure μ such that $-\Delta u = f + \mu$ in Ω, and $-\Delta u = f$ in $\Omega \setminus I$, both equalities in the sense of distributions. Also, μ is such that $\mathrm{supp}(\mu) \subset I$. Putting together these facts along with (3.1.7), we can obtain the following distributional equalities

$$\begin{cases} \mu = 0 & \text{in } \Omega \setminus I, \\ \mu = -\Delta\psi - f & \text{in } I. \end{cases} \tag{3.1.8}$$

The latter can be reduced to

$$\mu = (-\Delta\psi - f)\vartheta(u - \psi) \text{ in the sense of distributions in } \Omega, \text{ where } \quad \vartheta(t) = \begin{cases} 1 & t \leq 0 \\ 0 & t > 0. \end{cases} \tag{3.1.9}$$

Recall that, formally, $\Delta u = \Delta\psi$ in I. Since u is a supersolution of $L - f$, then $-\Delta u - f \geq 0$ in I. So we can replace $(-\Delta\psi - f)$ by $(-\Delta\psi - f)^+$ in equation (3.1.9). The positivity of the new right-hand side of (3.1.9) will be a key ingredient in the proof of Lemma 3.1.4. All of the above allows us to envision the parameterized family of problems (3.1.5) by substituting μ by $-\Delta u - f$ in (3.1.9), and by approximating ϑ by a family of functions $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, as in Remark 3.1.3.

Now, given $\varepsilon > 0$, we can focus in the problem of finding a solution for each problem (3.1.5). The following lemma gives us the desired existence, uniqueness and some regularity. We will use the theory from subsection 1.1.4, in particular: definitions 1.1.10, 1.1.12, 1.1.17, also we will use Corollary 1.1.17.1.

Lemma 3.1.4. *(i) Let ψ be as in (3.1.1) and let f be as in (3.1.2). Let $\vartheta : \mathbb{R} \longrightarrow [0,1]$ be a non-increasing uniformly Lipschitz function. Then, there exists a unique weak solution (as stated in Definition 1.3.3) $w \in H^1_0(\Omega) \cap H^{2,p}(\Omega)$ of the problem*

$$\begin{cases} -\Delta w = \max(-\Delta\psi - f, 0)\vartheta(w - \psi) + f & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.10}$$

(ii) Consider the map $(f, \vartheta) \mapsto w$, where w is the solution given by (i). Such a map is continuous, i.e. there exists a constant c independent of the pair (f, ϑ) such that

$$\|w\|_{H^{2,p}(\Omega)} \leq c\left(\|f\|_{L^p(\Omega)} + \|\max(-\Delta\psi - f, 0)\|_{L^p(\Omega)}\right) \tag{3.1.11}$$

for all $f \in L^2(\Omega) \cap L^p(\Omega)$ and for all non-increasing uniformly Lipschitz function $\vartheta : \mathbb{R} \longrightarrow [0,1]$.

Proof.
Since ϑ is bounded, the function $\vartheta(w-\psi) \in L^\infty(\Omega)$ for any $w \in H_0^1(\Omega)$. By hypothesis, both f and $\Delta\psi$ belong to $L^2(\Omega) \cap L^p(\Omega)$. Thus, $\max(-\Delta\psi - f, 0) \in L^2(\Omega) \cap L^p(\Omega)$. Let $w \in H_0^1(\Omega)$. By the previous comments, the operator $Lw : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\langle Lw, \varphi \rangle = \int_\Omega \underbrace{\nabla w}_{\in L^2(\Omega)} \cdot \nabla\varphi - \underbrace{(\max(-\Delta\psi - f, 0)\,\vartheta(w-\psi) + f)}_{\in L^2(\Omega)} \varphi \, dx$$

belongs to $H^{-1}(\Omega)$ (see Theorem 6.0.5). We have thus defined the operator $L : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$. We want to prove the existence of a weak solution of (3.1.10) by means of Corollary 1.1.17.1, in the closed convex non-empty set $H_0^1(\Omega)$. We now verify the remaining hypotheses of this corollary. First we prove that L is a strictly monotone and coercive operator. For $u, v \in H_0^1(\Omega)$

$$\begin{aligned}\langle Lw - Lv, w - v \rangle &= \|w - v\|^2_{H_0^1(\Omega)} - \int_\Omega \max(-\Delta\psi - f, 0)\,[\vartheta(w-\psi) - \vartheta(v-\psi)]\,(w-v)\, dx \\ &\geq \|w - v\|^2_{H_0^1(\Omega)}, \qquad\qquad (*)\end{aligned}$$

where the latter inequality is true given that ϑ is a non-increasing function, which implies that $[\vartheta(w-\psi) - \vartheta(v-\psi)]\,(w-v) \leq 0$ a.e. in Ω. Inequality in $(*)$ shows that L is both, monotone and coercive (see Definition 1.1.10 and Definition 1.1.17). Also, the equality in $(*)$ shows that if $\langle Lw - Lv, w - v\rangle = 0$, then $\|w - v\|^2_{H_0^1(\Omega)} = 0$, i.e. $u = v$ in $H_0^1(\Omega)$. Therefore, L is strictly monotone (see Definition 1.1.10).

Now we show that L is continuous on finite dimensional subspaces of $H_0^1(\Omega)$ (see Definition 1.1.12). Actually, we will prove something more general: strong convergence in $H_0^1(\Omega)$ implies weak convergence in $H^{-1}(\Omega)$. Let $\{w_n\}_n \subseteq H_0^1(\Omega)$ be a sequence such that $w_n \to w \in H_0^1(\Omega)$. Let $\varphi \in H_0^1(\Omega)$, then

$$|\langle Lw_n - Lw, \varphi\rangle| \leq \int_\Omega |\nabla(w_n - w) \cdot \nabla\varphi| + c\int_\Omega \max(-\Delta\psi - f, 0)\,|w_n - w|\,|\varphi| dx, \qquad (3.1.12)$$

where $c > 0$ is the Lipschitz constant associated to ϑ, i.e. $|\vartheta(x) - \vartheta(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$. The first term in the right-hand side of (3.1.12) is bounded by $\|w_n - w\|_{H_0^1(\Omega)}\,\|\varphi\|_{H_0^1(\Omega)}$, which converges to 0 as $n \to \infty$. For the second term in the right-hand side of (3.1.12), we apply the generalized Hölder's inequality to obtain

$$c\int_\Omega \max(-\Delta\psi - f, 0)\,|w_n - w|\,|\varphi| dx \leq c\,\|\max(-\Delta\psi - f, 0)\|_{L^p(\Omega)}\,\|w_n - w\|_{L^q(\Omega)}\,\|\varphi\|_{L^q(\Omega)},$$

where q is such that $p^{-1} + 2q^{-1} = 1$, i.e. $q = 2p/(p-1)$. The fact that $w_n - w$ and φ belong to $L^q(\Omega)$, comes from the Sobolev's embedding (see, e.g., Appendix 6): if $N > 2$ then $H_0^1(\Omega) \subseteq L^r(\Omega)$ for any $r \in [1, 2N/(N-2))$, or if $N \in \{1, 2\}$ then $H_0^1(\Omega) \subseteq L^r(\Omega)$ for any $r \in [1, \infty)$. Notice that $p > N$ implies that $q \in [1, 2N/(N-2))$. Such an embedding also implies that $w_n \to w$ in $L^q(\Omega)$ as $n \to \infty$. Therefore, Lw_n converges weakly to Lw in $H^{-1}(\Omega)$. In particular, L is continuous on finite dimensional subspaces of $H_0^1(\Omega)$. By Corollary 1.1.17.1, there exists $w \in H_0^1(\Omega)$ such that $\langle Lw, v - w\rangle \geq 0$ for all $v \in H_0^1(\Omega)$. In particular, for $\varphi \in H_0^1(\Omega)$, we can set $v = \varphi + w$. Therefore, $\langle Lw, v - w\rangle = \langle Lw, \varphi\rangle \geq 0$. Thus, w satisfies

$$\int_\Omega \nabla w \cdot \nabla \varphi - [\max(-\Delta\psi - f, 0)\,\vartheta(w-\psi) + f]\,\varphi\; dx = 0 \quad \forall \varphi \in H_0^1(\Omega),$$

which says that w is a weak solution of (3.1.10). Uniqueness is guaranteed by the fact that L is a strictly monotone operator: suppose that w_1 and w_2 are two weak solutions of (3.1.10) in $H_0^1(\Omega)$. For any $\varphi \in H_0^1(\Omega)$, by definition

$$\langle Lw_1, \varphi\rangle = 0 \quad \text{and} \quad \langle Lw_2, \varphi\rangle = 0.$$

Taking $\varphi = w_1 - w_2 \in H_0^1(\Omega)$, and subtracting the previous equations give us $\langle Lw_1 - Lw_2, w_1 - w_2\rangle = 0$. Thus, $w_1 = w_2$ in $H_0^1(\Omega)$. By part (i) of Theorem 1.3.9, it follows that $w \in H^{2,p}(\Omega)$.

Finally, we prove part (ii) of the lemma. For any $f \in L^2(\Omega) \cap L^p(\Omega)$, and for any non-increasing uniformly Lipschitz function $\vartheta : \mathbb{R} \longrightarrow [0,1]$, we have a solution $w \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta w = F & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.13}$$

where $F = \max(-\Delta\psi - f, 0)\vartheta(w-\psi) + f \in L^p(\Omega)$. By part (ii) of Theorem 1.3.9, there exists a constant c independent of F such that

$$\begin{aligned} \|w\|_{H^{2,p}(\Omega)} &\le c\,\|\max(-\Delta\psi - f, 0)\vartheta(w-\psi) + f\|_{L^p(\Omega)} \\ &\le c\left(\|\max(-\Delta\psi - f, 0)\|_{L^p(\Omega)}\,\|\vartheta\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)}\right) \\ &\le c\left(\|\max(-\Delta\psi - f, 0)\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right), \end{aligned}$$

for any $f \in L^2(\Omega) \cap L^p(\Omega)$, and for any non-increasing uniformly Lipschitz function $\vartheta : \mathbb{R} \longrightarrow [0,1]$.

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For $\varepsilon > 0$, let ϑ_ε as in (3.1.4). It is not difficult to show that the function ϑ_ε satisfies the hypotheses of Lemma 3.1.4. Thus, for ψ as in (3.1.1) and f as in (3.1.2), each problem (3.1.5) possesses a solution $u_\varepsilon \in H_0^1(\Omega) \cap H^{2,p}(\Omega)$. The next theorem asserts that the family of functions $\{u_\varepsilon\}_{\varepsilon>0}$ possesses a subsequence that weakly converges to a function $\tilde{u} \in H^{2,p}(\Omega)$. Using Lemma 1.1.15 (Minty's Lemma) we will conclude that $\tilde{u} = u$ a.e. in Ω, where u is the solution of O.P. We now prove the main theorem of this chapter.

Proof of Theorem 3.1.2. Let $\varepsilon > 0$. We begin by showing that $u_\varepsilon \in \mathbb{K}$: first, $u_\varepsilon \in H_0^1(\Omega)$. Second, if we set $\varphi = u_\varepsilon - \max(u_\varepsilon, \psi) = -\max(\psi - u_\varepsilon, 0)$, by Proposition 1.2.8, $\varphi \in H_0^1(\Omega)$. We will prove that $\varphi = 0$ a.e. in $x \in \Omega$. Which shows that $u_\varepsilon \in \mathbb{K}$. Notice that,

$$\int_\Omega \{\nabla u_\varepsilon \cdot \nabla\varphi - [\max(-\Delta\psi - f, 0)\vartheta_\varepsilon(u_\varepsilon - \psi) + f]\,\varphi\}\; dx = 0. \tag{3.1.14}$$

Recall $\psi \in H^{2,p}(\Omega)$. Thus, in particular, $\psi \in H^2(\Omega)$, where the weak derivatives of ψ in $L^2(\Omega)$ are just the classical derivatives of ψ. Since $\varphi \in H_0^1(\Omega)$, by Remark 6.0.2,

$$\int_\Omega (\Delta\psi\varphi + \nabla\psi \cdot \nabla\varphi)\; dx = 0. \tag{3.1.15}$$

After subtracting both equations (3.1.14) and (3.1.15), and moving some terms, we obtain

$$\int_\Omega \nabla(u_\varepsilon - \psi)\cdot\nabla\varphi \; dx = \int_\Omega \left[\max(-\Delta\psi - f, 0)\vartheta_\varepsilon(u_\varepsilon - \psi) + f + \Delta\psi\right]\varphi \; dx. \tag{3.1.16}$$

By definition, $\varphi = -\max(\psi - u_\varepsilon, 0) \le 0$ a.e. in Ω. From Lemma 1.2.7, we know that

$$\nabla\varphi = \begin{cases} \nabla\left(u_\varepsilon - \psi\right) & \text{a.e. in } \{\psi - u_\varepsilon > 0\},\\ 0 & \text{a.e. in } \{\psi - u_\varepsilon \le 0\}. \end{cases}$$

Up to a set of measure zero, the following sets are equal: $\{\psi - u_\varepsilon > 0\} = \{\varphi < 0\}$ and $\{\psi - u_\varepsilon \le 0\} = \{\varphi \ge 0\}$. Since $\varphi \le 0$ a.e. in Ω, $|\{\varphi > 0\}| = 0$. Using all of the previous in (3.1.16), we obtain

$$\int_\Omega |\nabla\varphi|^2 \; dx = \int_{\{\varphi<0\}} \left[\max(-\Delta\psi - f, 0)\vartheta_\varepsilon(u_\varepsilon - \psi) + f + \Delta\psi\right]\varphi \; dx, \tag{3.1.17}$$

where the right-hand side of the previous equation is the right-hand side of (3.1.16), after splitting the integral over the sets $\{\varphi < 0\}$, $\{\varphi = 0\}$ and $\{\varphi > 0\}$. Notice that $\vartheta_\varepsilon(u_\varepsilon(x) - \psi(x)) = 1$ a.e. $x \in \{\varphi < 0\}$, hence

$$\|\varphi\|^2_{H^1_0(\Omega)} = \int_{\{\varphi<0\}} \left[\max(-\Delta\psi - f, 0) + f + \Delta\psi\right]\varphi \; dx \le 0.$$

Proving that $\varphi = 0$ in $H^1_0(\Omega)$, which implies that $u_\varepsilon \ge \psi$ a.e. in Ω. Thus, $u_\varepsilon \in \mathbb{K}$.

Now, from Lemma 3.1.4, there exists a constant c independent of u_ε such that

$$\|u_\varepsilon\|_{H^{2,p}(\Omega)} \le c\left(\|f\|_{L^p(\Omega)} + \|\max(-\Delta\psi - f, 0)\|_{L^p(\Omega)}\right).$$

The above inequality says that $\{u_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H^{2,p}(\Omega)$. Reflexivity of the latter space, and Sobolev's embedding of $H^{2,p}(\Omega)$ into $C^{1,\alpha}\left(\overline{\Omega}\right)$, $\alpha = 1 - N/p$ (see, e.g., [1]), say that there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$, and a function $\tilde{u} \in H^{2,p}(\Omega) \cap C^{1,\alpha}\left(\overline{\Omega}\right)$ such that $u_{\varepsilon_k} \overset{w}{\to} \tilde{u} \in H^{2,p}(\Omega) \cap C^{1,\alpha}\left(\overline{\Omega}\right)$, where $\varepsilon_k \to 0$ as $k \to \infty$. Also, since $\mathbb{K}$ is convex and closed in $H^1_0(\Omega)$, $\tilde{u} \in \mathbb{K}$. We will use Minty's Lemma (Lemma 1.1.15) to show that $\tilde{u}$ solves (3.1.3), and by uniqueness $\tilde{u} = u$ a.e. in Ω, which would conclude the proof of Theorem 3.1.2. For simplicity we write $\{u_k\}_{k\in\mathbb{N}}$ instead of $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$.

Let $k \in \mathbb{N}$, and let us consider the operator $L_k : \mathbb{K} \longrightarrow H^{-1}(\Omega)$, defined by

$$\langle L_k w, \varphi\rangle = \int_\Omega \nabla w \cdot \nabla\varphi - \left(\max\left(-\Delta\psi - f, 0\right)\vartheta_{\varepsilon_k}(w - \psi) + f\right)\varphi \; dx,$$

and $L : \mathbb{K} \longrightarrow H^{-1}(\Omega)$, defined by

$$\langle Lw, \varphi\rangle = \int_\Omega \nabla w \cdot \nabla\varphi - f\varphi \; dx.$$

By the same arguments in the proof of Lemma 3.1.4, for each $k \in \mathbb{N}$, L_k is continuous on finite dimensional subspaces of $H^1_0(\Omega)$. Minty's Lemma says that $\langle L_k v, v - u_k\rangle \ge 0$ for all $v \in \mathbb{K}$, if and only if $\langle L_k u_k, v - u_k\rangle \ge 0$ for all $v \in \mathbb{K}$. Now, let us show that the latter inequality is true: let

$v \in \mathbb{K}$. Since u_k is a weak solution of problem (3.1.5), with $\vartheta_{\varepsilon_k}$ as in (3.1.4), and $v - u_k \in H_0^1(\Omega)$, then

$$\langle L_k u_k, v - u_k\rangle = \int_\Omega \nabla u_k \cdot \nabla(v - u_k) - [\max(-\Delta\psi - f, 0)\,\vartheta_{\varepsilon_k}(u_k - \psi) + f]\,(v - u_k)\; dx = 0.$$

Therefore, by Minty's Lemma,

$$\langle L_k v, v - u_k\rangle = \int_\Omega \nabla v \cdot \nabla(v - u_k)\; dx - \int_\Omega [\max(-\Delta\psi - f, 0)\vartheta_{\varepsilon_k}(v - \psi) + f]\,(v - u_k)\, dx \geq 0. \tag{3.1.18}$$

Observe that (3.1.18) is true for every $k \in \mathbb{N}$ and every $v \in \mathbb{K}$. From inequality (3.1.18), we will prove that $\langle Lv, v - \tilde{u}\rangle \geq 0$ for all $v \in \mathbb{K}$. First, suppose that $v \in \mathbb{K}$ is such that $v \geq \psi + \delta$ a.e. in Ω, for some $\delta > 0$. If $\varepsilon_k < \delta$, by Definition 3.1.4, then $\vartheta_{\varepsilon_k}(v - \psi) = 0$ a.e. in Ω. Letting $k \to \infty$ in (3.1.18), we obtain

$$\langle Lv, v - \tilde{u}\rangle = \int_\Omega \nabla v \cdot \nabla(v - \tilde{u})\; dx - \int_\Omega f\,(v - \tilde{u})\, dx \geq 0. \tag{3.1.19}$$

Notice that the above inequality is true for any $v \in \mathbb{K}$ such that $v \geq \psi + \delta$ a.e. in Ω, for some $\delta > 0$. We wish to extend it to any $v \in \mathbb{K}$, as we now do.

Let $v \in \mathbb{K}$. Let $\delta > 0$ be such that $\delta < -\max_{\partial\Omega}\psi$ (recall that $\psi < 0$ on $\partial\Omega$ and $\psi \in C(\overline{\Omega})$). Thus, $\psi + \delta < 0$ on $\partial\Omega$. The function $v_\delta = \max(v, \psi + \delta) \in H_0^1(\Omega)$ (by Proposition 1.2.8). Clearly, $v_\delta \geq \psi + \delta$ a.e. in Ω. Hence, it satisfies (3.1.19), i.e.

$$\int_\Omega \nabla v_\delta \cdot \nabla(v_\delta - \tilde{u})\; dx - \int_\Omega f\,(v_\delta - \tilde{u})\, dx \geq 0. \quad (*)$$

By Proposition 1.2.9, $v_\delta \to v \in H_0^1(\Omega)$ as $\delta \to 0^+$. So we can take the limit as $\delta \to 0^+$ in $(*)$. Concluding that (3.1.19) is true for any $v \in \mathbb{K}$. By using again Minty's Lemma, applied to the operator L, it follows that $\tilde{u}$ solves (3.1.3), and by uniqueness in Theorem 2.2.1, $\tilde{u} = u$ a.e. in Ω.

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3.2 Final comments on Chapter 3: $C^{1,1}_{loc}(\Omega)$ regularity of u

In this section we present, without proofs, a summary of further regularity results for the solution u of the obstacle problem. In particular, such a results show that under certain conditions over ψ, f and Ω, $u \in C^{1,1}_{loc}(\Omega)$. This results were first obtained by Frehse in 1972 (see [9]). However, here we follow the results presented in chapter 2 of [19]. The $C^{1,1}_{loc}$-regularity of u turns out to be the optimal regularity as shown by the Example 4.0.3 in Chapter 4.

Also, throughout this section, the set $\Omega \subseteq \mathbb{R}^N$ will denote a smooth bounded open connected set. We will consider an obstacle ψ such that $\psi \in C^2(\overline{\Omega})$ (this assumption is stronger than the one we needed for ψ in section 2.1). Let $h \in H^1(\Omega)$ be such that $h \geq \psi$ on $\partial\Omega$ in the sense of $H^1(\Omega)$ (see Section 1.2). Let $f \in L^\infty(\Omega)$. Set

$$\mathbb{K} = \{v \in H^1(\Omega) : v \geq \psi \text{ a.e. in } \Omega \text{ and } u - h \in H_0^1(\Omega)\}.$$

The variational form of the obstacle problem with non-null datum h consists in

$$\text{finding } u \in \mathbb{K} \text{ such that } \int_\Omega \nabla u \cdot \nabla(v-u)\ dx \geq \int_\Omega f(v-u)\ dx \quad \forall v \in \mathbb{K}. \tag{3.2.1}$$

From Remark 2.3.5, we know that a unique solution u of problem (3.2.1) exists. Now, set $\tilde{f} = f - \Delta\psi$, $h_1 = h - \psi$, and

$$\mathbb{K}_1 = \{v \in H^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega \text{ and } v - h_1 \in H^1_0(\Omega)\}.$$

Notice that $\tilde{f} \in L^\infty(\Omega)$ (since $\Delta\psi \in C(\overline{\Omega})$). It can be shown (see subsection 1.1.1.3 in [19]) that $w = u - \psi$ solves the problem of

$$\text{finding } w \in \mathbb{K}_1 \text{ such that } \int_\Omega \nabla w \cdot \nabla(v-w)dx \geq \int_\Omega \tilde{f}(v-w)dx \quad \forall v \in \mathbb{K}_1. \tag{3.2.2}$$

To avoid the use of new notation, we denote the solution of problem (3.2.2) with u, take $\tilde{f} = f$, and $h_1 = h$. It can be shown (see Subsection 1.3.2 in [19]) that such a solution u of problem (3.2.2) satisfies $u \geq 0$ and $u \in C^{1,\alpha}_{loc}(\Omega) \cap H^{2,p}_{loc}(\Omega)$ for any $p \in (1,\infty)$, and for any $\alpha \in (0,1)$. Also, u satisfies

$$\begin{cases} \Delta u = f & \text{a.e. } x \in \{u > 0\}, \\ \Delta u = 0 & \text{a.e } x \in \{u = 0\}. \end{cases} \tag{3.2.3}$$

Equations (3.2.3) say that u is a weak solution of the equation

$$\Delta u = f\ \chi_{\{u>0\}} \text{ in } \Omega. \tag{3.2.4}$$

Expression (3.2.4) shows that $\Delta u \in L^\infty(\Omega)$, one would like to obtain a similar result for D^2u. This, in turn, can be proven to be related to the $C^{1,1}_{loc}$-regularify of u. The key of the $C^{1,1}_{loc}$-regularity of u consists in showing that D^2u is uniformly bounded on compact sets in Ω: the following result about the growth of u near $\partial\{u = 0\}$ is the first step to prove such a fact.

Theorem 3.2.1 (Quadratic Growth). *Let u be the solution of problem (3.2.2). Let $\Gamma = \partial\{u = 0\}$. Let $x^0 \in \Gamma$ and let $R > 0$ be such that $B_{2R}(x^0) \subset \Omega$. Then,*

$$\sup_{B_R(x^0)} u \leq C_N \|f\|_{L^\infty(\Omega)} R^2,$$

where C_N is a constant depending only on the dimension.

See [19] for a proof of this result. The following Theorem extends the previous result to points that are closer to the coincidence set I than to the boundary $\partial\Omega$.

Theorem 3.2.2. *Let u be the solution of problem (3.2.2). Let $x \in \Omega$. If $2\text{dist}(x, I) < \text{dist}(x, \partial\Omega)$, then*

$$u(x) \leq C_N \|f\|_{L^\infty(\Omega)} (\text{dist}(x, I))^2,$$

where C_N is a constant depending only on the dimension.

The fact that $f \in L^\infty(\Omega)$ is essential to obtain the $C^{1,\alpha}_{loc}(\Omega)$ regularity of the solution for each $\alpha \in (0,1)$, and it is crucial for the bounds in Theorem 3.2.1 and Theorem 3.2.2. However, one needs a stronger assumption on f to obtain the $C^{1,1}_{loc}(\Omega)$ regularity. We assume that there exists $\xi \in C^{1,1}(\overline{\Omega})$ such that

$$f = \Delta\xi \quad \text{a.e. } x \in \Omega. \tag{3.2.5}$$

Recall that $\xi \in C^{1,1}(\overline{\Omega})$ implies that $\Delta\xi \in L^\infty(\Omega)$ (see Theorem 4 of page 279 in [5]). The last theorem of this chapter is the following.

Theorem 3.2.3 ($C^{1,1}_{loc}$ regularity). *Let f be as in (3.2.5). Let $h \in H^1(\Omega)$ be such that $h \geq 0$ on $\partial\Omega$ in $H^1(\Omega)$. Let u be the solution of problem (3.2.2). Then $u \in C^{1,1}_{loc}(\Omega)$ and*

$$\|u\|_{C^{1,1}(K)} \leq C\left(\|u\|_{L^\infty(\Omega)} + \left\|D^2\xi\right\|_{L^\infty(\Omega)}\right),$$

for any set $K \subset\subset \Omega$, where $C = C(N, \operatorname{dist}(K,\partial\Omega))$ is a constant depending on N and $\operatorname{dist}(K,\partial\Omega)$.

See Theorem 2.3 in [19] for a proof.

Chapter 4

The obstacle problem: The one-dimensional case

In this chapter we show an alternative approach to obtain some of the regularity of the solution of the obstacle problem in the one-dimensional setting. We use several properties that are only true in the one-dimensional case. Throughout this chapter, Ω will be an open bounded interval of the form $\Omega = (\alpha, \beta) \subseteq \mathbb{R}$. We begin with a proposition that says that the functions in $H^1(\Omega)$ are essentially the absolutely continuous functions u on $[\alpha, \beta]$ whose derivative u' is in $L^2(\Omega)$.[1]

Proposition 4.0.1. *Let $u : [\alpha, \beta] \longrightarrow \mathbb{R}$ be a function. Then, $u \in H^1(\Omega)$ if and only if there exists a function $\tilde{u} : [\alpha, \beta] \longrightarrow \mathbb{R}$ such that $\tilde{u}$ is absolutely continuous on $[\alpha, \beta]$, $\tilde{u}' \in L^2(\Omega)$, and $\tilde{u} = u$ a.e. in Ω.*

See Theorem 8.2 in [3] for a proof of this proposition. From now on, when we take a given $v \in H_0^1(\Omega)$, we are going to identify v with its absolutely continuous representative $\tilde{v}$ given by the above proposition.

In this chapter we are going to assume the following: $\psi \in C(\overline{\Omega})$ is such that $\max_\Omega \psi > 0$, and $\psi(\alpha)$, $\psi(\beta) < 0$. As above, $\mathbb{K} = \{v \in H_0^1(\Omega) : v \geq \psi \text{ in } \Omega\}$. We define

$$a(u, v) = \int_\alpha^\beta u'(x)v'(x)\, dx \quad u, v \in H^1(\Omega),$$

which corresponds to $A(x) = 1$ for all $x \in \Omega$ in (2.1.3). Given any $f \in H^{-1}(\Omega)$, by Theorem 2.2.1, we know that there exists a unique $u \in \mathbb{K}$ solving the variational inequality

$$\int_\alpha^\beta u'(x)\left(v'(x) - u'(x)\right)\, dx \geq \langle f, v - u\rangle \quad \forall v \in \mathbb{K}. \tag{4.0.1}$$

Given that Ω is bounded, $f \in H^{-1}(\Omega)$ can be represented (see, e.g., Proposition 8.14 in [3]) by a

[1] A function $f : [a, b] \longrightarrow \mathbb{R}$ is absolutely continuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$ and any disjoint collection of subintervals $\{(a_1, b_1), \cdots, (a_n, b_n)\}$ in $[a, b]$, it holds that if $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_k |f(b_k) - f(a_k)| < \varepsilon$.

function $F \in L^2(\Omega)$ as follows

$$\langle f, \zeta \rangle = -\int_\Omega F\zeta' \, dx, \quad \forall \zeta \in H_0^1(\Omega). \tag{4.0.2}$$

By Theorem 2.3.12, there exists a non-negative Radon measure μ on Ω such that $supp(\mu) \subseteq I$, and

$$Lu = f + \mu \quad \text{in the distributional sense in } \Omega.$$

The above means that

$$\underbrace{\int_\Omega u'\zeta' \, dx}_{\langle Lu, \zeta \rangle} = \langle f, \zeta \rangle + \int_\Omega \zeta \, d\mu, \quad \forall \zeta \in C_0^\infty(\Omega). \tag{4.0.3}$$

Since $u \in H_0^1(\Omega)$, $u(\alpha) = 0 = u(\beta)$ (see Theorem 8.12 in [3]). By continuity of u and ψ, and the fact that $\psi(\alpha) < 0$ and $\psi(\beta) < 0$, $\text{dist}(I, \partial\Omega) > 0$. Therefore, I is a closed bounded set in $\mathbb{R}$. Thus, I is compact. The latter implies, since μ is a Radon measure, that $\mu(I) < \infty$. Since $\mu(\Omega \setminus I) = 0$, $\mu(\Omega) = \mu(I)$. This shows that μ is a finite measure.

Let us define the function $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}$ as $\varphi(\cdot) = \mu([\alpha, \cdot))$. By definition φ is a positive non-decreasing function, and we also proved that $\varphi(\beta) = \mu(\Omega) < \infty$, which implies that $\varphi \in L^\infty(\Omega)$. The latter implies that the distribution given by

$$(\varphi', \zeta) := -\int_\Omega \varphi\zeta' \, dx, \quad \zeta \in C_0^\infty(\Omega), \tag{4.0.4}$$

is well-defined (see section 1.3 in [11]). We will use Fubini's Theorem (see, e.g., [20]) to show that $\mu = \varphi'$ in the distributional sense in Ω, i.e.

$$\int_\Omega \zeta \, d\mu = (\varphi', \zeta) \quad \forall \zeta \in C_0^\infty(\Omega). \tag{4.0.5}$$

Let $\zeta \in C_0^\infty(\Omega)$. First, notice that both μ and the Lebesgue measure are σ-finite in Ω, in fact, they are finite measures in Ω. For $(x, y) \in \Omega \times \Omega$, consider the function $\zeta'(x)\chi_{[\alpha,x)}(y)$. The latter function is equal to $\zeta'(x)\chi_{(y,\beta]}(x)$. Let us see why this function is Borel measurable: for $a \in \mathbb{R}$, let

$$V_a := \left\{(x, y) \in \Omega \times \Omega : \ \zeta'(x)\chi_{(y,\beta]}(x) \geq a\right\}.$$

We use the following notation $[x \leq y] := \{(x, y) \in \Omega \times \Omega : \ x \leq y\}$, and $[x > y] := \{(x, y) \in \Omega \times \Omega : \ x > y\}$. It follows that

$$V_a = \begin{cases} [x \leq y] \cup \Big([x > y] \cap (\zeta')^{-1}([a, \infty)) \times \Omega\Big) & \text{if } a \leq 0, \\ [x > y] \cap (\zeta')^{-1}([a, \infty)) \times \Omega & \text{if } a > 0. \end{cases}$$

In any case, V_a is a measurable set in $\Omega \times \Omega$. Thus, the function $\zeta'(x)\chi_{[\alpha,x)}(y)$ is Borel measurable. Also, this function satisfies

$$\int_\Omega \int_\Omega \left|\zeta'(x)\chi_{[\alpha,x)}(y)\right| \, dx d\mu_y \leq \|\zeta'\|_{L^\infty(\Omega)} |\Omega| \, \mu(\Omega) < \infty.$$

The latter fact will allows us to use Fubini's Theorem (see below). Now,

$$\int_\Omega \varphi(x)\zeta'(x)\ dx = \int_\Omega \left(\int_{[\alpha,x)} d\mu_y\right)\zeta'(x)\ dx$$

$$= \int_\Omega \int_{[\alpha,x)} \zeta'(x)\ d\mu_y dx = \int_\Omega \int_\Omega \chi_{[\alpha,x)}(y)\zeta'(x)\ d\mu_y dx$$

$$= \int_\Omega \int_\Omega \chi_{(y,\beta]}(x)\zeta'(x)\ d\mu_y dx = \int_\Omega \int_\Omega \chi_{(y,\beta]}(x)\zeta'(x)\ dx d\mu_y,$$

latter equality by virtue of Fubini's Theorem. Therefore,

$$\int_\Omega \varphi(x)\zeta'(x)\ dx = \int_\Omega \int_\Omega \chi_{(y,\beta]}(x)\zeta'(x)\ dx d\mu_y$$

$$= \int_\Omega \int_y^\beta \zeta'(x)\ dx d\mu_y = \int_\Omega [\zeta(\beta) - \zeta(y)]\ d\mu_y$$

$$= -\int_\Omega \zeta(y)\ d\mu_y. \qquad \text{(Since } \zeta(\beta) = 0\text{)}.$$

Thus, proving (4.0.5). Now, combining (4.0.2), (4.0.3), and (4.0.5) we get that

$$\int_\Omega (u' + F + \varphi)\,\zeta'\ dx = 0, \quad \forall \zeta \in C_0^\infty(\Omega).$$

Therefore, (using Lemma 8.1 of [3]) there exists $c \in \mathbb{R}$ such that

$$-u'(x) = F(x) + \varphi(x) + c \text{ for a.e. } x \in \Omega. \tag{4.0.6}$$

At this point, let us assume further that $F \in L^p(\Omega)$ for some $p \in (1, \infty)$. Recall that $\varphi \in L^\infty(\Omega)$. Thus, equation (4.0.6) says that $u \in H^{1,p}(\Omega)$ and, by Sobolev's embedding (see, e.g., [1] and [3]), $u \in C^{0,\alpha}\left(\overline{\Omega}\right)$, where $\alpha = 1 - 1/p$. Also, if $F \in L^\infty(\Omega)$ then (by Proposition 8.4 in [3]) $u \in C^{0,1}(\Omega)$. We have proved the following theorem.

Theorem 4.0.2. *Let $f \in H^{-1}(\Omega)$. Let $F \in L^2(\Omega)$ be such that (4.0.2) is satisfied. Let $u \in \mathbb{K}$ be the solution to problem (4.0.1). The following two holds:*

(i) If $F \in L^p(\Omega)$, for some $p \in (1, \infty)$, then $u \in H^{1,p}(\Omega) \cap C^{0,\alpha}\left(\overline{\Omega}\right)$, where $\alpha = 1 - 1/p$.

(ii) If $F \in L^\infty(\Omega)$, then $u \in C^{0,1}(\Omega)$.

We finish this chapter with a one-dimensional example of the O.P. This example allows us to see that we cannot expect C^2-regularity for the solution.

Example 4.0.3. Let $\Omega = (-3, 3)$. Let $\psi(x)$ be the obstacle given by $\psi(x) = 1 - x^2$ and let $f = 0$. Let $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ be defined as

$$a(u, v) = \int_{-3}^{3} u'(x)v'(x)\ dx.$$

Let $\mathbb{K} = \{v \in H_0^1(\Omega) : v \geq \psi \text{ in } \Omega\}$. We already know that the problem of finding

$$u \in \mathbb{K} \text{ such that } a(u, v - u) \geq 0 \quad \forall v \in \mathbb{K},$$

has a unique solution in $H_0^1(\Omega) \cap H^{2,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for any $p \in (1, \infty)$, and for any $\alpha \in (0, 1)$. By part (a) in Theorem 2.3.12, we have that

$$\int_{-3}^{3} u'\varphi' \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega \setminus I).$$

Let O be a connected component of $\Omega \setminus I$, which is known to be a maximal open interval contained in $\Omega \setminus I$. The latter equality implies (by Lemma 8.1 of [3]) that there exists $c = c(O) \in \mathbb{R}$ such that $u' = c(O)$ a.e. in O. Continuity of u' implies that $u'(x) = c(O)$ for all $x \in O$. Therefore, there exists $d = d(O) \in \mathbb{R}$ such that

$$u(x) = c(O)x + d(O) \text{ for all } x \in O.$$

Let us observe that given $x \in \partial\Omega$, $u(x) = 0 > \psi(x)$. In particular, $\operatorname{dist}(I, \partial\Omega) > 0$. Additionally, $u = \psi$ in I, and $u \geq \psi$ in Ω. Thus, $u - \psi$ achieves its minimum value in Ω at every point of I, which implies that $u' = \psi'$ in I.

Therefore, at $x = -3$, $u(x) - \psi(x) > 0$. By continuity there exists $\epsilon_1 > 0$ such that $u - \psi > 0$ in $(-3, -3 + \epsilon_1)$. Let $O_1 = (-3, x_1)$ be the connected component of $\Omega \setminus I$ that contains $(-3, -3 + \epsilon_1)$. In this component $u(x) = c(O_1)x + d(O_1)$, where $c(O_1)$, $d(O_1)$ and $x_1 > -3$ are such that

$$u(-3) = 0, \;\; u(x_1) = \psi(x_1) \text{ and } u'(x_1) = \psi'(x_1).$$

The solution of the latter system of equations gives us the following

$$u(x) = 2\left(3 - 2\sqrt{2}\right)(x + 3) \text{ for all } x \in O_1 = (-3, x_1) = (-3, 2\sqrt{2} - 3). \tag{4.0.7}$$

By a similar reasoning around $x = 3$, we obtain that there exists $x_2 = 3 - 2\sqrt{2}$ such that

$$u(x) = -2\left(3 - 2\sqrt{2}\right)(x - 3) \text{ for all } x \in O_2 = (x_2, 3) = (3 - 2\sqrt{2}, 3). \tag{4.0.8}$$

Also,

$$\text{we claim that for every } x \in [x_1, x_2], \;\; u(x) = \psi(x). \tag{4.0.9}$$

Assume by contradiction that there exists $z \in (x_1, x_2)$ such that $u(z) > \psi(z)$. By continuity, $u - \psi > 0$ in a neighborhood of z. Let $O_z = (z_1, z_2)$ be the connected component of $\Omega \setminus I$ containing z. We now that there are $c_z, d_z \in \mathbb{R}$ such that $u(x) = c_z x + d_z$ for every $x \in O_z$. Since $z_1, z_2 \in \partial(\Omega \setminus I)$, $u(z_1) = \psi(z_1)$ and $u(z_2) = \psi(z_2)$. Thus,

$$u(x) - \psi(x) = (x - z_1)(x - z_2) \text{ for all } x \in (z_1, z_2).$$

The latter contradicts the fact that $u(z) - \psi(z) > 0$. Thus, claim (4.0.9) holds. Combining (4.0.7), (4.0.8) and (4.0.9), the solution of the obstacle problem is given by

$$u(x) = \begin{cases} 2\left(3 - 2\sqrt{2}\right)(x + 3) & \text{if } -3 < x < x_1 \\ \psi(x) & \text{if } x_1 \leq x \leq x_2 \\ -2\left(3 - 2\sqrt{2}\right)(x - 3) & \text{if } x_2 < x < 3, \end{cases} \tag{4.0.10}$$

where $x_2 = -x_1 = 3 - 2\sqrt{2}$.

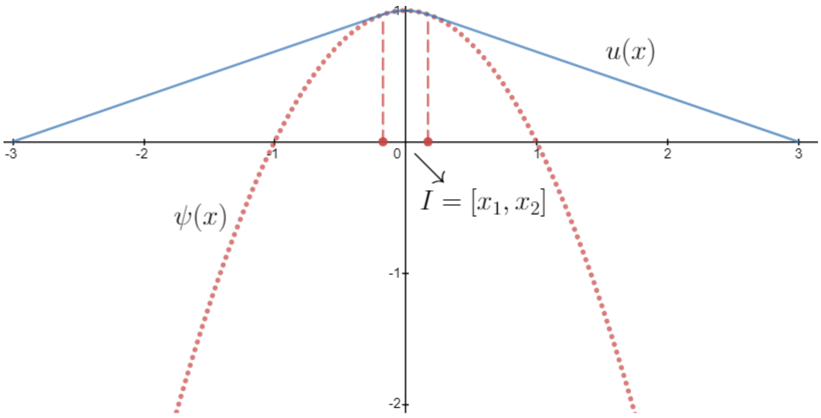

Figure 4.1: Solution of the O.P. of example (4.0.3).

Notice that indeed $u \in C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$. However, $u \notin C^2(\Omega)$. This means that even in the case when $f = 0$ and $\psi \in C^\infty(\Omega)$, we can not expect to have C^2-regularity for the solution.

Chapter 5

Regularity of free boundaries

The goal of this part of the work is to study the regularity of the *free boundary* (the boundary of the coincidence set). More precisely, we are going to see that if $\Gamma \subset \partial I$ is a C^1-hypersurface, then it is a C^∞-hypersurface. This fact is known since 1976, and was first proved by David Kinderlehrer and Louis Nirenberg (see [12]). Almost at the same time, Luis A. Caffarelli (see [4]) proved that near some special points the free boundary is a $C^{1,\gamma}$-hypersurface, for a given $\gamma \in (0, 1)$.

The result by Kinderlehrer and Nirenberg is proved essentially in two steps: the first consists, basically, in using a technical device (the Legendre transform) to translate the regularity of the free boundary into the regularity of a solution of a fully nonlinear differential equation. The second step consists in a subtle (non-direct) application of regularity theory for nonlinear elliptic operators.

Also, for simplicity we will limit ourselves to the study of the result from Kinderlehrer and Nirenberg when the dimension $N = 2$. However, the results presented in this chapter are still valid, with some additional technicalities, for $N > 2$ (see, e.g., [12] and [19]). Such a result uses a geometric tool known as *«Legendre transform»*, which we now present.

Before going to the details, we remark that Lewy and Stampacchia showed in [15] for the 2 dimensional case, that if Ω is bounded, convex and satisfies certain regularity; $\psi \in C^2(\overline{\Omega})$, $\psi < 0$ on $\partial\Omega$, ψ is strictly concave in Ω, and it is such that $\max_{x\in\Omega} \psi(x) = \psi(\overline{x}) = m > 0$; then the set $\Omega \setminus I$ is homeomorphic to an annulus. In particular, I is a non-empty set.

5.1 Legendre Transform

Given an interval $U \subset \mathbb{R}$, for any convex function $f : U \longrightarrow \mathbb{R}$ one can define a transformation of f called the *Legendre transform*. For simplicity in the exposition, we will limit ourselves to state the definition of such a transformation for strictly convex functions $f \in C^2(U)$.

Let $U \subset \mathbb{R}$ be an interval. Let $f : U \longrightarrow \mathbb{R}$ be a strictly convex function in $C^2(U)$. For any $x \in U$, $f''(x) > 0$. Thus, the function f' is invertible as a function from U to $Im(f')$. In this way, for any

$m \in Im(f')$ there exists a unique $x_m \in U$ such that $f'(x_m) = m$. Let $m \in Im(f')$, we define the *Legendre transform* of f at m, as

$$f^*(m) := mx_m - f(x_m).$$

Geometrically, $f^*(m)$ can be thought as the negative of the y-intercept of the line with slope m that is tangent to the graph of f at the point $(x_m, f(x_m))$ (see Figure 5.1).

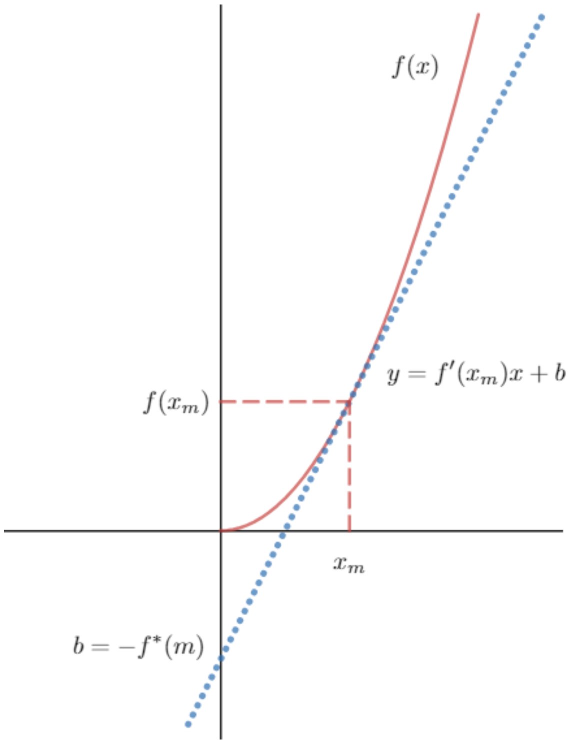

Figure 5.1: Geometric interpretation of f^*

Finally notice that, for every $m \in Im(f')$ it holds $m = f'(x_m)$, which implies that

$$\begin{aligned} \frac{df^*}{dm}(m) &= m\frac{dx_m}{dm} + x_m - f'(x_m)\frac{dx_m}{dm} \\ &= x_m. \end{aligned} \tag{5.1.1}$$

The latter equality shows one important property of f^*, its derivative at m is given by x_m. Thus, allowing us to write

$$(f^*)' = (f')^{-1}. \tag{5.1.2}$$

5.2 C^1 implies C^∞

From now on, we will assume that $\Omega = B_1(0) \subseteq \mathbb{R}^2$, and that the obstacle ψ is such that $\psi \in C^\infty(\overline{\Omega})$, $\psi < 0$ on $\partial\Omega$, and $-\Delta\psi > 0$ in Ω. Set

$$\mathbb{K} = \{v \in H^1_0(\Omega) : v \geq \psi \text{ a.e. in } \Omega\},$$

and let us consider the problem of

$$\text{finding } u \in \mathbb{K} \text{ such that } \int_\Omega \nabla u \cdot \nabla(v-u)dx \geq 0 \quad \forall v \in \mathbb{K}. \tag{5.2.1}$$

At this point, we already know that a unique solution u of problem (5.2.1) exists and it satisfies all of the following (see Chapter 3): $u \in C^{1,\alpha}(\overline{\Omega}) \cap H_0^1(\Omega) \cap H^{2,p}(\Omega) \cap C^{1,1}_{loc}(\Omega)$, for any $\alpha \in (0,1)$ and for any $p \in (1,\infty)$. A normalized solution of the obstacle problem is a solution $u \in C^{1,1}(B_1(0))$ of (5.2.1) satisfying

(a) $u \geq \psi$ in $B_1(0)$,

(b) $\Delta u = 0$ a.e. in $\Omega \setminus I = \{x \in B_1(0) : \; u > \psi\}$, and

(c) $0 \in \partial I$, i.e. 0 is a free boundary point.

Inequality in (a) is understood in the usual sense, since u and ψ are continuous. Notice that (b) holds, since $u \in H^{2,p}(\Omega)$, and it satisfies $Lu = 0$ in $\Omega \setminus I$ (see Theorem 2.3.12). The fact that $u \in C^{1,1}_{loc}(\Omega)$ implies that for each $x \in \Omega$ there exists $r > 0$ such that $u \in C^{1,1}(B_r(x))$, we take the ball $B_1(0)$ to simplify the proof.

Additionally, we assume that

$$u_{ij} \in C(\Gamma \cup (\Omega \setminus I)), \text{ for } 1 \leq i,j \leq 2 \text{ (see remark below).} \tag{5.2.2}$$

We assume that a portion of the free boundary is a 1-manifold of class C^1 (see [18] for a definition), i.e.

$$\Gamma \subseteq \partial I \text{ is a 1-manifold of class } C^1, \text{ with } 0 \in \Gamma. \tag{5.2.3}$$

Finally, we assume a technical condition:

$$\text{the inward unit normal to } \Omega \setminus I \text{ at } 0 \text{ is parallel to the positive } x_1 \text{ axis.} \tag{5.2.4}$$

Notice that such a vector exists since we are assuming that $0 \in \Gamma$ and Γ is a 1-manifold of class C^1.

Remark 5.2.1. Under the latter assumptions, we aim to prove that Γ is a 1-manifold of class C^1. First, let us discuss assumption (5.2.2). The facts that $u \in C^{1,1}(B_1(0))$ and that $\Delta u = 0$ a.e. in $\Omega \setminus I$ imply that $u \in C^\infty(\Omega \setminus I)$ (see Remark 1.3.10). Thus, justifying $u_{ij} \in C(\Omega \setminus I)$ for $1 \leq i,j \leq 2$. The justification of the continuity up to Γ is not at all trivial. In fact, this constitutes one of the major results of Luis A. Caffarelli in [4] (see Theorem 3 in [4]).
One of the implications of hypothesis (5.2.4), by virtue of the implicit function theorem, is that (see, e.g., [7] and [18]) there are numbers $\epsilon > 0$, $s > 0$, and a C^1-real valued function $h : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ such that

$$\Gamma \cap B_s(0) = \{(h(x_2), x_2) : \text{ for all } x_2 \in (-\epsilon, \epsilon)\}. \tag{5.2.5}$$

We begin by proving that the free boundary is contained in the domain Ω.

Lemma 5.2.2. *Let u be a normalized solution of the obstacle problem (as stated above). Let I be the coincidence set of the obstacle problem (as in Definition 2.3.10). Then $\partial I \subseteq \Omega$.*

Proof. Clearly, $\partial I \subseteq \overline{\Omega}$. Reasoning by contradiction, let us assume that $\partial\Omega \cap \partial I \neq \emptyset$. Let $x_0 \in \partial\Omega \cap \partial I$. Since $u \in H_0^1(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$, then we have that $u(x_0) = 0$. On the other hand, by hypothesis, $\psi(x_0) < 0$. The continuity of $u - \psi$ implies that there exists $\epsilon > 0$ such that

$$(u - \psi)\,|_{B_\epsilon(x_0)\cap\overline{\Omega}} > 0.$$

In particular, $u - \psi > 0$ in the set $B_\epsilon(x_0) \cap I$, which is a contradiction since $u = \psi$ in I. Therefore, $\partial\Omega \cap \partial I = \emptyset$.

☑

We now state and prove the main theorem of this chapter.

Theorem 5.2.3. *Let u be a normalized solution of the obstacle problem that satisfies (5.2.2). Let Γ be as in (5.2.3) and satisfying (5.2.4). Then there are a C^∞-real valued function g, and numbers $\delta > 0$, $r > 0$ such that*

$$\Gamma \cap B_r(0) = \{(g(x_2), x_2) : \text{ for all } x_2 \in (-\delta, \delta)\}. \tag{5.2.6}$$

This theorem says that in a neighborhood around 0 in $\mathbb{R}^2$, Γ can be viewed as the graph of a C^∞-function. The result in this theorem is still valid for every point of Γ: for a given $x \in \Gamma$ hypothesis (5.2.4) can be fulfilled after applying a rotation in $\mathbb{R}^2$ to the set Γ, and after applying a translation in $\mathbb{R}^2$ to Γ, we may always assume $x = 0$.

The proof of Thereom 5.2.3 follows the scheme from Petrosyan et. al. (see the proof of Theorem 6.17 in [19]). First, we show that the regularity of Γ in a neighborhood of 0, is related to the regularity of certain function v (defined in (5.2.10) below). Second, in the same way that in [19], we show (with more details than Petrosyan et. al.) that v satisfies certain fully nonlinear partial differential equation (see (5.2.16) below). Finally, one has to show that some regularity results are applicable to this differential equation.

Proof.

Step 1

Set $w = u - \psi$. Let us begin by observing that w satisfies the following

$$\begin{cases} \Delta w = -\Delta\psi & \text{in } \Omega \setminus I, \\ w = 0 & \text{on } \Gamma, \\ w_i = 0 & \text{on } \Gamma, \quad \text{for } i \in \{1, 2\}, \end{cases} \tag{5.2.7}$$

where w_i represents the $i-$partial derivative of w. The first equality in (5.2.7) follows from the fact that $\Delta u = 0$ in $\Omega \setminus I$. The second equality is just the fact that $u = \psi$ in I. The third equality holds because $u - \psi \in C^1(\Omega)$ achieves its minimum value over Ω at any point of I (recall $u \geq \psi$ in Ω, and $u = \psi$ in I). Thus, $u_i = \psi_i$ for any $i \in \{1, 2\}$ in I. In particular, the third equality in (5.2.7) holds in Γ.

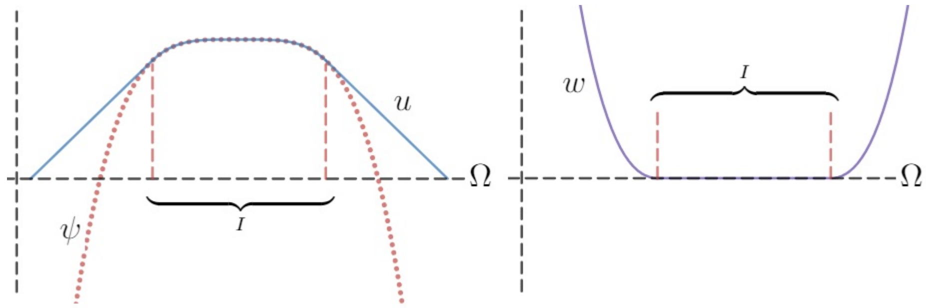

Figure 5.2: Graph of u, ψ and w

By hypothesis we already have that there are a C^1-real valued function h, and numbers $\epsilon > 0$, $s > 0$ such that (5.2.5) holds. In order to find the function g in the statement of the Theorem 5.2.3, we will use the Legendre transform.

There is a "natural" way of «*linearizing*» the portion Γ of the free boundary through the following transformation: for $x = (x_1, x_2) \in \Omega$, let us define

$$T(x) := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} w_1(x) \\ x_2 \end{bmatrix}. \tag{5.2.8}$$

By (5.2.7), for any $x \in \Gamma$

$$T(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}.$$

Thus, T maps Γ into a subset of $\{(y_1, y_2) \in \mathbb{R}^2 | \ y_1 = 0\}$.

Let us focus in the effect that T has over the region $\Omega \setminus I$. We claim that there exists $r \in (0, s)$ such that if $B_1 = B_r(0) \cap (\Omega \setminus I)$, then $T : B_1 \longrightarrow T(B_1)$ is a C^∞-diffeomorphism: recall that for $i \in \{1, 2\}$, $w_i = 0$ on Γ. In particular

$$\Gamma \subseteq \{x \in \Omega | w_i(x) = 0\}.$$

Therefore, by hypothesis (5.2.2), $\nabla w_i(0)$ is well defined and it is orthogonal to any tangent vector to Γ at 0, provided that $\nabla w_i(0) \neq 0$. Let us see that for $i = 1$, $\nabla w_i(0) \neq 0$. By (5.2.7) and the fact that the second derivatives of u belong to $C(\Gamma \cup (\Omega \setminus I))$, $\Delta w(0) = -\Delta \psi(0) > 0$ (positiveness of $-\Delta \psi(0)$ is given by hypothesis). In this way, if $w_{22}(0) = 0$ then $w_{11}(0) = -\Delta \psi(0) > 0$. Assume that $w_{22}(0) \neq 0$, then $\nabla w_2(0) \neq 0$, and

$$\nabla w_2(0) = k(1, 0), \text{ for a given } k \in \mathbb{R}.$$

The latter equation contradicts the fact that $w_{22}(0) \neq 0$. Arguing similarly it follows that $w_{12}(0) = 0$. We obtain that

$$DT(0) = \begin{bmatrix} w_{11}(0) & w_{12}(0) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\Delta\psi(0) & 0 \\ 0 & 1 \end{bmatrix}.$$

The fact that $-\Delta\psi(0) > 0$, implies that $\det(DT(0)) > 0$. Continuity of $\Delta\psi$, and hypothesis (5.2.2), imply that there exists $r \in (0, s)$ such that if $B = B_r(0) \cap (\Gamma \cup (\Omega \setminus I))$, then $\det(DT(\cdot)) > 0$ and $w_{11}(\cdot) > 0$ in B. The Inverse Function Theorem applied in $B_1 = B_r(0) \cap (\Omega \setminus I)$ says that $T(B_1)$ is open, and that $T : B_1 \longrightarrow T(B_1)$ is a C^∞-diffeomorphism (recall $w \in C^\infty(\Omega \setminus I)$), which proves the claim. Additionally, hypothesis (5.2.2) implies that $T : B \longrightarrow T(B)$ is a C^1-function.

Now we make use of the Legendre transform. Recall that $w_{11}(\cdot) > 0$ in B. In this way, the Legendre transform of $w(\cdot, x_2)$ is well-defined for each x_2 in $P_2(B) = \{x_2 \in \mathbb{R} | \ (x_1, x_2) \in B \text{ for some } x_1 \in \mathbb{R}\}$ (the domain of definition for each x_2 is explained below). Actually, to simplify even more the remaining part of the proof, inside $B_r(0)$ we can choose a neighborhood of 0 of the form $(\underline{x}_1, \overline{x}_1) \times (\underline{x}_2, \overline{x}_2)$. We rename $B_1 = (\underline{x}_1, \overline{x}_1) \times (\underline{x}_2, \overline{x}_2) \cap (\Omega \setminus I)$, $B_2 = (\underline{x}_1, \overline{x}_1) \times (\underline{x}_2, \overline{x}_2) \cap \Gamma$, and $B = B_1 \cup B_2$ (see Figure 5.3 and Figure 5.4).

For each $x_2 \in (\underline{x}_2, \overline{x}_2)$ there exists an interval $I_{x_2} := [x_1(x_2), \overline{x}_1) = [h(x_2), \overline{x}_1)$, (see condition (5.2.5)). In particular, for each $x_2 \in (\underline{x}_2, \overline{x}_2)$, $I_{x_2} \times \{x_2\} \subseteq B$.

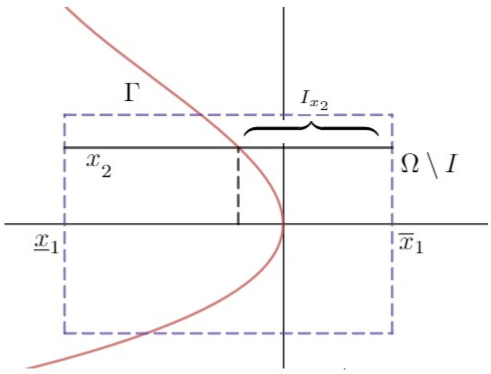

Figure 5.3: Portion of the free boundary Γ

Let $x_2 \in (\underline{x}_2, \overline{x}_2)$. By the previous discussion, we have an interval I_{x_2} such that for each $m \in Im\,(w_1(\cdot, x_2) : I_{x_2} \longrightarrow \mathbb{R})$ there is a unique point $x_1 = x_1(m, x_2) \in I_{x_2}$ such that $w_1(x_1, x_2) = m$. The Legendre transform of $w(\cdot, x_2)$ is given by

$$v(m, x_2) := m x_1(m, x_2) - w\big(x_1(m), x_2\big). \tag{5.2.9}$$

Using the definition of T in (5.2.8), and setting $x_1 = x_1(m, x_2)$, we get that $y_1 = m$ and $y_2 = x_2$. Thus, leaving us with

$$v(y_1, y_2) = y_1 x_1 - w(x_1, y_2), \qquad (y_1, y_2) \in T(B). \tag{5.2.10}$$

The latter shows that v is a well-defined function in $T(B)$. Also, for a given $x_2 \in (\underline{x}_2, \overline{x}_2)$, recalling that $x_1 = x_1(m, x_2)$ depends on m, the Legendre property (5.1.1) is translated into

$$\frac{\partial v}{\partial y_1}(y_1, y_2) = x_1. \tag{5.2.11}$$

We claim that there exists an open set $U \subset T(B_1)$ such that $v \in C^\infty(U \cup T(B_2))$, where U is such that a portion of its boundary is given by $T(B_2)$. However, in this step, we will only show in full detail how to find the set U and how to prove that $v \in C^\infty(U)$. The C^∞-extension of v up to $T(B_2)$ is proved at the end of step 2. Now, if such a claim were true, since for any $(x_1, x_2) \in B_2$, $y_1 = 0$ and $y_2 = x_2$, then by taking g as

$$g(x_2) := \frac{\partial v}{\partial y_1}(0, x_2) = x_1, \tag{5.2.12}$$

we would obtain the desired C^∞ function g.

Recall that $w \in C^\infty(B_1)$, and $w_{11}(\cdot) > 0$ in B. Thus, for any $x_2 \in (\underline{x}_2, \overline{x}_2)$, the function $w_1(\cdot, x_2) : I_{x_2} = [h(x_2), \overline{x}_1) \longrightarrow \mathbb{R}$ is strictly increasing, which implies

$$\max_{I_{x_2}} w_1(\cdot, x_2) = w_1(\overline{x}_1, x_2) > w_1(h(x_2), x_2) = 0.$$

We set $\overline{m} := \min_{[\underline{x}_2, \overline{x}_2]} w_1(\overline{x}_1, \cdot)$. Let us consider the three variables function $F : B_1 \times (0, \overline{m}) \longrightarrow \mathbb{R}$ given by $F(x_1, x_2, m) = w_1(x_1, x_2) - m$. By the same arguments we used to derive (5.2.9), for a fixed $(m', x_2') \in (0, \overline{m}) \times (\underline{x}_2, \overline{x}_2)$, there exists a unique $x_1 = x_1(m', x_2') \in I_{x_2}$ such that $F(x_1, x_2', m') = 0$. The implicit function theorem says that there exists a neighborhood V of (m', x_2') and a unique C^∞-function $x_1 : V \longrightarrow \mathbb{R}$ such that $w_1(x_1(m, x_2), x_2) - m = 0$ for all $(m, x_2) \in V$. Thus, the function $x_1 : (0, \overline{m}) \times (\underline{x}_2, \overline{x}_2) \longrightarrow \mathbb{R}$ locally given as above is C^∞. We set $U = (0, \overline{m}) \times (\underline{x}_2, \overline{x}_2)$. It follows that $v \in C^\infty(U)$.

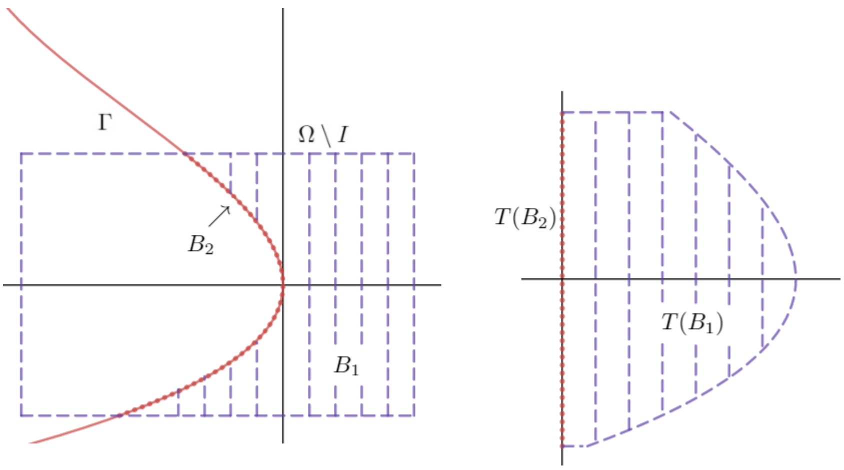

Figure 5.4: The effect of T on B_1 and B_2

Step 2

We now prove that v satisfies a fully nonlinear partial differential equation. Let us begin by noticing that by construction $U \subset T(B_1)$, and since T is a diffeomorphism of class $C^\infty(B_1)$, we know that

$$DT^{-1}(y) = \begin{bmatrix} \frac{1}{w_{11}(x)} & -\frac{w_{12}(x)}{w_{11}(x)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}, \tag{5.2.13}$$

for all $y \in U$, where we use the convention $y = T(x)$, i.e. $x = T^{-1}(y)$. Using the latter equality along with (5.2.11), we will prove that v can be seen as the solution of a partial differential equation. First, we adopt the following convention: subscripts for v represent the partial derivatives of v with respect to y_1 and/or y_2, and subscripts for w represent the partial derivatives of w with respect to x_1 and/or x_2. Similarly, the arguments of v and its derivatives are understood to be (y_1, y_2), and the arguments of w and its derivatives are understood to be (x_1, x_2), for the sake of simplicity we something omit these arguments. We obtain all of the following: in U (5.2.11) is the same as $v_1 = x_1$, which implies

$$v_{11} = \frac{\partial x_1}{\partial y_1} = \frac{1}{w_{11}}, \text{ and} \tag{5.2.14}$$

$$v_{12} = \frac{\partial x_1}{\partial y_2} = -\frac{w_{12}}{w_{11}} = -w_{12}v_{11}. \tag{5.2.15}$$

Recall that $v(y_1, y_2) = y_1 x_1 - w(x_1, y_2)$, from which we obtain (using the chain rule)

$$\begin{aligned} v_2(y_1, y_2) &= y_1 \frac{\partial x_1}{\partial y_2} - w_1(x_1, y_2)\frac{\partial x_1}{\partial y_2} - w_2(x_1, y_2) \\ &= -w_2(x_1, x_2) \qquad \text{(since } y_1 = w_1 \text{ and } y_2 = x_2\text{).} \end{aligned}$$

Also, using again the chain rule

$$\begin{aligned} v_{22}(y_2, y_2) &= -w_{12}(x_1, x_2)\frac{\partial x_1}{\partial y_2} - w_{22}(x_1, x_2) \\ &= -w_{12}v_{12} - w_{22} = \frac{v_{12}}{v_{11}}v_{12} - w_{22} \qquad \text{(by (5.2.11) and (5.2.15))} \\ &= \frac{v_{12}^2}{v_{11}} + w_{11} + \Delta\psi \qquad \text{(by (5.2.7), } \Delta w = -\Delta\psi\text{)} \\ &= \frac{v_{12}^2}{v_{11}} + \frac{1}{v_{11}} + \Delta\psi \qquad \text{(by (5.2.14)).} \end{aligned}$$

Observe that in the previous formula, $\Delta\psi$ is evaluated at $x = T^{-1}(y)$. After rewriting the latter expression we obtain

$$v_{11}v_{22} - v_{12}^2 - \Delta\psi\ v_{11} - 1 = 0,$$

i.e. v satisfies

$$\begin{aligned} v_{22} - \frac{v_{12}^2}{v_{11}} - \frac{1}{v_{11}} - \Delta\psi = 0 \text{ in } U, \\ v = 0 \text{ on } T(B_2), \end{aligned} \tag{5.2.16}$$

where the second identity is a consequence of the fact $T(B_2) \subseteq \{(y_1, y_2) \in \mathbb{R}^2 | \; y_1 = 0\}$, definition (5.2.10) and (5.2.7).

We now show the C^∞-regularity of v in $U \cup T(B_2)$. To do so, we enunciate adapted versions of Theorems 11.1' and 11.1 from Agmon, Douglis and Nirenberg in [2]. By the properties of the Legendre transform, for any $y \in U$, $x = T^{-1}(y) = (v_1(y_1, y_2), y_2)$. If we define the function G as

$$G(v_{11}, v_{12}, v_{22}, v_1, y_2) := v_{22} - \frac{v_{12}^2}{v_{11}} - \frac{1}{v_{11}} - \Delta\psi(v_1, y_2),$$

then G is a C^∞-function of the variables $(v_{11}, v_{12}, v_{22}, v_1, y_2)$ in the region where $v_{11} \neq 0$ (recall that $v_{11} > 0$ in $U \cup T(B_2)$). The equation $G(v_{11}, v_{12}, v_{22}, v_1, y_2) = 0$ in U is nonlinear and elliptic with respect to the solution v that we have found (see, e.g., the proof of Theorem 6.17 in [19] and Lemma 2.1 in [12] for a proof of the ellipticity).

Theorem 5.2.4 (Adaptation of Theorem 11.1' in [2]). *Let $v \in C^2(U \cup T(B_2))$. Assume that v is a solution of problem (5.2.16). Assume that G has continuous derivatives with respect to all of its arguments. Then $v \in C^{2,\alpha}(U \cup T(B_2))$ for any $\alpha \in (0, 1)$.*

Theorem 5.2.5 (Adaptation of Theorem 11.1 in [2]). *Let $\alpha \in (0, 1)$. Let v be a solution of problem (5.2.16) such that $v \in C^{2,\alpha}(U \cup T(B_2))$. Assume that G has continuous derivatives of any order with respect to all of its arguments. Then $v \in C^\infty(U \cup T(B_2))$.*

In order to apply Theorem 5.2.4, we have to show that $v \in C^2(U \cup T(B_2))$. Notice that for any $(y_1, y_2) \in U$, $v_1(y_1, y_2) = x_1 = T_1^{-1}(y)$ and $v_2(y_1, y_2) = -w_2(v_1(y_1, y_2), y_2)$. From (5.2.13) and (5.2.2), $DT^{-1}(y_1, y_2)$ can be extended continuously up to $T(B_2)$. Thus, T^{-1} is continuously differentiable in U up to $T(B_2)$. It follows that v_1 can be extended to a C^1 function in $U \cup T(B_2)$. The latter fact along with hypothesis (5.2.2) imply that v_2 can be extended to a C^1-function in $U \cup T(B_2)$. It follows that both, v_1 and v_2 belong to $C^1(U \cup T(B_2))$. Therefore $v \in C^2(U \cup T(B_2))$. Theorem 5.2.4 implies that in fact $v \in C^{2,\alpha}(U \cup T(B_2))$ for every $\alpha \in (0, 1)$. Next, Theorem 5.2.5 says that $v \in C^\infty(U \cup T(B_2))$.

☑

Chapter 6

Appendix

Throughout this appendix, Ω will denote a bounded open subset of $\mathbb{R}^N$. For $k \in \mathbb{N}$, the space of $k-$differentiable functions up to the boundary is defined as

$$\begin{aligned} C^k\left(\overline{\Omega}\right) = \big\{ f : \overline{\Omega} \longrightarrow \mathbb{R},\ &\text{such that there exists an open set } S, \text{ with} \\ &\overline{\Omega} \subseteq S, \text{ and a function } \phi \in C^k(S) \text{ such that } \phi\,|_{\overline{\Omega}} = f \big\}. \end{aligned} \tag{6.0.1}$$

For $\lambda \in (0,1)$, we define the space of $\lambda-$Hölder continuous functions as

$$C^{0,\lambda}\left(\overline{\Omega}\right) = \left\{ u : \overline{\Omega} \longrightarrow \mathbb{R},\ \text{such that}\ \ [u]_\lambda = \sup_{x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\lambda} < \infty \right\}. \tag{6.0.2}$$

The previous definition can be extended to $k-$differentiable functions as follows: for a given $k \in \mathbb{N}$, and $\lambda \in (0,1)$

$$C^{k,\lambda}\left(\overline{\Omega}\right) = \left\{ f \in C^k(\overline{\Omega}),\ \text{whose partial derivatives of order } k \text{ are in } C^{0,\lambda}\left(\overline{\Omega}\right) \right\}. \tag{6.0.3}$$

In what follows, we will state some definitions and theorems from Sobolev's spaces. Some key references are [1] and [3].

Definition 6.0.1. Let $m \in \mathbb{N}$. Let $u \in C^m\left(\overline{\Omega}\right)$. The following norm can be defined

$$\|u\|_{H^{m,s}(\Omega)} \coloneqq \sum_{0\le|\alpha|\le m} \|D^\alpha u\|_{L^s(\Omega)}\,,\ 1 \le s < \infty, \tag{6.0.4}$$

where D^α stands for $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}$ is a differential operator, $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}^N$ is a multi-index, and $|\alpha| = \sum_i \alpha_i$ is the norm of $\alpha \in \mathbb{N}^N$.

Let $m \in \mathbb{N}$, and let $s \in [1,\infty)$. We will denote with $H^{m,s}(\Omega)$ the completion of the space $C^m\left(\overline{\Omega}\right)$ with respect to the norm (6.0.4). This space is called the Sobolev space of order m with weak derivatives in $L^s(\Omega)$ (we will clarify soon what we mean by weak derivatives). The space $H^{m,s}(\Omega)$ is a Banach space (see Theorem 3.3 in [1]). Additionally, if we restrict $s \in (1,\infty)$, $H^{m,s}(\Omega)$ is reflexive (see Theorem 3.6 in [1]). For simplicity, we denote $H^m(\Omega) = H^{m,2}(\Omega)$.

Remark 6.0.2. We say that $\partial\Omega$ is Lipschitz, if for every point in $\partial\Omega$ there are a neighborhood U of the point in $\mathbb{R}^N$, an open subset $V \subseteq \mathbb{R}^{N-1}$, and a Lipschitz continuous function $f : V \longrightarrow \mathbb{R}$ such that $\partial\Omega \cap U$ can be seen as the graph in $\mathbb{R}^N$ of the function f. Let $m \in \mathbb{N}$, and let $s \in [1, \infty)$. By Theorem 3.17 in [1], if $\partial\Omega$ is Lipschitz, then $H^{m,s}(\Omega)$ is characterized as the set of functions

$$u \in L^s(\Omega) \text{ such that for every } \alpha \in \mathbb{N}^N,\ |\alpha| \leq m \text{ there exists } g_\alpha \in L^s(\Omega),\ \text{ satistying}$$
$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega g_\alpha(x)\varphi(x) dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{6.0.5}$$

The function g_α is called the weak derivative of u of order α. It can be proved that such a function g_α is unique in $L^s(\Omega)$, and so it is denoted as $D^\alpha u$.

The space $H^m(\Omega)$ is a Hilbert space with inner product defined as: for $u, v \in H^m(\Omega)$

$$(u, v)_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \left(\int_\Omega D^\alpha u(x) D^\alpha v(x)\ dx \right). \tag{6.0.6}$$

We also introduce two mores spaces: $H_0^{m,s}(\Omega)$ is defined as the completion of $C_0^\infty(\overline{\Omega})$ with respect to the norm (6.0.4), and $H^{m,\infty}(\Omega)$ is defined as the class of $C^{m-1}\left(\overline{\Omega}\right)$ functions whose derivatives of order $m-1$ are Lipschitz in $\overline{\Omega}$. When $s = 2$, we denote $H_0^m(\Omega) = H_0^{m,2}(\Omega)$.

Remark 6.0.3. In the definition of $H^{1,s}(\Omega)$ the space $C^1\left(\overline{\Omega}\right)$ can be replaced by $C^{0,1}\left(\overline{\Omega}\right) = H^{1,\infty}(\Omega)$ (see chapter II, section 4 of [13] for further details).

A very useful result in Sobolev spaces is about how to compare the L^2-norms of a function and its weak derivatives.

Theorem 6.0.4 (Poincaré's Inequality). *There exists $\beta > 0$ such that*

$$\int_\Omega |\varphi(x)|^2\, dx \leq \beta \int_\Omega |\nabla\varphi(x)|^2\, dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

See Theorem 6.30 in [1] for a proof. The latter theorem shows that the norm (6.0.4), when $m = 1$ and $s = 2$, is equivalent to the norm given by $\|\varphi\|_{H_0^1(\Omega)} := \|\nabla\varphi\|_{L^2(\Omega)}$ for any $\varphi \in H_0^1(\Omega)$, since

$$\|\varphi\|_{H^{1,2}(\Omega)} \leq \left(\sqrt{\beta} + 1\right) \|\nabla\varphi\|_{L^2(\Omega)} \leq \left(\sqrt{\beta} + 1\right) \|\varphi\|_{H^{1,2}(\Omega)}.$$

Let $s \in (1, \infty)$. Let $H^{-1,s'}(\Omega) = (H_0^{1,s}(\Omega))'$ denote the normed dual space of $H_0^{1,s}(\Omega)$, where s and s' are such that $1/s + 1/s' = 1$. It is possible to characterize this dual space using the notion of weak derivatives as follows.

Theorem 6.0.5. *Let $s \in (1, \infty)$. Let $T \in H^{-1,s'}(\Omega)$. Then there exists $f_0, \cdots, f_N \in L^{s'}(\Omega)$ such that*

$$T(\varphi) = \int_\Omega f_0 \varphi - \sum_{i=1}^N \int_\Omega f_i \frac{\partial\varphi}{\partial x_i}, \qquad \varphi \in H_0^{1,s}(\Omega). \tag{6.0.7}$$

For a proof of the previous theorem see Proposition 9.20 in [3]. Finally, we enunciate some Sobolev embedding theorems (see Theorem 4.12 and Theorem 6.3 in [1]).

Theorem 6.0.6. *Let Ω be a smooth domain. Let $m \in \mathbb{N}$, and $j \geq 0$ be integers. Let $s \in [1, \infty)$. All of the following hold true.*

(i) If either $ms > N$ or $m = N$ and $s = 1$, then

$$H^{m,s} \subseteq L^q(\Omega), \quad \text{for any } q \in [s, \infty].$$

(ii) If $ms > N > (m-1)s$, then

$$H^{j+m,s}(\Omega) \subseteq C^{j,\lambda}(\overline{\Omega}), \text{ for any } \lambda \in \left(0, m - \frac{N}{s}\right].$$

(iii) If $N = (m-1)s$, then

$$H^{j+m,s}(\Omega) \subseteq C^{j,\lambda}(\overline{\Omega}), \text{ for any } \lambda \in (0, 1)\,.$$

(iv) If $ms > N \geq (m-1)s$, then the following embedding is compact

$$H^{j+m,s}(\Omega) \subseteq C^{j,\lambda}(\overline{\Omega}), \text{ for any } \lambda \in \left(0, m - \frac{N}{s}\right).$$

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