Kinetic wealth-exchange model of economic growth

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Abstract

This thesis proposes a Kinetic wealth-exchange model of economic growth (KWEMEG) which explores the effects of saving of production, tax benefits with redistribution, and exchange aversion, over the wealth and the money distributions. The model is achieved extending the microeconomic formalism of the Chakraborti and Chakrabarti model (CCM), based on the Cobb-Douglas utility functions describing the preferences for consumption. The maximization of this functions, subjected to certain constraints over consumption leads to general rules of interaction between economic agents that induces the emergence of a conservative regime, where the total wealth of the system remains constant in time, and the interactions between agents are reduced to mere monetary exchanges; and a non-conservative regime, where the wealth and income increase exponentially, inducing an effect of economic growth with a constant rate. The particular cases of the KWEMEG set three new models in the context of econophysics that extend the CCM. All the cases are presented separately in the thesis, and their macroscopic dynamics are studied using the Boltzmann kinetic equation and Monte Carlo simulations, which allows to fit the emergent distributions to the gamma probability density function and to establish analytical relations for its parameters and the evolution of the economic inequality in terms of the Gini index. In addition, it is possible to study a mean field approach for self-similar distributions that allows to discuss the possibility of Pareto tails in the non-conservative regime. As a general result, the model leads to Piketty’s second fundamental law of capitalism as an emergent property, by adding an effect of income due to salary. The ideas developed in this work tie some of the problems of modern economics with the traditional speech of kinetic exchange models of markets, and propose an effective bridge between the microfoundation based on the maximization of the utility function and the non-conservative models. Both results are new in the context of econophysics.

Keywords: Kinetic exchange models of markets, economic growth, Boltzmann equation, mean field approximation, Piketty, Solow model
Resumen

En esta tesis se propone un modelo cinético de intercambio de riqueza con crecimiento económico (KWEMEG) en el que se explora el efecto del ahorro de la producción, los impuestos con beneficio tributario y la aversión al intercambio, sobre las distribuciones de riqueza y de dinero. Este modelo se obtiene a partir de una extensión al formalismo microeconómico del modelo de Chakraborti y Chakrabarti (CCM), donde las preferencias de consumo se describen mediante funciones de utilidad tipo Cobb-Douglas. Al maximizar estas funciones, sujetas a ciertas restricciones sobre el consumo, se obtienen reglas generales de interacción entre agentes económicos, caracterizadas por la emergencia de un régimen conservativo, donde la riqueza total se mantiene constante en el tiempo y las interacciones entre agentes se reducen a meros intercambios monetarios; y un régimen no conservativo, en el cual la riqueza crece exponencialmente, induciendo un efecto de crecimiento económico a tasa constante. Como casos particulares del KWEMEG se presentan por separado en esta tesis tres nuevos modelos en el contexto de la econofísica, que extienden el CCM. Su dinámica macroscópica se estudia analíticamente en todos los casos, utilizando la ecuación cinética de Boltzmann; y numéricamente por medio de simulaciones de Monte Carlo. Esto permite ajustar las distribuciones emergentes a densidades de probabilidad tipo gamma, y establecer relaciones analíticas para sus parámetros, así como para el índice de Gini, con el cual se estudia el nivel de desigualdad en las distribuciones. Adicionalmente, en el régimen no conservativo es posible estudiar distribuciones auto-semejantes utilizando una aproximación de campo medio, que abre la puerta a la discusión sobre la posible existencia de colas de Pareto. Como un resultado general, al agregar en el modelo un efecto relacionado con el ingreso por salario, se obtiene como una propiedad emergente la Segunda ley fundamental del capitalismo, de Piketty. Las ideas propuestas en este trabajo atan algunos de los problemas de la economía moderna con el discurso tradicional de los modelos cinéticos de intercambio y proponen un puente que conecta de forma efectiva la microfundamentación basada en la maximización de la función de utilidad y los modelos no conservativos. Ambos resultados son novedosos en el contexto de la econofísica.

Keywords: Modelos cinéticos de intercambio, crecimiento económico, ecuación de Boltzmann, aproximación de campo medio, Piketty, modelo de Solow
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1. Introduction

There is a great temptation to consider the exchanges of money which occur in economic interactions as analogous to the exchanges of energy which occur in physical shocks between molecules. In the loosest possible terms, both kinds of interactions should lead to similar states of equilibrium. That is, one should be able to explain the law of income distribution by a model similar to that used in statistical thermodynamics: many authors have done so explicitly, and all the others of whom we know have done so implicitly.


Since the late XIXth century, it is generally acknowledged that the tail of an income distribution follows a power law, i.e. the income of the richest portion of the population is well fitted by the function \( f(m) = Am^{-\gamma} \), where \( m \) is the level of personal income, \( A \) is a positive constant, and the power \( \gamma \) is related to the fractal dimension in the context of fractal geometry [1]. This result was first presented in 1897 by the italian economist Vilfredo Pareto, as an empirical regularity satisfied by the whole of the distribution of personal income in many countries [2]. However, modern data shows evidence of power laws only for the 5 % – 10 % of the population with highest personal income [3, 4], in spite of observations made by Pareto.

Beyond Pareto’s law, there exist a strong disagreement about the bulk of the distribution, which is frequently studied using distributions coming from the exponential family, as the Boltzmann-Gibbs, the Gompertz and the gamma [4, 5, 6]. As an attempt to shed light on this discussion, many theoretical studies in econophysics have put efforts towards the development of agent-based models inspired by the kinetic theory of gases [7, 8, 9, 10, 11, 12]. From this complex systems perspective, the mechanisms driving the emergence of wealth and money distributions are described analogous to the energy exchanges between pairs of molecules of a gas. This approach was first suggested by Mandelbrot in 1960 [13] and it is labeled in the econophysics literature as kinetic exchange models of markets.

According to this analogy, an economic transaction between any pair of agents \( i \) and \( j \), owning an amount of wealth \( w_i(t) \equiv w_i \) and \( w_j(t) \equiv w_j \), is expressed mathematically as \( w_i^* = w_i + \Delta w_i, \ w_j^* = w_j + \Delta w_j \), where \( \Delta w_i \) and \( \Delta w_j \) describe the change of the individual wealth...
at every time $t$ that a trading occurs. The amounts $\Delta w_{i,j}$, can be equal or different to each other, depending on the mathematical form of the microeconomic process that is modeled. Therefore, their definitions take into account different exogenous and endogenous economic factors, as well as the individual wealth that each agent possesses before the interaction. A general scheme for this dynamics is presented in figure 1-1.

In particular, for the case $\Delta w = \Delta w_i = -\Delta w_j$, the economic process is similar to an elastic collision, which implies that total wealth is conserved in time. The validity of this simplification is a very controversial issue from the point of view of economics [14]. On the one hand, it is possible to define a law of conservation over the total physical money of an economic system, under this assumption the gas-like dynamics are reduced to mere monetary exchanges. Nevertheless, the notion of wealth is wider than simple money, it includes different types of tangible and intangible goods which are produced and consumed, but also that rent and lose their value in time. Therefore, it is clear that this quantity is not conserved at any significantly long time scale.

The first agent-based model in the spirit of Mandelbrot’s hypothesis was proposed far from the scenario of econophysics, by sociologist John Angle, who studied the emergence of the economic inequality as a consequence of the interactions between economic agents competing against each other for wealth [15]. This microscopic dynamics, known as “Inequality process”, are modeled in a similar way of the random interactions due to stochastic processes in the kinetic theory of gases. Using Monte Carlo simulations, Angle shows that the model

![Fig. 1-1.: Gas-like mechanism of exchange between economic agents in kinetic models of markets. An economic system is idealized as a set of N agents interacting by pairs in an analogous way to the molecules of a gas that exchange energy via elastic and inelastic collisions. The variation on the individual wealth is a consequence of the microeconomic activity at every time-step, this can be related to a conservative or a non-conservative process.](image-url)
reaches an equilibrium state, where the data follows a gamma distribution. According to the author, this distribution pattern is able to reproduce properly the stylized facts of the wealth inequality, in spite of the lack of characteristic power law behavior in the richest fraction of the population.

In the context of econophysics, a similar approach to that of Angle was independently introduced, first by Ispolatov et al. in 1998 [16], and extended parallelly in 2000 by Dragulescu and Yakovenko, and by Chakraborti and Chakrabarti [8, 9]. The Dragulescu and Yakovenko Model (DYM) studies the money distribution in a closed economic system where total money and total population remain constant in time. The amount exchanged is defined as a random fraction of the money possessed by each pair of economic agents involved in a trading. This interactions lead to the Boltzmann-Gibbs distribution, the same equilibrium distribution obtained from the elastic collisions between pairs of molecules in an isolated gas, such that the economic temperature is defined as the average money per agent.

On the other hand, the Chakraborti and Chakrabarti model (CCM) introduces a “saving propensity parameter” $\lambda \in [0,1]$, which limits the amount of money available for each agent to exchange, according to the equation: $\Delta w = (1-\lambda) [\epsilon(w_i + w_j) - w_i]$ [9]. The total number of economic agents and the total money of the system are set to be conserved quantities, as in the DYM. Nevertheless, the Boltzmann-Gibbs distribution is not obtained in the model, because the process is not reversible [17]. In contrast, it is shown by Patriarca that for fixed values of $\lambda$ the money distributions are well fitted by the gamma probability density function, in such a way that the level of inequality decreases for lower values of $\lambda$ [18].

In the limit case $\lambda = 0$, the dynamics of the CCM reduces to the DYM. In addition, its isomorphism to the Inequality Process was shown by Angle in 2010 [19]. The mechanisms of money exchange in these three models have attracted criticism from some economist who described them as inaccurate from the point of view of economics [14, 20]. According to Thomas Lux, their rules of interaction preserve the “theft” spirit from Angle’s Surplus theory of social stratification [15], ignoring the “genuinely voluntary exchange”, which is more accurate in modern economies. As a proposal for supplement the dynamics of these models, Lux reviews a model of two goods general equilibrium derived by Silver et al. [21], where the distributions are obtained from the dynamic of an economy with stochastic exchange preferences, that are introduced through a Cobb-Douglas utility function [22].

This microeconomic formalism was extended by Chakrabarti et al. in 2009, achieving a powerful microfoundation for the pairwise dynamics behind the CCM and the DYM [23, 24]. In this framework, the economic activity is assumed to be a consequence of the production of perishable goods, in such a way that every agent tries to buy a fraction of the other agent’s production by selling a portion of their own production, and using part of their money. The preferences for these goods are introduced in the same manner of the Silver’s model, by means of a Cobb-Douglas utility function. After every transaction, both agents consume the
whole amount of goods. This fact guarantees the conservation of wealth in time, due to the fact that the individual wealth of the agents is changed only by the amount of money traded, but not by an amount of goods added to their individual stock.

According to Gallegati et al., this conservation law is a natural consequence of models essentially focused on exchange without taking into account production, which inherently leads to a confusion between the economic concepts of transaction and income [14]. In the microeconomic framework developed by Chakrabarti, the goods are assumed to be perishable, therefore, the only source of income is, in fact, the money exchanged, in spite of the production of goods at every time-step when a trading occurs. Nevertheless, this microeconomic formulation constitutes an extremely powerful result aiming to connect the macro and micro spheres of economics, which is exploited in the this thesis with a non-conservative perspective, as it is discussed later.

Out of the conservation laws hypothesized in the previous models, there also exist important non-conservative approaches to the study of wealth distributions in the context of econophysics. On the whole, the non-conservative dynamics of this class of models are introduced using stochastic mutiplicative terms which induce the exponential increasing of the average wealth in time [10, 11, 12]. The effects of these increasing dynamics on the wealth distribution have been extensively investigated since 2000 by different studies which include additional elements as taxation, rent and inflation [25, 26, 27, 28]. However, the microeconomics behind these models are not clear, and are apparently disconnected to the previous discussion.

A first approach in this context was proposed by Bouchaud and Mézard in 2000. In their model, the evolution of the individual wealth of an economic agent $i$ is given by the differential equation:

$$\frac{dw_i}{dt} = \eta_i w_i + \sum_{j(\neq i)} J_{ij} w_j - \sum_{j(\neq i)} J_{ji} w_i,$$

where $\eta_i$ is a Gaussian stochastic variable with mean $\mu$ and variance $\sigma^2$, describing the spontaneous increasing or decreasing of wealth in time due to investment, and the terms $J_{ij}$ and $J_{ji}$ sketch the interactions between economic agents, where an agent $j$ spends an amount $J_{ij}$ of their money trading with an agent $i$, and vice-versa. Note that the two deterministic terms of the equation evoke the kinetic nature described in the DYM and the CCM.

The dynamics of the Bouchaud and Mézard model (BMM) are solved using a mean field approach. Assuming that an agent $i$ feels an average influence from their environment $\langle w \rangle = \frac{1}{N} \sum_i w_i$ and setting $J/N \equiv J_{ij} = J_{ji}$ it is found that the average wealth of the system behaves as the exponential function $\langle w(t) \rangle = \langle w_0 \rangle \exp (\mu + \sigma^2) t$. Additionally, by normalizing the individual wealth using the average $\tilde{w} = \frac{w}{\langle w \rangle}$, the differential equation for $\tilde{w}$ describes a conservative diffusive-like process, which can be studied by associating a Fokker Planck equation with equilibrium solution $f_{eq}(\tilde{w}) = \left( \frac{\gamma - 1}{\Gamma[\gamma]} \right) \left( \exp \left[ -\frac{\gamma - 1}{\tilde{w}^{\gamma}} \right] \right)$, where $\gamma = 1 + \frac{J}{\sigma^2}$.

For large values of $\tilde{w}$, the equilibrium distribution behaves as a power-law. This emerging behaviour from the BMM is a characteristic feature of models considering the wealth growth
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as a consequence of linear stochastic terms. An analogous model of inelastic scattering based on this basic scheme was proposed by Slanina in 2004 [11]. In this model, the interactions are characterized by a kinetic-like process plus a stochastic term proportional to the individual wealth which drives its increasing in time. Thus, the economic agents interact following the symmetric rules: 

\[
\Delta w_i = \beta [w_i - w_j] + gw_i, \quad \Delta w_j = -\beta [w_i - w_j] + gw_j.
\]

Note that the first term in both equations corresponds to a conservative wealth exchange \(\Delta w = \beta [w_i - w_j]\), transferred from one agent to the other, while the term \(gw_{i,j}\) corresponds to the return over their individual wealth.

The evolution in time of the distribution is studied by associating a spatially homogeneous Boltzmann equation for the two molecule distribution, where the time is normalized as \(\tau = \frac{2t}{N}\). This normalization is a consequence of the fact that at every time-step, only two agents from the set of \(N/2\) couples are interacting following the rules \(\Delta w_{i,j}\). In the limit of continuous trading, the solutions of the Slanina Model (SLM) are the same distributions of the BMM, where the power law behavior emerges for higher values of individual wealth. A mathematical generalization for this Boltzmann-type approach was proposed by Toscani et al. in 2006 using a weak formulation of the Boltzmann kinetic equation [29, 30].

The connection between the SLM and the BMM obeys the nature of the stochastic multiplicative processes behind them, due to the fact that the emergence of Pareto tails is a general feature of this kind of processes [31]. A first extension aimed to introduce this ideas into the framework of the CCM was proposed in 2004 by Chatterjee et al. [32]. Their work extends the additive dynamics of the CCM to the case of a random saving propensity parameter \(\lambda_t\), uniformly distributed in the domain \([0, 1]\), in such a way that the top levels of the money distribution reproduce a power law. After this approach, a non-conservative extension was sketched by Basu and Mohanty in 2008, where it is proposed a spontaneous growing market driven by a white noise \(\eta\), analogous to the SLM, added to the term \(\Delta w\) [33]. The stochastic noise guarantees the emergence of a power law behavior in the higher levels of wealth, preserving the gamma distribution of the original model.

This non-conservative dynamics are implicitly related to the concept of economic growth, due to the fact that the increasing of wealth in time is driven by the economic activity of the agents, which corresponds to some kind of income. The study of the economic growth is essential to understand the dynamics behind the economic inequality [34]. The conventional economics textbook perspective claims that the inequality is good for generating initiatives as drivers of the economic growth. However, in a large series of empirical works based on regressions of GPD series for different countries, the results show a negative correlation between the income inequality and the growth of the GPD [35]. Furthermore, the long run behavior of the economic growth has important implications on the convergence/divergence of the wealth distribution. According to Thomas Piketty, the ratio between wealth and income increases in a scenario of low economic growth, which implies a higher concentration...
of wealth in few hands [36].

In recent years, the development of new models introducing basic facts from the theory of economic growth has attracted the attention of some studies in econophysics [37, 38, 39, 40]. However, these models preserve the stochastic nature of the non-conservative approach, without a clear microfoundation governing the spontaneous growth. In macroeconomics, the economic growth is a measurement of the increasing of production in time caused by the microeconomic activity of the entities [41]. Therefore, a gas-like model capturing the dynamics of the economic growth should take into account the increasing of production as a driver of the increasing of income and wealth.

The main objective of this thesis is to propose a kinetic wealth-exchange model of economic growth (KWEMEG) by extending the microeconomic formalism of the CCM with a more general non-conservative perspective. This goal is achieved by introducing two additional parameters governing the preferences for goods in the utility functions of the CCM. First, a saving factor $s$ limiting the amounts of goods consumed after every transaction, in such a way that a fraction of total production is saved by the economic agents. And second, a control parameter $q$ over the power of preferences for goods, which is reflected on the emergent distributions as a taxation rate that induces redistribution of wealth, and reduces the level inequality.

The rules of interaction of the KWEMEG are obtained by maximizing the utility of the agents, using the Lagrange multipliers method. In general, the results depend on $q$ and $s$, but also on the parameter $\lambda$ introduced in the CCM, which is redefined as the exchange aversion of the agents. The saving of production $s$ separates the KWEMEG into two different regimes. First, the dynamics are completely conservative if $s = 0$, under this condition the wealth is reduced to mere money. Conversely, if $s \neq 0$, the wealth is not conserved in time, and it increases exponentially, as in the models described above. This result proposes an effective bridge between the microfoundation based on the utility function and the non-conservative models, that has not been shown in econophysics until now.

In the frame of the conservative and non-conservative regimes defined by $s$. The limit cases $\lambda = 0$ and $q = 0$ allows to differentiate between four models, that are studied separately in the thesis. The simplest case is the CCM where $q = s = 0$, and a conservative extension of this is obtained by introducing $q \neq 0$, which induces the effect of taxation and redistribution. This model is new in the context of the kinetic exchange models of markets. On the other hand, the non-conservative cases set the conditions to study the effect of economic growth, characterized by a constant rate $g$, which is estimated in terms of the exogenous parameters $s$ and $\lambda$, and is not affected by the effect of the taxation rate $q$.

Introducing an additional source of income, related to a salary paid to the agents, in the non-conservative regime, the behavior of the ratio between total wealth and total income
tends, in the long term, to the ratio $s/g$. This result shows a strong connection between the dynamics of the KWEMEG and the so-called, Piketty’s second fundamental law of capitalism [36], which in fact, establishes the same conclusion as a regularity for modern economies, based on empirical observations, and on the Solow theory of economic growth [42]. This original result can be consulted in references [43, 44], its relevance obeys the fact that the KWEMEG proposes an effective way to tie new ideas coming from modern economics to the regular speech of the kinetic exchange models of market.

In all the cases described above, the emergent distributions are studied numerically using Monte Carlo simulations. In most of the cases, the simulated data is well fitted by the gamma probability density function. Furthermore, the macroscopic dynamics are studied analytically using a Boltzmann-type approach based on the weak formulation of the Boltzmann equation proposed by Toscani et al. [29, 30]. The latter allows to compute the moments of the emergent distributions, that are used to obtain analytical relations for the parameters of the gamma. Only for the cases where $s = 0$, the system reaches to a steady state, characterized by the maximization of the Shannon entropy. However, in the non-conservative models, it is possible to study quasi-steady distributions, where the entropy is maximized, by normalizing the wealth dividing by the average wealth. Under certain conditions, such normalization opens the possibility of studying self-similar distributions with Pareto tails. However this results are proposed as a forthcoming research, following references [10, 29].

The previous ideas are presented along this thesis in the following structure. First, the microfoundation of the KWEMEG and its particular conservative and non-conservative cases are discussed in chapter 2. On the other hand, the analytical macroscopic approach, based on the Boltzmann equation, to the emergent distributions of money and wealth from both regimes, is presented in chapter 3. The study of the conservative regime is proposed in chapter 4, while chapter 5 is devoted to the non-conservative regime. In both cases, the dynamics of the model are approached numerically using Monte Carlo simulations, and analytically by means of the Boltzmann kinetic equation. In particular, for the non-conservative cases it is possible to study certain quasi-steady state for self-similar distributions, by means of a mean field approximation. The implications in the long term macroeconomics of the economic growth are presented in chapter 6, where it is considered an additional source of income related to salary. Finally, conclusions and remarked ideas are discussed in chapter 7.
2. Kinetic wealth-exchange model of economic growth (KWEMEG)

In the context of economics, the economic growth is suggested by many authors as one of the key factors governing social inequality. A period of high economic growth can contribute to reduce inequality, but also to increase it, depending on the policies behind it. In general, the measurement of the economic growth is made through the Gross Domestic Product (GDP), which corresponds both to the total output of goods and services of a nation, and its total income [41]. Thus, this variable is computed by summing over all the expenditure of a nation, or over its total production. An accurate calculation takes into account different factors as the salaries paid to labor, the return on wealth, the firms' profits and the negative effect of depreciation on wealth, and subtracts the foreign investment. However, in a very simplified closed economy, the economic growth can be measured as the mere increasing of total production \( Y \).

The production of a nation obeys the microeconomic activity of the entities, such as firms or any kind of producers, that compose the economy. This entities transform different production factors, in order to increase their total output. In neoclassical economics, this process is modeled by means of an aggregate production function, which establish the technological ability to transform different inputs into services and goods. The most important production factors are the capital or wealth \(^1\) \( W \) that is allocated for the production process, and the labor \( L \) of the workers involved in this. In most of the cases, the neoclassical aggregate production function is defined as a monotonic increasing function \( Y = F(W, L) \). When it satisfies the property of constant returns to scale: \( F(zL, zW) = zF(L, W) \), it is possible to re-scale the macroeconomic production as a function of one unit of labor, by dividing by \( L \).

A noteworthy mathematical approach to the analysis of the relation between \( W \) and \( L \) and the production of a nation was developed by Solow in 1956 [42]. This simple model of economic growth is addressed from a pure macroeconomic perspective where the accumulation of wealth and the increasing of labor units are measured over the whole of the entities that

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\(^1\)For simplicity, the words capital and wealth are used as synonyms along this work, in accordance with reference [36]. Formally, the capital is interpreted as a fraction of wealth directly employed in the production process. However, in the context of the KWEMEG all the wealth possessed by the economic agents can be used to produce goods.
comprise the system. In the main, the increasing of production obeys the supply and demand for goods that comes from the consume and investment, and the later is driven by the saving rate of production. Under this macro-dynamics a high saving rate induces a higher level of economic growth until a steady state in the economics, by means of the increasing of wealth. In the long term, the ratio between total production and total wealth $W/Y$ tends to saturate into the ratio between the saving rate an the rate of economic growth $s/g$. The results from the Solow growth model have important implication in modern economics and their validity for the analysis of empirical data has a wide range, even in present days [36].

The study of the economic growth is addressed in this thesis from the perspective of kinetic exchange models of markets, where interactions between agents constitute the cause for the emergence of the economic growth, in contrast with the solow model, where the use of the neoclassical aggregation function with constant returns to scale allows “to make contact with microeconomics” through the notion of aggregation over all the labor units [45]. In order to introduce this phenomenon in that context, it is proposed a general scheme of a simple economy with a fixed number of economic agents $N$, which produce and trade goods in a market. The consumption of goods is restricted by a saving parameter $s$ in such a way that a fraction of the total production must be added to the individual wealth of the agents. In absence of population growth, the economic growth is governed by the increasing of wealth in time, in a similar way as in the Solow model, due to the fact that there is no increment in the number of workers, and only if $s = 0$ the wealth is conserved in time.

### 2.1. Microeconomic foundation of the gas-like models of money and wealth exchange

The formulation of the Kinetic wealth-exchange model of economic growth (KWEMEG) starts from a microeconomic scheme based on the models with stochastic preferences introduced in references [21, 23, 24]. At every time-step, two agents $i$ and $j$ are randomly selected to trade in the market, from the set of $N$ economic agents. Before trading, the agent $i$ produces an amount $X$ of certain good, and the agent $j$ produces an amount $Z$ of another kind of good. In addition, the initial wealth possessed by each agent is determined by the variables $w_i = w_i(t)$ and $w_j = w_j(t)$. When meeting in the market, both agents buy an amount of the other’s agent good by selling a fraction of their own good, and also using a portion of their own wealth. At the end of the trading, the agent $i$ ends with $x_i$ and $z_i$ of their good and the other’s agent good, and total wealth $w_i^* = w_i(t + 1)$. On the other hand, the agent $j$ ends with $x_j$, $z_j$ and $w_j^* = w_j(t + 1)$.

The remaining amounts after trading $x_{i,j}$ and $z_{i,j}$ are not automatically added to the wealth of the agents. Before that, they consume certain fractions $A = 1 - s$ of these goods, defined
as the complement of the saving of production $s$, according to their preferences, which are
modeled using the Cobb-Douglas utility functions:

$$U_i(x_i, z_i, w^*_i) = [Ax_i]^{\theta - q}[Az_i]^{\phi + q}[w^*_i]^\lambda, \quad (2-1)$$

$$U_j(x_j, z_j, w^*_j) = [Ax_j]^{\theta + q}[Az_j]^{\phi - q}[w^*_j]^\lambda. \quad (2-2)$$

For simplicity, the powers $\theta - q$, $\phi + q$ and $\lambda$ are normalized to 1 as $\theta + \phi + \lambda = 1$. Thus, the utility functions are constant return to scale. This fact allows to introduce the stochastic nature of each transaction as it is shown latter. The parameter $q$ was introduced without modifying the normalization condition, however its meaning, as well as the meaning of $\theta$, $\phi$ and $\lambda$ make sense when the dynamics between agents are obtained.

The output of the utility functions 2-1, 2-2 can be interpreted as a number that corresponds to the level of benefit or any kind of profit gained by each agent while consuming an amount of each commodity [46, 47]. Note that, these functions determine the preferences for consuming goods, but also govern the amounts traded in the market and the remaining wealth of the agents. On the other hand, the restrictions over consumption are completely determined by the initial budget that each agent possesses before trading. These budget constraints are defined by summing the amount of wealth $w_{i,j}(t)$ and the total production of each agent $X$ and $Z$, multiplied by their corresponding selling prices $p_x$ and $p_z$, as follows:

$$Ap_x x_i + Ap_z z_i + w^*_i = w_i + p_x X, \quad (2-3)$$

$$Ap_x x_j + Ap_z z_j + w^*_j = w_j + p_z Z. \quad (2-4)$$

At every trading, both agents pursuit to maximize their utility subjected to the budget constraints. Thus, it is possible to define the following Lagrangian functions in terms of the utility functions 2-1, 2-2 and constraints 2-3, 2-4, introducing the Lagrange multipliers $\mu_i$ and $\mu_j$:

$$\mathcal{L}_i(x_i, z_i, w^*_i, \mu_i) = [Ax_i]^{\theta - q}[Az_i]^{\phi + q}[w^*_i]^\lambda - \mu_i [Ap_x x_i + Ap_z z_i + w^*_i - w_i - p_x X], \quad (2-5)$$

$$\mathcal{L}_j(x_j, z_j, w^*_j, \mu_j) = [Ax_j]^{\theta + q}[Az_j]^{\phi - q}[w^*_j]^\lambda - \mu_j [Ap_x x_j + Ap_z z_j + w^*_j - w_j - p_z Z]. \quad (2-6)$$

This maximization problem is solved by computing the partial derivatives equal to zero

$$\frac{\partial \mathcal{L}_i}{\partial x_i} = \frac{\partial \mathcal{L}_j}{\partial z_j} = \frac{\partial \mathcal{L}_j}{\partial w^*_i} = \frac{\partial \mathcal{L}_j}{\partial \mu_{i,j}} = 0.$$

The demand functions $x_{i,j}$, $z_{i,j}$ and $w^*_{i,j}$ are obtained solving the equations system for each agent. Thus, for the case of the agent $i$ the solution is:

$$x_i = \frac{\theta - q}{\lambda Ap_x} w^*_i, \quad (2-7)$$

$$z_i = \frac{\phi + q}{\lambda Ap_z} w^*_i, \quad (2-8)$$
\[ w^*_i = \lambda [w_i + p_x X]. \] (2-9)

On the other hand, the demand functions for the agent \( j \) are:

\[ x^*_j = \theta + q \frac{w^*_j}{\lambda p_x}, \] (2-10)

\[ z^*_j = \phi - q \frac{w^*_j}{\lambda p_x}, \] (2-11)

\[ w^*_j = \lambda [w_j + p_z Z]. \] (2-12)

Note that the demand functions 2-7, 2-8, 2-10 and 2-11 decrease as the prices of the goods increase, following the law of supply and demand, i.e. the higher the price of a commodity the smaller is the quantity that is bought by consumers. Additionally, the wealth of both agents is proportional to their total production and their initial wealth.

Under the assumption that demand perfectly matches supply, the market conditions are \( x_i + x_j = X \) and \( z_i + z_j = Z \). The clearing prices of the goods \( \hat{p}_x, \hat{p}_z \) are obtained by solving these equations using the demand functions 2-7, 2-8, 2-10, 2-11, the result is:

\[ \hat{p}_x = \frac{A\theta [w_i + w_j] - q A [w_i - w_j] + 2 q (\theta + \phi) w_i}{X (\lambda + A - 1)(A + 2q)}, \] (2-13)

\[ \hat{p}_z = \frac{A\phi [w_i + w_j] + q A [w_i - w_j] + 2 q (\theta + \phi) w_j}{Z (\lambda + A - 1)(A + 2q)}. \] (2-14)

The evolution of individual wealth in terms of the parameters \( A, \theta, \phi, \lambda \) and \( q \) can be obtained by replacing the prices in equations 2-9 and 2-12. This expressions are deterministic when the values of all the parameters are fixed, but, in the main, the uniqueness of transactions in a market is better captured through a stochastic process. This microeconomic dynamic is captured using the normalization condition \( \theta + \phi + \lambda = 1 \) and defining the variable \( \varepsilon = \frac{\theta}{\theta + \phi} \).

For fixed values of \( \lambda \), if \( \theta \) is uniformly distributed over the domain \( [0, 1 - \lambda] \), then \( \varepsilon \) is uniformly distributed over the domain \( [0, 1] \). The resulting expressions are:

\[ \hat{p}_x = \frac{A\varepsilon (1 - \lambda) [w_i + w_j] - q A [w_i - w_j] + 2 q (1 - \lambda) w_i}{X (\lambda + A - 1)(A + 2q)}, \] (2-15)

\[ \hat{p}_z = \frac{A(1 - \varepsilon)(1 - \lambda) [w_i + w_j] + q A [w_i - w_j] + 2 q(1 - \lambda) w_j}{Z (\lambda + A - 1)(A + 2q)}. \] (2-16)

The previous formalism was first introduced in econophysics by Chakrabarti et al.[23, 24], inspired by the Silver’s model with stochastic preferences [21]. Its main strength is the possibility of modeling non-trivial interactions between economic agents leading to well known emergent distributions patterns, from a clear microeconomic perspective. In the context of the Chakrabarti and Chakrabarti model (CCM), the whole production of goods is consumed at every transaction, i.e. \( A = 1 \). This restriction avoids the possibility of economic growth.
and implies that the total wealth of the system is conserved in time. Therefore, the production only constitutes a motivation to trade in a market, without any impact in the long term behavior of the system.

This conservation law imposed in the CCM is contrary to the expected effect of savings in economics. In general, the capital stock of a system must be altered by the total production of its agents in presence of a rate of saving different to zero [48]. Note that the parameter $\lambda$ is introduced in the dynamics of the CCM as the “saving propensity” of the agents, although, it only limits the amount of wealth available to exchange at every transaction. For the case $\lambda = 1$ there is no interaction between agents. In this order of ideas, $\lambda$ is renamed as the exchange aversion of the economic agents, and the effect of savings are captured by introducing the parameter $s = 1 - A$ that restricts the consumption over the production.

### 2.2. Conservative models of money exchange

Until now, the concept wealth have been used generically, to make reference to the stock of money and goods possessed by the agents before and after each transaction. However, when the production is assumed to be perishable, each agent consumes the whole amount of goods traded in the market and the saving of production is set $s = 0$. Therefore, the only source of income added to their stock at every transaction is the amount of money corresponding to the exchanged goods, which is fixed by means of the clearing prices 2-29, 2-30. Note that, in this case a conservation law is applying over the total money of the system, which is valid from the point of view of economics. On the other hand, if $s \neq 0$, a fraction of total production is saved, and the wealth does not remain constant in time.

In this order of ideas, the behavior of the parameter $s$ in the limit $s = 0$ and outside this limit allows to separate the model into a conservative and a non-conservative regime, which are studied independently in the thesis. Nevertheless, despite this distinction, the variables $w$ is used both to represent money and wealth, and keeping in mind that in the case $s = 0$ the wealth of the agents reduces to mere money.

#### 2.2.1. Chakraborti and Chakrabarti model, the limit case $s = q = 0$

In particular, the limit case $s = q = 0$ induces to the following simplification of the clearing prices 2-15, 2-16, which leads to the exchange rule of the Chakraborti and Chakrabarti model (CCM) [9]

$$
\hat{p}_x = \frac{\varepsilon(1 - \lambda)}{X\lambda}[w_i + w_j],
$$

(2-17)
\[ \dot{p}_z = \frac{(1 - \varepsilon)(1 - \lambda)}{Z\lambda}[w_i + w_j]. \] 

(2-18)

The evolution in time of the individual money is obtained replacing these result in equations 2-9, 2-12:

\[ w_i^* = \lambda w_i + \varepsilon(1 - \lambda)[w_i + w_j], \] 

(2-19)

\[ w_j^* = \lambda w_j + (1 - \varepsilon)(1 - \lambda)[w_i + w_j]. \] 

(2-20)

These equations can be expressed in the form \( w_i^* = w_i + \Delta w \) and \( w_j^* = w_j - \Delta w \) by adding and subtracting \( w_i \) and \( w_j \) in the corresponding expression. Computing the algebraic steps it is found:

\[ \Delta w = (1 - \lambda) \left[ \varepsilon(w_i + w_j) - w_i \right]. \] 

(2-21)

The dynamics of this model have been extensively investigated in a series of papers published along 20 years, setting a paradigm in the study of wealth inequality form the perspective of econophysics. Its main result, concerning to the economic inequality, is that the money follows a gamma distribution, such that, if \( \lambda \) becomes higher, then the level inequality in the distribution decreases. This macroscopic dynamics are approached in chapter 4 from the perspective of the Boltzmann equation.

### 2.2.2. Taxation scheme for money re-distribution

Another interesting limit case of the KWEMEG, is the taxation scheme introduced by the parameter \( q \) in the conservative regime, inducing an effect of money redistribution. In this context, the clearing prices 2-15, 2-16 are reduced considering \( s = 0 \) and \( q \neq 0 \) as follows:

\[ \dot{p}_x = \frac{\varepsilon(1 - \lambda)[w_i + w_j] - q[w_i - w_j] + 2q(1 - \lambda)w_i}{X\lambda(1 + 2q)}. \] 

(2-22)

\[ \dot{p}_z = \frac{(1 - \varepsilon)(1 - \lambda)[w_i + w_j] + q[w_i - w_j] + 2q(1 - \lambda)w_j}{Z\lambda(1 + 2q)}. \] 

(2-23)

Thus, the evolution of the individual money is given by:

\[ w_i^* = \lambda w_i + \frac{\varepsilon(1 - \lambda)[w_i + w_j] - q[w_i - w_j] + 2q(1 - \lambda)w_i}{1 + 2q}, \] 

(2-24)

\[ w_j^* = \lambda w_j + \frac{(1 - \varepsilon)(1 - \lambda)[w_i + w_j] + q[w_i - w_j] + 2q(1 - \lambda)w_j}{1 + 2q}. \] 

(2-25)
2.3 Non-conservative models of wealth exchange

Operating the same computations as in the CCM, i.e. adding and subtracting $w_i$ and $w_j$ in the corresponding expression, it is obtained:

\[ w_i^* = w_i + \frac{1 - \lambda}{1 + 2q} \varepsilon (w_i + w_j) - w_i - \frac{q}{1 + 2q} [w_i - w_j], \quad (2-26) \]

\[ w_j^* = w_j + \frac{1 - \lambda}{1 + 2q} [-\varepsilon (w_i + w_j) + w_i] + \frac{q}{1 + 2q} [w_i - w_j]. \quad (2-27) \]

Note that the effect of the taxation rate $q$ does not affect the conservativeness of total money, due to the fact that equations 2-26 and 2-27 hold the relation $w_i^* + w_j^* = w_i + w_j$. Therefore, the conservative dynamics of the model are expressed in the fashion $w_i^* = w_i + \Delta w$, $w_j^* = w_j - \Delta w$, where $\Delta w$ is given by:

\[ \Delta w = \frac{1 - \lambda}{1 + 2q} \varepsilon (w_i + w_j) - w_i - \frac{q}{1 + 2q} [w_i - w_j]. \quad (2-28) \]

The first term of the exchange rules is analogous to the dynamics of the CCM, i.e. $\Delta w = (1 - \lambda) [\varepsilon (w_i + w_j) - w_i]$, where the increasing of $\lambda$ reduces the level of inequality of the emergent distributions by reducing the propensity of the agents to exchange in the market. On the other hand, the second term corresponds to a redistribution effect controlling the money inequality, where a portion of the difference of money between agents is given to the poorest one. This redistribution scheme has the nature of a taxation on private wealth (in this case money), and is similar to the taxation scheme introduced by During et al. in the context of non-conservative models with multiplicative stochastic growth of wealth [27].

2.3. Non-conservative models of wealth exchange

In contrast with the previous cases, the non-conservative regime rises when the saving of production is different to zero ($s \neq 0$). Assuming $q = 0$ and recalling that $A = 1 - s$, the clearing prices 2-15, 2-16 become:

\[ \hat{p}_x = \frac{\varepsilon (1 - \lambda)}{X(\lambda - s)} [w_i + w_j], \quad (2-29) \]

\[ \hat{p}_z = \frac{(1 - \varepsilon)(1 - \lambda)}{Z(\lambda - s)} [w_i + w_j]. \quad (2-30) \]

The change on the individual wealth is written in the same way of previous models, replacing this prices in equations 2-9, 2-12, and reorganizing the resulting expression by adding and subtracting $w_i$ and $w_j$:

\[ w_i^* = w_i + \frac{1 - \lambda}{\lambda - s} \varepsilon (w_i + w_j) - w_i + \frac{1 - \lambda}{\lambda - s} s w_i, \quad (2-31) \]
\[ w^*_j = w_j + \frac{1 - \lambda}{\lambda - s} \lambda [-\varepsilon (w_i + w_j) + w_i] + \frac{1 - \lambda}{\lambda - s} s w_j. \]  

(2-32)

Therefore, the amounts \( \Delta w_i \) and \( \Delta w_j \), governing the evolution of individual wealth in time, are identified as:

\[ \Delta w_i = \frac{1 - \lambda}{\lambda - s} \lambda (\varepsilon (w_i + w_j) - w_i) + \frac{1 - \lambda}{\lambda - s} s w_i, \]  

(2-33)

\[ \Delta w_j = \frac{1 - \lambda}{\lambda - s} \lambda [-\varepsilon (w_i + w_j) + w_i] + \frac{1 - \lambda}{\lambda - s} s w_j. \]  

(2-34)

Note that \( \Delta w_i \neq \Delta w_j \), unless \( s = 0 \). The first term in both equations correspond to an exchanged amount between agents \( \Delta w = \frac{1 - \lambda}{\lambda - s} \lambda \{\varepsilon [w_i + w_j] - w_i\} \), while the second term is analogous to the return over the individual wealth of the Slanina model [11], where \( g = \frac{1 - \lambda}{\lambda - s} s \) is the rate of increasing of wealth which is related to the economic growth, due to the fact that the only production factor considered in the model is the individual wealth. In the case \( s = 0 \), equations 2-33, 2-34 reduce to the dynamics of the CCM, where total wealth is conserved in time, i.e. \( g = 0 \). Conversely, for \( s \neq 0 \) the increasing of personal wealth, by means of savings [43, 44], is reflected in the total wealth of the system \( W \) as:

\[ W(t + 1) = \sum_{k=1}^{N} w_k(t + 1) = \sum_{k=1}^{N} w_k(t) + [\Delta w_i + \Delta w_j] = W(t) + g[w_i(t) + w_j(t)]. \]  

(2-35)

A similar effect on total wealth is obtained for \( q \neq 0 \). In this case, the change on the individual wealth can be written by computing the same algebra than in previous cases as:

\[ \Delta w_i = \frac{(1 - \lambda)(1 - s)}{(\lambda - s)(1 - s + 2q)} \{\varepsilon \lambda (w_i + w_j) - \lambda w_i + s w_i\} + \frac{q}{(\lambda - s)(1 - s + 2q)} \{\lambda (w_j - w_i) + s [2w_i - \lambda (w_i + w_j)]\}. \]  

(2-36)

\[ \Delta w_j = \frac{(1 - \lambda)(1 - s)}{(\lambda - s)(1 - s + 2q)} \{-\varepsilon \lambda (w_i + w_j) + \lambda w_i + s w_j\} + \frac{q}{(\lambda - s)(1 - s + 2q)} \{\lambda (w_i - w_j) + s [2w_j - \lambda (w_i + w_j)]\}. \]  

(2-37)

The first term in both equations is analogous to the dynamics described by 2-33 and 2-34, which induce the increasing of wealth in time. On the other hand, the second term has the nature of a taxation acting over the difference of wealth, as in equation 2-28, but also over the increasing of wealth due to saving of production. This relations satisfy the equation for the evolution of total wealth 2-35, which implies that, in fact, the only factor governing the wealth growth is the saving of production \( s \).

The increasing of the production of the agents is directly related to the wealth growth in the non-conservative regime. At every time, total production is computed as \( Y = p_x X + p_z Z \).
Replacing equations 2-15 and 2-16 it is obtained:

\[
Y = \hat{p}_x X + \hat{p}_z Z \\
= \frac{\varepsilon(1-s)(1-\lambda)[w_i + w_j] - q(1-s)[w_i - w_j] + 2q(1-\lambda)w_i}{(\lambda-s)(1-s+2q)} \\
+ \frac{(1-\varepsilon)(1-s)(1-\lambda)[w_i + w_j] + q(1-s)[w_i - w_j] + 2q(1-\lambda)w_j}{(\lambda-s)(1-s+2q)} \\
= \frac{1-\lambda}{\lambda-s}[w_i + w_j].
\] (2-38)

Therefore, the economic growth is expected to be a consequence of the increase of wealth in time, in the non-conservative regime. This discussion is revisited in chapter 6.

In comparison with the models of stochastic growth [10, 11, 12], it is expected for the cases \( q = 0 \) and \( q \neq 0 \), that the average wealth of the system increases as \( \langle w \rangle \propto e^{\gamma \tau} \), where \( \tau = 2t/N \). Note that the increasing of wealth in time is defined in this class of models as an analogy with an energy flux from the outside of the system. This idea is accurate as a physico-mathematical approach to the modeling of wealth growth, but does not reflect any microeconomic process. In contrast, the dynamics of the KWEMEG have a clear microeconomic foundation based in the utility functions 2-1, 2-2.

### 2.4. Monte Carlo algorithm for the dynamics of the KWEMEG

The study of the dynamics described in the previous sections is tackled in this thesis in two ways. First, an analytical approach is achieved by associating a Boltzmann equation to the economic dynamics, which allows to establish approximated mathematical relations over the behavior of the moments of the distribution and its asymptotic behavior. This method has been extensively studied by Toscani et al. [29, 30], and it takes relevance in the next chapter. On the other hand, all the models are studied using numerical methods based on the Monte Carlo Algorithm, which allows to extract the statistical properties of the emergent distributions of the models.

In general, the simulation strategy applied for all the models obeys the following logic. At every time-step, two economic agents \( i \) and \( j \) are randomly selected from a pool of \( N = 1000 \) agents. The production of the agent \( i \) is computed as \( p_x X \) and the production of the agent \( j \) is \( p_z Z \), where the values \( p_{x,z} \) are the clearing prices 2-15 and 2-16. The sum over both goods multiplied by them respective prices corresponds to total production at every time-step, which according to the macroeconomic theory is equivalent to the total income of the system. Note that, in the non-conservative models, due to the growing dynamics of the...
individual wealth moving the prices of goods, it is expected that this quantity increases its value in time, proportionally to the average wealth.

The individual wealth of each agent changes following equations 2-36 and 2-37, and their particular cases. In order to guarantee the uniqueness of every economic trading, a different value of the stochastic variable $\varepsilon$ is generated at every Monte Carlo time-step. This variable, as well as the random selection of the economic agents, is computed using a pseudo-random number generator based on the Mersenne Twister algorithm with a period of $2^{19937} - 1$ [49]. All the results presented in this work were obtained taking an average of 1000 ensembles, which satisfy the initial condition, $w_k(0) = 1$ for $k = 1, 2, \ldots, N$, i.e. every economic agent starts with one unity of initial wealth. Subsequently, the system evolves following the dynamics described above for fixed values of $\lambda$, $q$ and $s$, in such a way that $\lambda > s$. This constraint excludes the state of debt for any transaction, but also avoids the divergence induced by the factor $\frac{1}{(\lambda-s)}$ over the variables $\Delta w_{i,j}$ and $p_{x,z}$. Thus, the domains considered for each parameter in this thesis are: $q, \lambda \in [0, 1]$ and $s \in [0, \lambda)$. 
3. Boltzmann-type approach to the evolution of money and wealth distributions

The pairwise dynamics between economic agents, introduced in the previous chapter, share, in a broad sense, the same nature with the collisions occurred in dilute gases. Therefore, it is expected to use a similar description for the statistical properties of both systems. For the case of a dilute gas, the average distance between molecules \((\frac{V}{N})^{\frac{1}{3}}\) is sufficiently larger in comparison with the DeBroglie wave-length \(\lambda_{BD}\), that any interaction between molecules can be neglected [50]. Under these conditions, the gas describes a classical motion where each molecule is moving free until it collides into another. Every collision event, changes the velocity of both molecules and the direction of their motion instantaneously. After that, each one describes again a free motion, before a new binary collision, involving another molecule, occurs.

The description of such system is made through the probability density function \(f\), which is defined in a \(6N\)-dimensional phase space, where each molecule has associated three freedom degrees related to its coordinates \(x_i\) and three more, corresponding to their velocities \(\xi_i\). Therefore, the product \(f(x,\xi,t)d^3xd^3\xi\) describes the average number of molecules in the infinitesimal volume \(dV = d^3xd^3\xi\). In the case of a completely free motion, the evolution in time of the density distribution obeys the Liouville’s theorem, where \(\frac{df}{dt} = 0\). However, this statement is not true in presence of binary collisions between molecules. Thus, it is expected that the evolution of \(f\) depends on an additional collision term:

\[
\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}}. \tag{3-1}
\]

In the most general case, under the influence of an external force \(F\), the total derivative reads \(\frac{df}{dt} = \frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_\xi f\), which gives place to the Boltzmann Transport Equation:

\[
\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_\xi f = \left(\frac{\partial f}{\partial t}\right)_{\text{collision}}. \tag{3-2}
\]
However, for a free dilute gas, 3-2 reduces to:
\[
\frac{\partial f}{\partial t} = -\xi \cdot \nabla_x f + \left( \frac{\partial f}{\partial t} \right)_{\text{collision}}.
\] (3-3)

This equation describes the total change of the distribution function in a given point of the phase space. The term \(dV \xi \cdot \nabla_x f\) corresponds to the decrease per unit time of the number of molecules in the infinitesimal volume \(dV\) [50]. On the other hand, the increasing and decreasing of the density function, as consequence of a collision where the velocities of two molecules are transformed from \([\xi, \xi_1]\) into \([\xi^*, \xi_1^*]\), is described by \(dV \left( \frac{\partial f}{\partial t} \right)_{\text{collision}}\). This factor is named as collision operator (or collision integral) and labeled using \(Q(f, f)\).

In the context of Kinetic Theory, the derivation of this operator can be achieved based on the characteristics and principles of the dilute gases [50]. On the whole, it is defined in terms of the differential cross-section \(d\Omega = \frac{\beta[\xi, \xi_1] \to [\xi^*, \xi_1^*]}{||\xi - \xi_1||} \xi_1^* d\xi^*\), where \(\beta[\xi, \xi_1] \to [\xi^*, \xi_1^*]\) is the collision rate per unit time. However, in this chapter, its formulation is adapted for kinetic exchange models of markets, exploiting the similarity between the physical and the economic phenomena.

When a trading occurs in the market, the individual wealth of a pair of economic agents is transformed as \([w, v] \to [w^*, v^*]\), in analogy to the change of the velocities due to a binary collision. This process can be described in terms of the following equations:
\[
w^* = p_1 w + q_1 v, \tag{3-4}
\]
\[
v^* = p_2 w + q_2 v, \tag{3-5}
\]

where the variables \(w \equiv w_i(t), \) \(v \equiv w_j\) have been introduced replacing the index notation, with the purpose of simplifying the equations along this chapter, and \(p_{1,2}, q_{1,2}\) give the detail of the transaction.

The coefficients \(p_{1,2}, q_{1,2}\) depend on a set of constant and stochastic variables \(\zeta\). Thus, the distribution density is defined as an explicit function \(f(w, \zeta, t)\) of the individual wealth \(w\), the time, and the vector \(\zeta\), such that the product \(f(w, \zeta, t) dw\) describes the total number of agents with certain level of wealth \(dw\), where \(w \in \mathbb{R}_+\). Note that, the associated phase space for \(f\) is the N-dimensional space set by the freedom degrees \(w\), where \(dw\) constitutes an infinitesimal volume, due to the fact that the parameters \(\zeta\) are exogenous. In this order of ideas, the total derivative in equation 3-3 is expanded, in terms of the partial derivatives of \(w\) and \(t\). The interpretation of the collision operator is still valid, in this framework, i.e. the product \(Q(f, f) dw\) corresponds to the increasing of the total number of agents with a level of wealth \(dw\) due to transactions.

Considering an interaction between two agents with individual wealth \(w\) and \(v\), in the ranges \(dw, dv\), acquiring the values \(w^*\) and \(v^*\), the total number of such transactions per unit time
and volume $dw$ is proportional to the product between the number of agents in the volume $dw$, i.e. $f(w, \zeta, t)dw$, and the probability that a transaction occurs. Similarly, the probability of the transactions is proportional to the number of agents per unit volume $f(v, \zeta, t)dv$ multiplied by the ranges of the individual wealth acquired after the transaction $dw^*$ and $dv^*$. Thus, the total number of transactions $[w, v] \to [w^*, v^*]$ may be written as:

$$\beta_{[w,v] \to [w^*,v^*]} f(w, \zeta, t) f(v, \zeta, t) dw dv dw^* dv^*. \quad (3-6)$$

The coefficient $\beta_{[w,v] \to [w^*,v^*]}$ is the rate of transactions per unit time, analogous to the collision rate for a dilute gas in one dimension. In general, the variation of the probability density $f(w, \zeta, t)$ depends on two kinds of transactions. On the one hand, a transaction of the type $[w, v] \to [v^*, w^*]$, where an agent with individual wealth $w$ leaves the phase space volume $dw$, and the opposite transaction $[w^*, v^*] \to [w, v]$, that brings into $dw$ an agent with original wealth outside that level. The first kind of transactions are referred in the context of the Kinetic Theory as losses and the total number of such losses per unit time in the volume $dw$ are computed as:

$$L = dw \int \beta_{[w,v] \to [w^*,v^*]} f(w, \zeta, t) f(v, \zeta, t) dv dw^* dv^*. \quad (3-7)$$

In addition, the second kind corresponds to the gains, which are computed as:

$$G = dw \int \beta_{[w^*,v^*] \to [w,v]} f(w^*, \zeta, t) f(v^*, \zeta, t) dv dw^* dv^*. \quad (3-8)$$

Using equations 3-7 and 3-8, the increase of the density distribution due to transactions can be computed as $Q(f,f) dw = G - L$, therefore, the Boltzmann equation for models of market reads as follows:

$$\frac{df(w, \zeta, t)}{dt} = \int \left\{ \beta_{[w^*,v^*] \to [w,v]} f(w^*, \zeta, t) f(v^*, \zeta, t) \right. \\
\left. - \beta_{[w,v] \to [w^*,v^*]} f(w, \zeta, t) f(v, \zeta, t) \right\} dv dw^* dv^*, \quad (3-9)$$

If the trading satisfies the time-reversible symmetry, then the rates of transaction in both processes are the same:

$$\beta_{[w^*,v^*] \to [w,v]} = \beta_{[w,v] \to [w^*,v^*]}. \quad (3-10)$$

In addition, the principle of detailed balance establishes that the total number of agents 3-6 in both interactions must be equal:

$$\beta_{[w,v] \to [w^*,v^*]} f(w, \zeta, t) f(v, \zeta, t) dw dv dw^* dv^* = \beta_{[w^*,v^*] \to [w,v]} f(w^*, \zeta, t) f(v^*, \zeta, t) dw dv dw^* dv^*, \quad (3-11)$$
Therefore, at equilibrium state, the density distribution satisfies the condition:

\[ f(w, \zeta, t)f(v, \zeta, t) = f(w^*, \zeta, t)f(v^*, \zeta, t). \]  

This condition implies that the solution is given by the Boltzmann-Gibbs distribution. In the context of econophysics, the time reversal symmetry is satisfied by the Dragulescu and Yakovenko model (DYM), where the transactions between economic agents are defined as in terms of a proportional fraction of \( v + w \) that is transferred from one agent to the other. In fact, under this scheme, the distribution of money follows the function \( f(w) = c \exp(-w/\langle w \rangle) \), where the mean \( \langle w \rangle \) corresponds to the temperature of the system [8].

In contrast with the DYM, none of the underlying cases of the Kinetic wealth-exchange model of economic growth (KWEMEG), even the model of Chakraborti and Chakrabarti (CCM), satisfies the time reversal symmetry. This argument is clear in the non-conservative case, because of the fraction of wealth that is constantly added to the individual wealth of the agents. On the other hand, for the conservative case, the logic obeys the non-symmetric form of the interactions \( \Delta w \), where one of the agents gives a fraction \( l = \frac{(1-\lambda)}{(2q+1)} \) of their individual money to the other. Under this conditions their balances are \( (1-l)w \) and \( v + lw \). Conversely, if the process is reversed, the agent with individual money \( v + lw \) must give a fraction \( l \) of to the other, and the balances are \( w + l(v + lw) \) and \( (1-l)(v + lw) \), which implies that the system does not return to the same initial condition \([w, v]\) [17].

### 3.1. Weak formulation of the Boltzmann equation

In general, the Boltzmann equation 3-9 has exact solutions only for few cases, depending on the dynamics of the system. Nevertheless, it is possible to write a set of weak formulations with a wider range of analytical solutions given information about the stylized facts of the distributions, by studying the effect of the collision operator over smooth test functions \( \Phi(w) \):

\[
\int Q(f, f) \Phi(w) dw = \int \{ \beta_{[w^*, v] \rightarrow [w, v]} f(w^*, \zeta, t)f(v^*, \zeta, t) \} \Phi(w) dv dw* dv* \\
- \int \{ \beta_{[w, v] \rightarrow [w^*, v^*]} f(w, \zeta, t)f(v, \zeta, t) \} \Phi(w) dv dw* dv*, \tag{3-13}
\]

Note that, both integrals in the right-hand side of the equation are evaluated over all the variables \( w, v, w^* \) and \( v^* \). Hence, changing variables \( v, w \leftrightarrow v^*, w^* \) in the first integral:

\[
\int Q(f, f) \Phi(w) dw = \int \beta_{[w, v] \rightarrow [w^*, v^*]} f(w, \zeta, t)f(v, \zeta, t) [\Phi(w^*) - \Phi(w)] dv dw* dv*, \tag{3-14}
\]
A symmetrical formulation for this equation is achieved by changing variables $w, w^* \leftrightarrow v, v^*$ and summing the results divided by 2:

$$\int Q(f, f) \Phi(w) dw = \frac{1}{2} \int_{[w,v] \to [w^*, v^*]} \beta[w,v] f(w, \zeta, t) f(v, \zeta, t) [\Phi(w^*) + \Phi(v^*) - \Phi(w) - \Phi(v)] dwdvdw^*dv^*. \quad (3-15)$$

As it was remarked before, the rate of transactions per unit time keeps a close relation with the rate of collisions in a dilute gas, which depends on the differential cross-section of the problem. However, in the context of kinetic exchange models of markets this rate has a simpler mathematical form. On the one hand, the transactions describe a one-dimensional phenomena, where the results of the transformation are closely related with the values of the parameters $\zeta$. In the main, most of this variables are set as exogenous fixed parameters, but they can also be defined as stochastic variables in time. Therefore, it can be defined an analogous one-dimensional cross-section proportional to $d\zeta$. Furthermore, the dynamics of the KWEMEG are defined in such a way that, at every-time, only two agents are interacting in the market, i.e. one couple from the set of $\frac{N}{2}$ possible couples. Under both considerations, the rate of transactions per unit time can be modeled as $\beta_{[w,v] \to [w^*, v^*]} = \frac{2}{N} d\zeta$.

Replacing the rate of transaction, the non-symmetric weak formulation of the Boltzmann equation reads as follows:

$$\frac{d}{dt} \int f(w, \zeta, t) \Phi(w) dw = \frac{1}{N} \langle \int f(w, \zeta, t) f(v, \zeta, t) [\Phi(w^*) - \Phi(w)] dwdvdw^*dv^* \rangle. \quad (3-16)$$

Similarly, the symmetric form becomes:

$$\frac{d}{dt} \int f(w, \zeta, t) \Phi(w) dw = \frac{1}{N} \langle \int f(w, \zeta, t) f(v, \zeta, t) [\Phi(w^*) + \Phi(v^*) - \Phi(w) - \Phi(v)] dwdvdw^*dv^* \rangle, \quad (3-17)$$

where the average operates over the parameters $\zeta$.

The previous results are the equivalent to the equations introduced by Toscani et al. in [29, 30]. However, in that case the rate of transaction is assumed equal to 1, in the same spirit of the one-dimensional problems for Maxwell molecules in references [51, 52].
3.2. Evolution of the moments of the distribution

An interesting hierarchy for the evolution of the moments of money and wealth distributions can be obtained by replacing $\Phi(w) = w^r$ in equation 3-17:

$$\frac{d}{dt} \int f(w, \zeta, t) w^r dw = \frac{1}{N} \left( \int f(w, \zeta, t) f(v, \zeta, t) [(w^*)^r + (v^*)^r - w^r - v^r] dw dv dw^* dv^* \right).$$

(3-18)

Using relations 3-5, 3-4, and the binomial expansion of the terms $(w^*)^r$, $(v^*)^r$ the previous equation becomes:

$$\frac{d}{dt} \int f(w, \zeta, t) w^r dw =$$

$$\frac{1}{N} \left( \int f(w, \zeta, t) f(v, \zeta, t) \left[ \sum_{k=0}^{r} \binom{r}{k} (p_k^1 q_1^{r-k} + p_k^2 q_2^{r-k}) w^k v^{r-k} - w^r - v^r \right] dw dv dw^* dv^* \right).$$

(3-19)

Note that all products of the coefficients $p_{1,2}, q_{1,2}$ can be taken out of the integrals as follows:

$$\frac{d}{dt} \int f(w, \zeta, t) w^r dw =$$

$$\frac{1}{N} \left( \sum_{k=0}^{r} \binom{r}{k} (p_k^1 q_1^{r-k} + p_k^2 q_2^{r-k}) \int f(w, \zeta, t) f(v, \zeta, t) w^k v^{r-k} dw dv dw^* dv^* \right)$$

$$- \frac{1}{N} \left( \int f(w, \zeta, t) f(v, \zeta, t) w^r dw dv dw^* dv^* \right)$$

$$- \frac{1}{N} \left( \int f(w, \zeta, t) f(v, \zeta, t) v^r dw dv dw^* dv^* \right).$$

(3-20)

Thus, replacing the moments of the distribution $M_r(t) = \int f(w, \zeta, t) w^r dw$ and factorizing the terms $k = 0$ and $k = n$, it is obtained:

$$\frac{d}{dt} M_r(t) = (p_i^r + q_i^r - 1) + M_r(t) + \sum_{k=1}^{n-1} \binom{r}{k} \langle p_k^r q_1^{r-k} \rangle + M_k(t) M_{r-k}(t),$$

(3-21)

where the notation $\langle \psi(p_i, q_i) \rangle_+ := \frac{1}{N} \langle \psi(p_1, q_1) + \psi(p_2, q_2) \rangle$ have been introduced, as in reference [30], in order to abbreviate the expected values with respect to $\zeta$, operating over the coefficients $p_{1,2}, q_{1,2}$. 

It is clear from equations 3-5 and 3-4, that the condition necessary for the conservation of total wealth in time is \( p_1 + p_2 = q_1 + q_2 = 1 \), which implies \( w + v = w^* + v^* \). In this regime, the moments are constant in time and they can be easily computed using the recursive relation:

\[
M_r = \sum_{k=1}^{r-1} \binom{r}{k} \langle p_i^k q_i^{r-k} \rangle + M_k M_{r-k},
\]  
(3-22)

This relation is valid for \( r > 1 \), however, it is possible to assume that \( M_0 = M_1 = 1 \), where the first condition implies the normalization of the density distribution, and the second condition is based on the conservation of wealth. Conversely, if \( p_1 + p_2 \neq 1 \) and \( p_1 + p_2 \neq 1 \), the wealth is not conserved. In this case, it is necessary to estimate recursively the moments.

It is easy to show from 3-21, that if the wealth is not conserved in time, the first moment of the distribution behaves as the exponential function:

\[
M_1(t) = M_1(0) \exp[\langle p_i + q_i - 1 \rangle t].
\]  
(3-23)

The increasing or decreasing on this moment depends on the definitions of \( p_{1,2}, q_{1,2} \), therefore, this function increases if their linear combination satisfies \( p_1 + p_2 + q_1 + q_2 > 2 \).

The first moment is completely determined by knowing the initial condition \( M_1(0) \). Without loss of generality, this value is set as \( M_1(0) = 1 \) and this initial condition is kept for the numerical analysis proposer in the next chapters. Using 3-23 the high order moments \( r > 1 \) can be computed recursively. In particular, the recursive relation for the second moment of the distribution is:

\[
\frac{d}{dt} M_2(t) = \langle p_i^2 + q_i^2 - 1 \rangle M_2(t) + 2\langle p_i q_i \rangle M_1(t)^2.
\]  
(3-24)

The solution of this equation is computed using standard methods of ordinary differential equations:

\[
M_2(t) = \frac{2\langle p_i q_i \rangle}{(1 - p_i^2 - q_i^2)} \exp[2\langle p_i + q_i - 1 \rangle t] + \frac{c}{\exp[(1 - p_i^2 - q_i^2)t]},
\]  
(3-25)

where \( c \) is a constant of integration that can be neglected in the limit \( t \to \infty \) only if \( 1 - p_i^2 - q_i^2 > 0 \).

Relations 3-22, 3-23 and 3-25 are exploited in the following chapters with the aim of studying the stylized facts of the different particular cases of the KWEMEG. In addition, a numerical analysis using Monte Carlo simulations is proposed in order to test the validity of this approach. With this in mind, the following forms of the coefficients \( p_{1,2}, q_{1,2} \) are obtained by comparing equations 3-5, 3-4 and, 2-9, 2-12, 2-15, 2-16, recalling that \( w \equiv w_i(t), v \equiv w_j(t), w^* \equiv w_i(t+1), v^* \equiv w_i(t+1) \).

\[
p_1 = \frac{\lambda(1 - s)}{(\lambda - s)(1 - s + 2q)}[\lambda + \varepsilon(1 - \lambda) - s + q],
\]  
(3-26)
\begin{align*}
q_1 &= \frac{\lambda (1-s)}{(\lambda - s)(1-s+2q)}[\varepsilon (1-\lambda) + q], \\
p_2 &= \frac{\lambda (1-s)}{(\lambda - s)(1-s+2q)}[(1-\varepsilon)(1-\lambda) + q], \\
q_2 &= \frac{\lambda (1-s)}{(\lambda - s)(1-s+2q)}[\lambda + (1-\varepsilon)(1-\lambda) - s + q].
\end{align*}

In this case, \( p_1 + p_2 = q_1 + q_2 = \frac{1-s}{\lambda-s} \lambda \), which implies that the wealth is not conserved if \( s \neq 0 \). Nevertheless, for \( s = 0 \) the conservation condition \( p_1 + p_2 = q_1 + q_2 = 1 \) is satisfied.

In general, the calculation of the moments using the previous relations deals with the expected values over the products involving \( p_{1,2}, q_{1,2} \), which are expanded as polynomial functions of the exogenous parameters \( q, \lambda, s \) and the stochastic variable \( \varepsilon \). However, this average only affects \( \varepsilon \), due to the fact that the exogenous parameters are fixed values, determined outside the model. Thus, this expression are easily solved using the following relation:

\begin{equation}
\langle a^k \varepsilon \rangle = \int_0^1 a^k \varepsilon d\varepsilon = \frac{a}{k+1},
\end{equation}

where the probability density function of \( \varepsilon \) is \( f(\varepsilon) = 1 \), by definition, and \( a \) is any constant parameter.
4. Conservative models of money distribution

The existence of the conservative and the non-conservative regimes, in the context of the Kinetic wealth exchange model of economic growth (KWEMEG), was established in chapter 2 as a direct consequence of the saving of production. When the saving parameter is different to zero \((s \neq 0)\), the goods are partially consumed after trading, in such a way that the remaining portion increases the individual wealth by a rate \(g\), which induces a net growth in the total wealth of the system. In contrast, the limit case \(s = 0\) sets the conservation of total wealth in time, due to the fact that the goods produced by the agents are completely consumed after trading in the market.

In the conservative regime, the interactions between agents only depend on the exogenous parameters \(\lambda\) and \(q\), which are associated, respectively, to the exchange aversion of the agents and the taxation over personal wealth; and on the stochastic variable \(\varepsilon\), that guarantees the uniqueness of the transactions. This interpretation of the parameters strictly obeys the mathematical definition achieved for the interaction rule \(\Delta w\) governing the evolution of the individual money. However, its effects on the emergent macroscopic properties have not been discussed so far. In this order of ideas, this chapter carries out an approach to the emergent dynamics through a numerical study based on the general strategy of simulation described in chapter 2, and the application of the analytical results of the Boltzmann equation introduced in the previous chapter.

In particular, the analytical approach puts effort in shedding light on the stylized facts of the distribution by solving the first and second moments in terms of the exogenous parameters. Using these results, it is possible to find analytical relations for the probability density functions of both models. In addition, the level of inequality is studied by means of the Gini index (see appendix C), which is also expressed as a function of the exogenous parameters.
4.1. Chakraborti and Chakrabarti model

Setting the parameters $s = q = 0$, the KWEMEG reduces to the model of Chakraborti and Chakrabarti (CCM). Under this simplification, the coefficients $p_{1,2}$, $q_{1,2}$, introduced in the Botzmann-type approach, reads as follows:

\begin{align}
p_1 &= \lambda + \varepsilon(1 - \lambda), \\
qu &= \varepsilon(1 - \lambda), \\
p_2 &= (1 - \varepsilon)(1 - \lambda), \\
q_2 &= \lambda + (1 - \varepsilon)(1 - \lambda).
\end{align}

By definition, the first moment of the distribution must be conserved at every transaction, therefore, it is necessary to set an initial condition, in order to solve the recursive relations for higher moments. According to the simulation scheme described in chapter 2, at $t = 0$, all the agents starts with $w_k(0) = 1$, where $k = 1, 2, \ldots, N$. This implies that for the conservative models $M_1(t) = 1$. In addition, it was shown that the second moment of the distributions in the conservative regime is computed through the general relation:

\[ M_2 = \frac{2\langle p_i q_i \rangle_+}{\langle 1 - p_i^2 + q_i^2 \rangle_+} (M_1)^2. \]  

Replacing the coefficients $p_{1,2}$, $q_{1,2}$ defined above, and using $\langle \varepsilon^k \rangle = \frac{1}{k+1}$, it is obtained:

\[ M_2 = \frac{\lambda + 2}{1 + 2\lambda}. \]

In general, the emergent money distributions of the CCM are well fitted using the gamma probability density function [18, 53]:

\[ f(w) = \frac{1}{a^b \Gamma(b)} \left( \frac{w}{a} \right)^{b-1} \exp \left( - \frac{w}{a} \right) dw, \]

where $a$ is the scale of the distribution and $b$ is the shape parameter. In addition the first and second moments around zero of the gamma distribution satisfy the relations $M_1 = ab$ and $M_2 = a^2 b(b+1)$ (see appendix B). Thus, using equation 4-6 and the initial condition $M_1 = 1$, the parameters of the distribution can be computed in terms of $\lambda$. The result obtained for the scale parameter is:

\[ a = \frac{1 - \lambda}{1 + 2\lambda}. \]
On the other hand, the shape is easily computed as $b = 1/a$, which leads to:

$$b = \frac{1 + 2\lambda}{1 - \lambda}. \quad (4-9)$$

Adding and subtracting 1, the previous expression reads as:

$$b = 1 + \frac{3\lambda}{1 - \lambda}. \quad (4-10)$$

This relation is exactly the same proposed by Patriarca et al. in reference [18], based on an analogy between the dynamics of the model and a isolated gas in $D$ dimensions. In that case, the gamma distribution is the analytical energy distribution for the thermal system in $D \neq 2$ dimensions, which apparently leads to the conclusion that the gamma distribution is the analytical emergent money distribution of the CCM. However the validity of this conjecture has been discussed by different authors, who conclude the opposite statement [54, 53], due to the fact that the higher moments of the distribution, computed in terms of $a$ and $b$, using the relation $M_r = a^rb(b+1)$ do not match with the moments computed through the Boltzmann equation for $r \geq 4$.

In this order of ideas, the using of the gamma probability density function is implemented in the thesis based on statistical arguments for fitting the emergent distribution patterns of the simulated data. It is important to recall that the estimation method applied for obtaining the parameters of the gamma uses the analytical expressions for the moments achieved by solving the Boltzmann equation, which, in fact, are emergent macroscopic properties from the microscopic dynamics. Nevertheless, this fact does not allow to conclude that the gamma distribution is the analytical emergent distribution of the models, in accordance with the previous arguments.

The emergent distributions reproduced for $\lambda = \{0, 0.2, 0.4, 0.6, 0.8\}$ are shown in figure 4-1a). The results were obtained running over $20 \times 10^3$ Monte Carlo time-steps and taking an average over $10^4$ ensembles, following the simulation method described in chapter 2. All the cases were fitted using relations 4-7, 4-8 and 4-9, and the goodness of fit was verified using the Kolmogorov-Smirnov test. In the main, the analytical expressions as well as the gamma distribution are accepted by the criteria, with a level of confidence $\alpha = 0.05$. Note that all the results were normalized to 1, in order to show them in the same scale, by dividing by the amount of money of the riches agent ($\max(w_k(t))$). Under this normalization, the scale parameter is $a/\max(w_k(t))$.

In general, the distribution of money is more equitable for higher values of $\lambda$, due to the fact that the bulk moves to the upper levels of money and the distribution becomes sharper, as $\lambda$ increases. The simulated data is fitted using the gamma distribution described by equations 4-7 and 4-9. Note that, for the case $\lambda = 0$, $a = b = 1$, the gamma distribution reduces to:

$$f(w) = \exp \left(-\frac{w}{\langle w \rangle}\right), \quad (4-11)$$
Fig. 4-1.: Distribution of money in the CCM. **a)** The emergent distribution patterns, reproduced using Monte Carlo simulations, are well fitted by the gamma probability density function. **b)** The level of inequality of the distributions is lower as \( \lambda \) increases, therefore the Gini Index decreases for higher values of \( \lambda \). The dots in the graphic correspond to the value of this metric computed from the simulated data, while the curve was calculated using analytical relation 4-12.

where \( \langle w \rangle = ab = M_1 = 1 \). Thus, the Boltzmann-Gibbs distribution of the model of Dragulescu and Yakovenko (DYM) is obtained in the particular case \( \lambda = 0 \).

The inequality of the distributions can be studied explicitly by means of the Gini index. Broadly, this index quantifies the level dispersion of a distribution of data with respect to a hypothetical perfectly egalitarian distribution (see appendix C). Thus, a Gini index tending to 1 implies a very egalitarian distribution, while a value about 0 approaches the perfectly egalitarian distribution. Its value can be computed directly from the simulated data but also using the analytical relation for the gamma probability density function [55]:

\[
G = \frac{1}{\sqrt{\pi}} \frac{\Gamma(b + \frac{1}{2})}{\Gamma(b + 1)}.
\] (4-12)

Replacing the analytical relation for \( b \), the value of the Gini index can be predict for each distribution obtained for any fixed value of \( \lambda \). In figure 4-1 **b)**, the evolution of Gini index is plotted as a function of \( \lambda \). Note that the values computed from the simulated data are well fitted by analytical expression 4-12. In addition, the maximum value of the index, \( G = 0.5 \), is reached for \( \lambda = 0 \). In this point, the gamma distribution reduces into the Boltzmann-Gibbs, this result is shown analytically in the appendix B.

The usual range covered by Gini index from empirical data is \([0, 0.65]\), [56]. Thus, the scope of the CCM is limited when reproducing some important facts around economic inequality. However, if the value of \( \lambda \) is set as a random number defined in the domain \([0, 1]\), the Gini
index reaches values near to 0.9 [57]. A similar effect occurs as cause of the saving parameter inducing the non-conservative regime, Gini index can reach higher values of This dynamics are discussed in detail in the next chapter.

### 4.2. Taxation and redistribution of money

The parameter $q$ introduced in the powers of the utility function modifies the dynamics of the CCM by an extra term proportional to the difference of money between agents $-\frac{q}{1+2q}(w_i - w_j)$. Thus, if the agent $j$ is richer than the agent $i$ this amount is added to the stock of the agent $i$, and similar for the case in which $i$ is richer that $j$. In addition, the transaction rule defined in the CCM is re-scaled by the constant $\frac{1}{1+2q}$, which also changes the dynamics of the model. The effect of this modification is studied using the same procedure for the results presented in the case of the CCM. Setting $s = 0$ and $q \neq 0$, the coefficients $p_{1,2}$, $q_{1,2}$ reads as follows:

\[ p_1 = \frac{1}{1 + 2q}[\lambda + \varepsilon(1 - \lambda) + q], \]  
(4-13)

\[ q_1 = \frac{1}{1 + 2q}[\varepsilon(1 - \lambda) + q], \]  
(4-14)

\[ p_2 = \frac{1}{1 + 2q}[(1 - \varepsilon)(1 - \lambda) + q], \]  
(4-15)

\[ q_2 = \frac{1}{1 + 2q}[\lambda + (1 - \varepsilon)(1 - \lambda) + q]. \]  
(4-16)

The second moment of the distribution is computed in the same way than for the CCM. Putting the coefficients into equation 4-5, and organizing the expression, it is obtained:

\[ M_2 = \frac{(1 - \lambda)(\lambda + 2) + 6q(q + 1)}{(1 - \lambda)(1 + 2\lambda) + 6q(q + 1)}. \]  
(4-17)

In this case, the data still follows the gamma distribution pattern, as it is shown in figures 4-2 a-b), where the distributions of money are presented for fixed $\lambda = \{0.5, 0.8\}$ and $q = 0.0, 0.2, 0.1, 0.3, 0.5$. For all the cases the Kolmogorov-Smirnov criteria is satisfied with a level of confidence $\alpha = 0.05$. Thus, the distributions are well fitted using the relation for the shape parameter:

\[ b = \frac{(1 - \lambda)(1 + 2\lambda) + 6q(q + 1)}{(1 - \lambda)^2}, \]  
(4-18)

which is obtained using the second moment of the distribution and the relations for the moments of the gamma distribution presented in the analysis of the CCM. This relation is
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Fig. 4-2.: Emergent money distributions for \( q \neq 0 \). a-b) The money distributions follows the gamma probability density function, where the parameters \( a \) and \( b \) are estimated in terms of \( \lambda \) and \( q \) using relations 4-18 and \( a = \frac{1}{b} \). In general, the distributions tend to right-skew and to increases their peakedness for higher values of \( q \). c-d) This effect is reflected by the Gini index, which decreases as the taxation rate increases. In all the cases the Gini index is well predicted by the theoretical function 4-12.
4.3 Steady state in the conservative regime

a general formulation of the shape parameter suggested by Patriarca in the context of the CCM. Thus, setting \( s = 1 \) and adding and subtracting 1, 4-18 reduces to 4-9. It is important to recall that the implementation of the gamma distribution in this context obeys a pure numerical argument based on the goodness of fit criteria. In addition, it shows that the taxation scheme introduced leads to a solid extension of the CCM, where the stylized facts of the money distribution are still captured by the gamma distribution.

An interesting effect of the taxation on the model is the decreasing on the level of wealth inequality as the value of \( q \) increases. This fact is captured by the evolution of the Gini index. Again, in this case, it is possible to establish a comparison between the values of the Gini index computed directly from the simulated data, following the logic presented in the appendix B, and the value predicted using equation 4-12 and the parameter \( b \) corresponding to this model. This comparison is presented in figures 4-2 c-d). Note that for every case of fixed \( \lambda \), the value of the coefficient decreases as \( q \) increases, and their maximum is reached as \( q = 0 \). Furthermore, the upper limit of the Gini index is still \( G = 0.5 \), which is obtained as \( \lambda = q = 0 \), where the model reduces to the the DYM. This behavior of the distribution pattern induced by \( q \) constitutes an important extension of the CCM, due to the fact that it reproduces the effect of a progressive tax benefit emerging from a basic microeconomic scheme based on the utility function.

4.3. Steady state in the conservative regime

By direct analogy with an isolated gas, it is expected, that the kinetic-like dynamics, in the conservative regime, leads to an equilibrium state, such that the stylized facts of the distribution remain unaltered in spite of the transactions. A useful approach to this state is the calculation of the Shannon entropy of the system, defined as:

\[
H = \sum_{l=1}^{m} p_l \log p_l,
\]

where the index \( l \) runs over all the \( m \) levels of money in the system, and \( p_l \) is the probability to find an agent in the level \( l \).

The meaning of the Shannon Entropy is framed into the context of the Theory of Information [58]. For a non-deterministic system, where the micro-states of its components must be described in terms of probabilities, this function measures the missing information that would be necessary in order to establish a completely deterministic description. Thus, its value increases when the precise information of any set of freedom degrees describing a micro-state is lost, as consequence of the dynamics of the components.

At equilibrium, the Shannon entropy must remain constant, due to the fact that the probabi-
3.4 Conservative models of money distribution

**Fig. 4-3.** Evolution of the Shannon Entropy for the conservative models of money exchange. 

a) In the context of the CCM \((s = q = 0)\), the entropy is maximized faster for lower values of \(\lambda\). b) On the other hand, the equilibrium state is independent of the taxation \(q\), and therefore, the entropy is maximized at the same time for all the cases. The effect of \(\lambda\) and \(q\) on the equilibrium state is evident in the subplots of both figures, where the entropy is normalized with respect to its maximum value.

Probabilities of each level do not have big fluctuations. This fact implies that the probability density function does not change in time, and therefore, the gains and losses of the Boltzmann equation must be balanced. In particular, in the models considered above, it is expected that the Entropy maximizes, due to the fact that the initial condition for all the ensembles is the Dirac Delta, where all the agents starts with the same amount of money, and the probability of been in another level is equal to zero. Thus, this state of the system constitute an state of maximum information, which is lost at every time that a transaction occurs.

The Shannon entropy is computed using the simulated data of all the cases presented above, through the definition 4-19. For every fixed \(\lambda\) and \(q\), this metric is computed each 100 time-steps, in order to study its evolution in time. This process is achieved defining a fixed wide

\[
d = \frac{\max[w_k(t)] - \min[w_k(t)]}{m}
\]

where \(k = 1, 2, 3, ..., N\) and the terms \(\max/\min[w_k(t)]\) are, respectively, the money of the richest and the poorest agent during all the time interval studied. Note that, the \(m\) levels of money used for every \(\lambda\) and \(q\) must be the same at every time, thus, the maximum and minimum of \(w_k\) represent an absolute value, which is defined over the whole time interval considered. In particular, in this analysis, it was studied the time interval \([0, 10000]\).

The evolution of \(H\) is shown in figure 4-3, for the CCM and the model with taxation rate. In the first case, the parameter \(\lambda\) slows down the relaxation process of the system, due to the fact that the propensity of the agents to exchange in the market is lower for higher values
of $\lambda$. In contrast, the change of the taxation $q$ does not affect the relaxation process, and therefore, the equilibrium state is reached approximately at the same time for all the cases. In order to make evident the effect of $\lambda$ and $q$ on the entropy, the curves were normalized using the maximization value in every case. The results of this normalization are shown in the subplots of both figures. Note that, for the first case, it is clear that the entropy is maximized faster as $\lambda$ increases. On the other hand, all the curves of the second plot saturate into the same normalized curve, which implies that, in this case, the entropy is maximized at the same time.
5. Non-conservative models of wealth exchange

In contrast to the conservative dynamics discussed in the previous chapter, the non-conservative regime is governed by the exponential growth of the average wealth. In the Boltzmann approach to the dynamics of the Kinetic Wealth Exchange Model of Economic Growth (KWE-MEG), it was found that if $s \neq 0$, then the first moment of the distribution of wealth is:

$$M_1(t) = M_1(0) \exp[\langle p_i + q_i - 1 \rangle t]. \quad (5-1)$$

Recalling that the coefficients $p_{1,2}, q_{1,2}$ are defined as:

$$p_1 = \frac{\lambda(1 - s)}{(\lambda - s)(1 - s + 2q)}[\lambda + \epsilon(1 - \lambda) - s + q], \quad (5-2)$$

$$q_1 = \frac{\lambda(1 - s)}{(\lambda - s)(1 - s + 2q)}[\epsilon(1 - \lambda) + q], \quad (5-3)$$

$$p_2 = \frac{\lambda(1 - s)}{(\lambda - s)(1 - s + 2q)}[(1 - \epsilon)(1 - \lambda) + q], \quad (5-4)$$

$$q_2 = \frac{\lambda(1 - s)}{(\lambda - s)(1 - s + 2q)}[\lambda + (1 - \epsilon)(1 - \lambda) - s + q], \quad (5-5)$$

and replacing this expression, operating the algebraic steps, the first moment reduces to:

$$M_1(t) = \exp\left(\frac{(1 - \lambda)}{(\lambda - s)} \frac{2t}{N}\right), \quad (5-6)$$

where $M_1(0) = 1$, due to the initial condition set in the simulations. Defining $g = \frac{1 - \lambda}{\lambda - s} s$ and $\tau = \frac{2t}{N}$, the previous expression is, in fact, the relation introduced in chapter 2, by direct comparison with the models proposed by Bouchaud and Mézard in 2000 [10] and by Slanina in 2004 [11]. Note that, in the case $s = 0$ the rate of economic growth is $g = 0$, which implies that the first moment is conserved in time, and therefore the dynamics are the same as in the conservative regime, where the wealth of the agents is mere money.
5.1 Stylized facts of the emergent wealth distributions for $q = 0$

Similarly, the solution of the differential equation for the second moment of the distribution, obtained in chapter 3, is:

$$M_2(t) = \frac{2 \langle p_i q_i \rangle_+}{(1 - p_i^2 - q_i^2)_+ + 2 \langle p_i + q_i - 1 \rangle_+} \exp[2 \langle p_i + q_i - 1 \rangle_+ t] + \frac{c}{\exp[(1 - p_i^2 - q_i^2)_+ t]}.$$

(5-7)

In general, the constant of integration for this solution $c$ can not be obtained using the initial conditions, due to the fact that at $t = 0$, the distribution pattern can be defined arbitrarily, and the dynamics of the model does not keep in time the information of this distribution. Therefore, it is expected that the behavior of the moments is independent of the initial condition. However, at $t \to \infty$ this term can be neglected if the condition $\langle 1 - p_i^2 - q_i^2 \rangle_+ > 0$ is satisfied.

Evaluating this relation, by replacing coefficients 5-2,5-3,5-4,5-5, it is obtained:

$$\langle 1 - p_i^2 - q_i^2 \rangle_+ = \frac{1}{c} \langle 2 - p_i^2 - q_i^2 - p_i^2 - q_i^2 \rangle_+ = \frac{2}{3N} \frac{1}{(\lambda - s)^2(1 - s + 2q)^2} [3(\lambda - s)^2(1 - s + 2q)^2 - 2\lambda^2(1 - s)^2(1 - \lambda)^2 - 3\lambda^2(1 - s)^3(\lambda - s) - 6q\lambda^2(1 - s)^2(1 - s + q)].$$

(5-8)

The order of magnitude of the exogenous parameters is $\lambda, s, q \sim \mathcal{O}(10^{-1})$ and $\lambda > s$. Therefore, at least $3(\lambda - s)^2(1 - s + 2q)^2 \sim \mathcal{O}(10^{-4})$, $2\lambda^2(1 - s)^2(1 - \lambda)^2 \sim \mathcal{O}(10^{-6})$, $3\lambda^2(1 - s)^3(\lambda - s) \sim \mathcal{O}(10^{-6})$, and $6q\lambda^2(1 - s)^2(1 - s + q) \sim \mathcal{O}(10^{-6})$, which implies that:

$3(\lambda - s)^2(1 - s + 2q)^2 - 2\lambda^2(1 - s)^2(1 - \lambda)^2 - 3\lambda^2(1 - s)^3(\lambda - s) - 6q\lambda^2(1 - s)^2(1 - s + q) > 0,$

and $\langle 1 - p_i^2 - q_i^2 \rangle_+ > 0$. Using this result, the solution of the second moment can be simplified for $t \to \infty$ as follows:

$$M_2(t) = \frac{2 \langle p_i q_i \rangle_+}{(1 - p_i^2 - q_i^2)_+ + 2 \langle p_i + q_i - 1 \rangle_+} \exp[2 \langle p_i + q_i - 1 \rangle_+ t].$$

(5-10)

Note that this approximation, and the inequality 5-9, are still valid for the case $q = 0$. Thus, it is possible to obtain the second moment of the emergent distributions, in both cases, by direct replacing of the coefficients $p_{1,2}$ and $q_{1,2}$ in 5-10.

### 5.1 Stylized facts of the emergent wealth distributions for $q = 0$

A very interesting fact of the KWEMEG is that the behavior of the first moment of the distribution is exactly the same for the cases $q = 0$ and $q \neq 0$. In the case $q = 0$, the
Fig. 5-1.: Evolution in time of the moments $M_1(t)$ and $M_2(t)$, for $q = 0$. a-b) The moments computed using the simulated data follow the analytical expressions obtained through the Boltzmann kinetic approach for the distributions. In particular, for the second moment, the coefficient multiplying the exponential function reads as

$$m_2 = \frac{\lambda^2(\lambda+2-3s)}{3(\lambda-s)(\lambda+s-\lambda s)} - 2\lambda^2(1-\lambda),$$

in accordance with equation 5-15.

Coefficients $p_1,2, q_1,2$ are simplified as follows:

$$p_1 = \frac{\lambda}{\lambda - s} [\lambda + \varepsilon(1 - \lambda) - s],$$

$$q_1 = \frac{\lambda}{\lambda - s} [\varepsilon(1 - \lambda)],$$

$$p_2 = \frac{\lambda}{\lambda - s} [(1 - \varepsilon)(1 - \lambda)],$$

$$q_2 = \frac{\lambda}{\lambda - s} [\lambda + (1 - \varepsilon)(1 - \lambda) - s].$$

Putting this relations into equation 5-1 it is easy to show that the first moment of the distribution still satisfies expression 5-6.

Again, replacing coefficients 5-11,5-12,5-13,5-14 in the previous result, it is obtain the second moment for $q = 0$:

$$M_2(t) = \frac{\lambda^2(\lambda + 2 - 3s)}{3(\lambda-s)(\lambda+s-\lambda s)} - 2\lambda^2(1-\lambda) \exp\left(\frac{4}{N(\lambda-s)} st\right).$$

This result implies that $M_2(\tau) \propto \exp(2g\tau)$. The evolution of $M_1(t)$ and $M_2(t)$ for the cases $\lambda = \{0.5, 0.8\}$, $s = \{0.05, 0.1, 0.2\}$ is shown in figures 5-1. The dots correspond to the data obtained from the simulations while the curves are the fits obtained using analytical expression 5-6 and 5-15.
5.1 Stylized facts of the emergent wealth distributions for $q = 0$

Fig. 5-2.: Evolution in time of the emergent wealth distributions for $q = 0$. a-d) The simulated data is well fitted by the gamma probability density function. The parameters were computed, in all the cases, using the Maximum likelihood estimation method (MLE). e-f) The exponential behavior of the first moments is captured by the scale parameter $\alpha$, which follows the function $\alpha = (m_2 - 1) \exp(g \frac{2}{N})$, while the shape parameter remains constant in time. In all the cases studied, their values are well fitted by the theoretical functions obtained using the Boltzmann equation.
In contrast to the models described in the previous chapter, it is not expected to reach an equilibrium state for the case \( s \neq 0 \), due to the non-conservative dynamics of the model. The exponential growth of the moments has important effects on the stylized facts of the emergent wealth distributions (see figure 5-2). Broadly, the distribution moves along the \( w \)-axis as a consequence of the increasing in time of the mean \( \mu = M_1(t) \). Additionally, the behavior of the second moment around zero \( M_2(t) \) induces the exponential increasing of the variance \( \sigma^2 \propto \exp(\frac{4\mu t}{N}) \). This fact is clear using the definition \( \sigma^2 = M_2^\mu = M_2^0 - [\mu]^2 \), and relation 5-15.

In spite of the lack of equilibrium state, it is expected that the emergent wealth distributions for \( s \neq 0 \) hold the gamma probability density function, or a more general formulation of this, due to the fact that the dynamics for \( s \neq 0 \) constitute a generalization of the CCM. In this order of ideas, the evolution in time of the wealth distributions is studied using the gamma probability density function as in the previous chapter. The results of this approach are presented in figure 5-2 for the cases \( \lambda = \{0.5, 0.8\} \), \( s = \{0.05, 0.1, 0.2\} \). In general, the gamma distribution is accepted, for all the cases studied, by the Komogorov-Smirnov test, however, it is important to recall that there is no any argument to believe that this probability density function is the analytical distribution emerging from the microscopic dynamics of the system.

The parameters of the gamma distribution are obtained in the same way of the previous chapter, using the relations for the moments \( M_1(t) = ab \), \( M_2(t) = a^2b(b + 1) \), where the explicit contribution of the time is included by the moments. Thus, solving this equations, by replacing the moments 5-1 and 5-15, it is found that the shape parameter is:

\[
b = \frac{1}{m_2 - 1} = \frac{3(\lambda - s)(\lambda + s - \lambda s) - 2\lambda^2(1 - \lambda)}{(1 - \lambda)(\lambda^2 + 3s)}, \tag{5-16}
\]

where \( m_2 = \frac{\lambda^2(\lambda + 2 - 3s)}{3(\lambda - s)(\lambda + s - \lambda s) - 2\lambda^2(1 - \lambda)} \) is the coefficient multiplying the exponential in analytical expression for the second moment 5-15. On other hand, the scale parameter reads as:

\[
a = \frac{M_1(t)}{b} = (m_2 - 1) \exp\left(\frac{2t}{N}\right) = \frac{(1 - \lambda)(\lambda^2 + 3s)}{3(\lambda - s)(\lambda + s - \lambda s) - 2\lambda^2(1 - \lambda)} \exp\left(\frac{2t}{N}\right). \tag{5-17}
\]

Note that the exponential behavior of the first moment is captured by the scale parameter \( a \), which follows the function \( a(t) = (m_2 - 1) \exp\left(\frac{2t}{N}\right) \), while the shape \( b \) remains constant in time. In general, all the cases studied are well fitted using relations 5-16 and 5-17, in particular the results of this approximation are shown in figures 5-2 e-f) for the same cases previously studied.
5.2 Taxation and redistribution of wealth

As it was remarked before, the growth of the average wealth is not modified in the case $q \neq 0$. Therefore, relation 5-6 is still valid to predict the behavior of the first moment. Similarly, the second moment is computed replacing coefficients 5-2, 5-3, 5-4 and 5-5 in equation 5-10. The result obtained computing the algebraic steps and organizing the expression is:

$$M_2(t) = \frac{\lambda^2(1-s)^2[(1-\lambda)(\lambda+2-3s)+6q(1-s+q)]}{3(\lambda-s)[(1-s+2q)^2(\lambda+s-2\lambda s)-\lambda^2(1-s)^2]-\lambda^2(1-s)^2[2(1-\lambda)^2+6q(1-s+q)]} \exp \left[ \frac{(1-\lambda)}{(\lambda-s)} \frac{4t}{N} \right].$$

(5-18)

The results of $M_1(t)$ and $M_2(t)$ for $\lambda = 0.8$, $\lambda = 0.1$ and $q = \{0.1, 0.3, 0.5\}$ are shown in figures 5-3. It is clear that, for the case of the first moment, the simulated data follows the same function exponential function, in spite of the change on the taxation rate. However, the change on the second moment depends on the coefficient multiplying the exponential function. Note that in the semi-log representation of the data presented in figure 5-3b, the
exponential patterns constitute a set of parallel lines such that their intersection point goes up along the vertical axis as \( q \) decreases.

Due to the dynamics of the moments, the evolution in time of the emergent wealth distributions shows the same behavior that in the case \( q = 0 \). The exponential increasing is captured by the evolution in time of the scale parameter \( a \), while the shape \( b \) remains constant in time. Thus, both parameters can be estimated using the relations \( M_1 = ab \) and \( M_2 = a^2b(b + 1) \). The relation obtained for the shape \( b \) using results 5-6 and 5-18 reads as follows:

\[
b = \frac{1}{m_2 - 1} = \frac{3(\lambda - s) [(1 - s + 2q)^2(\lambda + s - 2\lambda s) - \lambda^2(1 - s)^3] - \lambda^2(1 - s)^2 [2(1 - \lambda)^2 + 6q(1 - s + q)]}{\lambda^2(1 - s)^2 [(1 - \lambda)(4 - \lambda - 3s) + 12q(1 - s + q)] - 3(\lambda - s) [(1 - s + 2q)^2(\lambda + s - 2\lambda s) - \lambda^2(1 - s)^3]},
\]

where \( m_2 \) is the coefficient multiplying the exponential function in equation 5 – 18.

Using the relation for \( b \), the scale \( a \) is obtained as:

\[
a = \frac{1}{b} \exp \left( \frac{2t}{N} \right)
\]

The evolution in time of the wealth distributions for the cases \( \lambda = 0.8, s = 0.1, q = \{0.1, 0.3, 0.5, 0.7\} \) is shown in figures 5-2. In subplots a-d, the simulated data is fitted using the MLE method. In general, the gamma distribution is accepted, in all the cases studied, by the Kolmogorov-Smirnov test with a level of significance \( \alpha = 0.5 \). In addition, theoretical relations 5-19 and 5-20 matches with a high level of accuracy the value of the parameters estimated by means of the MLE method. The comparison between both results is shown in subplots e-f).

In general, the expressions for the moments 5-6 and 5-18, and the parameters defined by relations 5-19 and 5-20 are the most general results of the KWEMEG. It is easy to check that setting \( s = 0 \) or \( q = 0 \), this relations reduced to the simplified results obtained in the conservative and the non-conservative regimes.In this order of ideas, it is important to remark that the KWEMEG constitutes a strong extension of the CCM, by introducing a parameter given account of the saving of production, which induces the emergence of a non-conservative regime of economic growth, but also a taxation rate, which has effects in both regimes.
5.2 Taxation and redistribution of wealth

Fig. 5-4: Evolution in time of the emergent wealth distributions for $q \neq 0$. a-d) The parameters of the gamma distributions were computed, in all the cases, using the Maximum likelihood estimation method (MLE). In general, the simulated data is well fitted by the gamma probability density function. e-f) The scale parameter $a$ follows the function $a = (m_2 - 1) \exp(g_{N}^{2N})$, while the shape parameter remains constant in time. In all the cases studied, their values are well fitted by the theoretical functions obtained using the Boltzmann equation.
Non-conservative models of wealth exchange

5.3. Quasy-steady distributions

An approximation to the quasi-stationary state of the non-conservative regime is achieved by studying the self-similar distributions for the normalized variable \( \tilde{w} = \frac{w}{\langle w \rangle} \), where \( \langle w \rangle = M_1(t) = \exp \left( \frac{g N t}{N} \right) \). Under this normalization the probability density function of the gamma distribution reads as:

\[
    f(\tilde{w}) = \frac{1}{\tilde{a}^b \Gamma(b)} \left( \frac{\tilde{w}}{\tilde{a}} \right)^{b-1} \exp \left( -\frac{\tilde{w}}{\tilde{a}} \right) d\tilde{w},
\]

where the scale parameter is \( \tilde{a} = \frac{a}{\langle w \rangle} \). Note that the value of \( \tilde{a} \) is constant in time, due to the fact that \( a(t) = a_0 \exp \left( \frac{g N t}{N} \right) \), and therefore \( \tilde{a} = a_0 \). In addition, the shape parameter
5.3 Quasy-steady distributions

is not altered by the effect of the normalization. Thus, the self-similar distributions give a good picture about the stylized facts of the distribution of wealth.

The results obtained for $\lambda = 0.8$ fixed and $s = \{0, 0.1, 0.2, 0.3\}$; and for $\lambda = 0.8$, $s = 0.1$ and $q = \{0.1, 0.3, 0.5, 0.7\}$ are shown in figures 5-5 a-b). In general, the distribution tends to increase its peakedness for higher values of $q$. Conversely, the increasing of $s$ causes the decreasing the peakedness, which implies that the distribution becomes less egalitarian. This effect is reflected by the Gini index in subplots c-d), which behaves directly proportional to $s$ and inversely proportional to $q$. In these cases, the values computed from the simulated data are well predicted by the theoretical expression for the Gini index of the gamma distribution [55]:

$$G = \frac{1}{\sqrt{\pi}} \frac{\Gamma(b + \frac{1}{2})}{\Gamma(b + 1)},$$

(5-22)

where $b$ is given by equations 5-16 and 5-19.

The emergent distributions are successfully characterized by means of the gamma probability density function, however, in the limit $s \to \lambda$ the goodness of fit is lost for the majority of the cases. In addition, this limit allows to obtain values of the Gini index greater that 0.5, which removes the restriction of the CCM imposed by the case $\lambda = 0$. This effect is shown in figures 5-6, assuming $q = 0$. Note that in both cases, the prediction made using the theoretical relation for the Gini index of the gamma distribution does not matches the values computed from the simulated data, however it still gives an idea for the behavior of the index.

The effect of dividing by $\langle w \rangle$ induces a quasi-stationary state, in which the entropy reaches a
maximum, as it is shown in figures 5-7. This state variable is computed from simulated data using the Shannon’s definition, as in the previous chapter. In particular, for the case \( q = 0 \), it is found that for higher values of \( \lambda \), the relaxation process takes more time, conversely, the system reaches faster the quasi-stationary state as \( s \) increases. Furthermore, the change of \( q \) can be neglected for the study of the quasi-stationary state, due to the fact that the increasing in time of the average wealth does not depend on the taxation value.

![Graph showing the quasi-stationary state](image)

**Fig. 5-7**: The quasi-stationary state of the self-similar distributions for \( \bar{w} = \frac{w}{\langle w \rangle} \) is studied using the Shannon entropy. In both cases, it was fixed \( q = 0 \). **a)** In general, the relaxation process is faster for higher values of \( s \). **b)** This process is slower for higher \( \lambda \). In order to make easier the comparison between curves, all the results were normalized to one by dividing by the maximum value of the entropy for every case.

### 5.3.1. Mean field approach to the behavior of the self-similar distributions

From a mesoscopic point of view, the expressions governing the dynamics of the individual wealth \( \Delta w_{i,j} \) can be simplified assuming that every economic agent feels an average influence form its environment \( \langle w \rangle = \frac{1}{N} \sum_{i=1}^{N} w_i \). This fact implies that any economic agent \( k \) interacts with a representative agent having an average wealth \( \langle w(t) \rangle \), in accordance with the following equations:

\[
\Delta w'_k = \frac{(1 - \lambda)(1 - s)}{(\lambda - s)(1 - s + 2q)} \{\varepsilon \lambda (w_k + \langle w \rangle) - \lambda w_k + sw_k\} \\
+ \frac{q}{(\lambda - s)(1 - s + 2q)} \{\lambda (\langle w \rangle - w_k) + s[2w_k - \lambda (w_k + \langle w \rangle)]\} 
\]  

(5-23)
5.3 Quasy-steady distributions

\[ \Delta w''_k = \frac{(1 - \lambda)(1 - s)}{(\lambda - s)(1 - s + 2q)} \left\{ -\varepsilon \lambda(\langle w \rangle + w_k) + \lambda \langle w \rangle + sw_k \right\} + \frac{q}{(\lambda - s)(1 - s + 2q)} \left\{ \lambda(\langle w \rangle - w) + s[2w_k - \lambda(\langle w \rangle + w)] \right\} \]  \tag{5-24}

Due to the dynamics of the model, at every time \( t \), every economic agent \( k \) has a probability \( \frac{1}{N} \) of increasing their wealth through \( \Delta w''_j \) and the same probability of increasing through \( \Delta w'_j \):

\[ w_k(t + 1) = w_k(t) + \frac{1}{N} \{ \Delta w'_k + \Delta w''_k \} .\]  \tag{5-25}

Thus, replacing expressions 5-23 and 5-24, it is obtained:

\[ w_k(t + 1) = w_k(t) + \frac{2}{N} \frac{(1 - \lambda)}{(\lambda - s)} sw_k(t) + \frac{1}{N} \frac{(1 - s)(1 - \lambda + 2q)}{(\lambda - s)(1 - s + 2q)} \lambda \left[ \langle w \rangle - w_k(t) \right] .\]  \tag{5-26}

Assuming a continuous time horizon, such that \( \Delta t \to 0 \), this equation is transformed into the following differential equation:

\[ \frac{dw_k}{dt} = g \frac{2}{N} w_k(t) + \frac{J}{N} \left[ \langle w \rangle - w_k(t) \right] ,\]  \tag{5-27}

where \( g = \frac{(1 - \lambda)}{(\lambda - s)} s \) is the rate of economic growth and \( J = \frac{(1 - s)(1 - \lambda + 2q)}{(\lambda - s)(1 - s + 2q)} \lambda \) is the rate of exchange between economic agents. Note that the last expression is analogous to the equation from the mean field approximation of Bouchaud and Mézard in reference [10]. Nevertheless, the factor \( g \) is a deterministic variable defined as a function of exogenous parameters \( s \) and \( \lambda \), and independent of \( q \). On the other hand, \( \frac{J}{N} \) contains the information of the average rate of transactions between agents.

Summing over \( k \), divided by \( N \), and integrating the result, it is found that the average wealth increases in time as:

\[ \langle w(t) \rangle = \langle w_0 \rangle \exp \left( \frac{2t}{N} \right) ,\]  \tag{5-28}

This result is exactly the same relation obtained for the first moment of the distributions through the Boltzmann kinetic equation. However, in this case, it is a consequence of simplifying the dynamics of the model using a mean field approximation, which is almost exactly because in the pairwise dynamics between agents there are no big fluctuations around the mean values of the variables.

The behavior of the self similar distributions can be approached by replacing the variable \( \tilde{w}_k = \frac{w_k}{\langle w \rangle} \), such that \( w_k = \langle w_0 \rangle \exp (g \frac{2t}{N}) \tilde{w}_k \), in equation 5-27. Computing the algebraic steps, it is obtained:

\[ \frac{d}{dt} \left( \langle w_0 \rangle \exp \left( g \frac{2t}{N} \right) \tilde{w}_k \right) = g \frac{2}{N} \left( \langle w_0 \rangle \exp \left( g \frac{2t}{N} \right) \tilde{w}_k \right) + \frac{J}{N} \left( \langle w \rangle - \langle w_0 \rangle \exp \left( g \frac{2t}{N} \right) \tilde{w}_k \right) .\]
\[ \langle w_0 \rangle \exp \left( g \frac{2t}{N} \right) \frac{d\tilde{w}_k}{dt} = J \frac{1}{N} \left( \langle w \rangle - \langle w_0 \rangle \exp \left( g \frac{2t}{N} \right) \right) \tilde{w}_k \]

\[ \Rightarrow \frac{d\tilde{w}_k}{dt} = J \frac{1}{N} (1 - \tilde{w}_k). \]  

(5-29)

The solution of this equation is given by the function:

\[ \tilde{w}_k = 1 - \exp \left( -\frac{J}{N} t \right). \]  

(5-30)

This result implies that, in fact, the wealth of the system remains constant in time of the normalized wealth \( \tilde{w}_k \).

In the context of the kinetic theory, under certain limits, the non-conservative dynamics lead to the rising of Pareto tails [29, 52]. The emergence of this distribution patterns is easier to check by studying the behavior of the self-similar distributions \( f(\tilde{w}) \). However, it requires special conditions over the rate of growth, as it is shown by Bouchaud and Mézard [10]. Supposing that \( g(t) \) is a stochastic variable in time, normally distributed with mean \( \mu \) and variance \( \sigma^2 \), the solution of equation 5-27 is given by:

\[ \langle w(t) \rangle = \langle w_0 \rangle \exp \left[ (n + \sigma^2) \frac{2t}{N} \right]. \]  

(5-31)

In this case, the increasing of the average wealth depends on the mean and the variance of \( g \), and not directly on it. Thus, replacing \( \tilde{w}_k \), equation 5-27 is transformed into:

\[ \frac{d\tilde{w}_k}{d\tau} = \left( g(\tau) - n - \sigma^2 \right)\tilde{w}_k + 2J (1 - \tilde{w}_k), \]  

(5-32)

where \( \tau = \frac{2t}{N} \). Note that the dependence on the rate \( g(t) \) was not vanished, in contrast to equation 5-29, because of its stochastic nature. Furthermore, this differential equation analogous to a diffusion process. Thus, it can be associated to the following Fokker-Planck equation, in accordance to reference [10]:

\[ \frac{\partial f(\tilde{w})}{\partial \tau} = \frac{\partial [2J(\tilde{w} - 1) + \sigma^2 \tilde{w}] f(\tilde{w})}{\partial \tilde{w}} + \sigma^2 \frac{\partial}{\partial \tilde{w}} \left[ \tilde{w} \frac{\partial f(\tilde{w})}{\partial \tilde{w}} \right]. \]  

(5-33)

At equilibrium (\( \frac{df_{eq}(\tilde{w})}{d\tau} = 0 \)), the solution of this equation is [10, 29]:

\[ f_{eq}(\tilde{w}) = \mathcal{L} \exp \left[ -\frac{\gamma - 1}{\tilde{w}^{1 + \gamma}} \right], \]  

(5-34)

where \( \mathcal{L} = \frac{(\gamma - 1)^{\gamma}}{\Gamma(\gamma)} \), and \( \gamma = \frac{2J}{\sigma^2} + 1 \). This result shows the evidence of power laws for high values of \( \tilde{w} \) in the stochastic regime. However, the stochastic behavior of \( g \) does not constitute an emergent property of the microeconomic dynamics. This fact is proposed as a further research problem underlying the KWEMEG.
6. Long-term behavior of the non-conservative regime

The variable $g$ has been identified so far as the rate of economic growth due to the increasing of wealth in time. According to the analysis presented using the Boltzmann kinetic equation, the average wealth of the agents follows the exponential function in time:

$$\langle w \rangle = \exp \left( \frac{g}{N} \right),$$

(6-1)

where $N$ is the total number of economic agents, and the factor $\frac{2}{N}$ corresponds to the normalization of the time due to the fact that, at every time-step, only two agents trade in the market.

Fig. 6-1.: a-b) The rate of economic growth $g = \frac{s(1-\lambda)}{\lambda-s}$ increases asymptotically with respect to the plane $s = \lambda$, in such a way that it diverges in the limit $s \to \lambda$. In general, $g \in \mathbb{R}$, however, only the domain $[0, 1]$ is considered in this work, because of its economic meaning.

The mathematical expression governing the rate of economic growth is:

$$g(\lambda, s) = \frac{l - \lambda}{\lambda - s} s.$$  

(6-2)

Note that $g$ is not altered by the effect of the taxation rate, and it vanishes as $s = 0$, limiting the model to the conservative regime. On the other hand, the case $\lambda = 0$ implies $g(\lambda = 0, s) = -1$, for any value of $s$. And moreover, if $s > \lambda$ then $g < 0$, which implies the
loss of wealth in time. From the point of view of economics, this loss can be caused by a depreciation effect on the wealth, or any decrease in the productive capacity of the system. However, this effect is not captured by the model, and therefore is excluded from the analysis. In this order of ideas, the parameters $s$ and $\lambda$ are restricted to the domain $\lambda \epsilon [0, 1]$, $s \epsilon [0, \lambda)$.

The behavior of $g$ is presented in figure 6-1 as function of $\lambda$ and $s$. Broadly, this rate increases asymptotically with respect to the plane $s = \lambda$, due to the fact that the coefficient $\frac{1}{\lambda - s}$ causes the divergence of $g$ for $s \to \lambda$.

![Graphs showing the behavior of $g$](image)

**Fig. 6-2.:** The total production of the system increases exponentially as $Y = 2\frac{1-\lambda}{\lambda - s} \exp\left(\frac{g^2 t}{N}\right)$. The square markers correspond to the simulated data, computed using the definition $p_x X + p_z Z$, and the solid lines are the theoretical form of $Y$. **a)** The increasing of total production is faster for higher values of $s$. **b)** On the other hand, the increasing of $\lambda$ diminishes the economic growth.

An important feature of the KWEMEG is that the relation between the parameters $s$ and $\lambda$ determines the increasing of production in time. In this context, total production $Y$ can be computed, at every time step, using the amounts of goods produced by every agent $X$ and $Z$ multiplied by the clearing prices $\hat{p}_x$, $\hat{p}_z$. In chapter 2, it was obtained that:

$$Y = \hat{p}_x X + \hat{p}_z Z = \frac{1 - \lambda}{\lambda - s} \left[w_i(t) + w_j(t)\right]. \quad (6-3)$$

The average increasing of this factor causes the exponential growth in time of total income as:

$$Y = 2\langle w_0 \rangle \frac{1 - \lambda}{\lambda - s} \exp\left(\frac{g^2 t}{N}\right). \quad (6-4)$$

On the whole, the economic growth is boosted by the saving of production and diminished by the exchange aversion. In addition, the increasing of production is completely independent
of the taxation rate. This relation reproduces, with a high level of accuracy, the behavior of the income computed from the simulated data using \( \hat{p}_z X + \hat{p}_z Z \). The results for the cases \( \lambda = 0.8, s = 0.0, 0.2, 0.4, 0.6 \) and \( s = 0.1, \lambda = \{0.2, 0.4, 0.6, 0.8\} \) are shown in Figure 6-2.

In chapter 2, it was shown that the evolution in time of total wealth is given by the equation:

\[
W(t + 1) = W(t) + \frac{1 - \lambda}{\lambda - s} s[w_i(t) + w_j(t)].
\] (6-5)

Comparing this equation with 6-3 it is obtained the following expression:

\[
W(t + 1) = W(t) + sY(t),
\] (6-6)

which leads to the following differential equation, assuming a continuous time horizon:

\[
\frac{dW(t)}{dt} = sY(t).
\] (6-7)

This result is the same obtained in neoclassical macroeconomics, for the accumulation of wealth, by assuming that the investment \( I \) perfectly matches the consumption \( C \) in the income, i.e., \( Y = C + I \) [41]. However, in this work, equation 6-7 is an emergent property of the microeconomic dynamics of the KWEMEG.

### 6.1. The labor income

It is clear from equation 6-3, that the only production factor increasing the income of the system is the wealth of the agents. However, in accordance with the discussion proposed in chapter 2, a more realistic description of an economy, should consider the labor as a production factor increasing total income. In order to introduce this factor, it is proposed a slight modification in the microeconomic dynamics of the KWEMEG. Thus, at every time step it supposed that all the economic agents get an average unity of salary \( \bar{Y}_l \) related to their labor in the economy, in such a way that the total income of the system is measured as:

\[
Y(t) = p_x X + p_z Z + N\bar{Y}_l.
\] (6-8)

Now, at every time-step, every economic agent saves a rate \( s \) of its salary. Therefore, the individual wealth increases as:

\[
w_k(t + 1) = w_k(t) + s\bar{Y}_l.
\] (6-9)
In general, the labor income $Y_l$ does not modify the rate of economic growth. Therefore, $Y$ can be fitted, in both cases, using relation $Y(t) = \left(\frac{2(1-\lambda)}{\lambda-s}\right) + N\bar{Y}_L \exp\left(g_2 t\right)$

Note that the pure kinetic-like dynamics introduced in the previous chapters has moved away to an heterogeneous model with a non-interacting term similar to the one proposed in reference [38]. Thus, the total wealth of the system at $t + 1$ is computed as:

$$W(t + 1) = W(t) + \frac{1 - \lambda}{\lambda - s} s[w_i(t) + w_j(t)] + sN\bar{Y}_L.$$  

(6-10)

The previous result still satisfies the conditions 6-7 and 6-6, which constitutes a central hypothesis in the Solow model of economic growth [42]. In addition, total production of the system increases as:

$$Y(t) = Y_0 \exp\left(\frac{2t}{N}\right),$$  

(6-11)

where $Y_0 = \frac{(1-\lambda)}{\lambda-s} 2\bar{w}_0 + N\bar{Y}_L$. This result can be rapidly checked by replacing $w_k(t) = w_k(t) + s\bar{Y}_l$ in the mean field approximation proposed in the previous chapter. In addition, it is reduced to equation 6-4 by setting $\bar{Y}_l = 0$. The simulated data obtained by adding dynamic 6-9 to the Monte Carlo algorithm described in chapter 2 is fitted using this result for the cases $\lambda = 0.8, s = 0.1, q = 0$, $\lambda = 0.8, s = 0.5, q = 0$; and considering $Y_l = 0$ and $Y_l \neq 0$, in figure 6-3.
6.2. Piketty’s second fundamental law of capitalism

The behavior of total income and total wealth in time leads to an important emergent property described by Thomas Piketty as the second fundamental law of capitalism [36]. According to the author, in the long-term, the ratio between wealth and income $W(t)/Y(t)$ tends to the ratio between the average saving rate of the population $s$, and the rate of economic growth $g$ [59]. Specifically, the ratio $W(t)/Y(t)$ is defined within the frame of Piketty’s works on economic inequality as the variable $\beta$, and its long-term behavior is induced from the macroeconomic result proposed in the Solow model of economic growth [60, 42].

Replacing the result 6-8 in the equation $\frac{dW(t)}{dt} = sY(t)$, and integrating the result it is obtained:

$$W(t) = \frac{s}{g}Y(t) + \left[ W_0 - \frac{s}{g}Y_0 \right].$$  \hfill (6-12)

where $\tilde{g} = \frac{2g}{N}$ is the rate of economic growth normalized by the transaction rate and $Y_0 = \frac{(1-\lambda)}{(\lambda-s)}2\bar{w}_0 + N\bar{Y}_l$ is the initial income.

This relation can be expressed in terms of the ratio $\beta(t) = \frac{W(t)}{Y(t)}$ by dividing by $Y(t)$ as follows:

$$\beta(t) = \frac{s}{\tilde{g}} + \left[ W_0 - \frac{s}{g}Y_0 \right] \frac{1}{Y(t)}. \hfill (6-13)$$

The previous result describes the behavior in time of the ratio $\beta(t)$. It is clear that, in the limit $t \to \infty$, the exponential behavior of $Y(t)$ tends to vanish the term inside the brackets, therefore equation 6-13 becomes:

$$\lim_{t \to \infty} \beta(t) = \frac{s}{\tilde{g}}. \hfill (6-14)$$

This results sets an interesting mathematical formalization of the second fundamental law of capitalism from the point of view of the microeconomics. The concept of infinity is introduced in the sense of a very large but reasonable period of time over which the economy reaches a steady state. The last result is a general property of the model which is also satisfied if $\bar{Y}_l = 0$. In that case it is found:

$$W_0 - \frac{s}{g} \left( \frac{(1-\lambda)}{(\lambda-s)}2\bar{w}_0 \right) = 0. \hfill (6-15)$$

Therefore, $\beta = \frac{s}{\tilde{g}}$ for every $t$. The asymptotic behavior of the simulated for $\beta(t)$ is shown in figures 6-4. Note that, in all the cases, the ratio saturates around the expected value, $s/\tilde{g}$. It was used $q = 0$ in all the cases, due to the fact that the economic growth does not depends on the taxation.
For $t \to \infty$ it is obtained that $\beta(t) = \frac{\kappa}{\theta}$. Thus, Piketty’s second fundamental law of capitalism is satisfied by all the cases studied.

This emergence of this fundamental law constitutes one the most important results of the KWEMEG, due to the fact that it is framed as one of the key factors governing the wealth inequality. In addition, it opens the possibility of modeling empirical facts, assumed as principles in the frame of macroeconomics, from the complex system perspective of the Kinetic exchange models of market.
7. Conclusions

The Kinetic wealth-exchange model of economic growth (KWEMEG) constitutes a strong extension of the Model of Chakraborti and Chakrabarti (CCM), achieved by introducing a taxation rate \( q \) and a parameter \( s \) imposing the saving of production, into the formalism of the Utility function. The maximization of the utility leads to a generalized dynamic, depending on three exogenous parameters: \( q, s \) and \( \lambda \), the original parameter of the CCM, which is redefined in this context as the exchange aversion of the agents, due to its effect on the dynamics of the KWEMEG. Thus, the combinations of the limit cases \( s = 0 \), and \( q = 0 \), leads to four particular cases, including the CCM, where \( s = q = 0 \), which are divided into a conservative and a non-conservative regime, defined by \( s = 0 \) and \( s \neq 0 \), respectively. An important fact is that, by definition, the conservative regime is related to the study of the distribution of money, while the non-conservative regime can be extended to wealth, in the sense of goods and services.

In the context of the kinetic exchange models of markets, the three cases, different to the CCM, can be considered as independent extensions that constitute three new models. On the one hand, the taxation scheme in the conservative regime, where \( s = 0 \) and \( q \neq 0 \) constitutes a first approach, with a clear microeconomic foundation, to the study of redistribution induced by tax benefit in the frame of the CCM. Similarly, the parameter \( s \neq 0 \) automatically extends the CCM to a non-conservative regime, characterized by the exponential growth of wealth and income, with a constant rate \( g \), which is independent of the taxation rate. And finally, the taxation scheme in the non-conservative regime induces a redistribution effect over wealth that does not affect the economic growth. This models are proposed separately in the thesis, as particular cases of the KWEMEG, and their original results can be consulted in references [43, 44].

In spite of the fact that the emergent macroscopic properties of the models are discussed separately in this work, all of them are studied numerically using the gamma probability density function. In addition, an analytical approach, crossed to these models, is presented using a weak formulation of the Boltzmann equation, which allows to predict the behavior of the moments of the distributions, but also to obtain mathematical relations for the gamma distribution. In most of the cases, the results from this analytical approach are highly accurate to the simulated data, showing the strength of the Boltzmann equation for the study of dynamics, out of the scenario of physics.
In general, the effects of the exogenous parameters on the distributions are similar in the four cases studied. The exchange aversion $\lambda$ and the rate of taxation decreases the level of inequality in the wealth and the money distributions. On the other hand, the saving of production increases the level of inequality. This behavior can be predicted with a high level of accuracy, in most of the cases, by computing the analytical relation for the Gini index, using the resulting parameters from the Boltzmann equation approach. However, the level of accuracy decreases in the limit $s \to \lambda$, where the Gini index of distributions goes over 0.5.

The non-conservative regime allows to tie important problems in the context of modern economics and agent based modeling, with the microeconomic formalism proposed in the CCM. First, the exponential dynamics of the average wealth are obtained with a clear microeconomic perspective, leading to the emergence of well known distribution patterns and macroscopic properties. And second, the explicit emergence of the economic growth, governing the exponential increasing of income in time, leads to the second fundamental law of capitalism, proposed by Piketty as one of the key factors governing the economic inequality [36]. Therefore, the macroscopic results open a macroeconomic perspective for the study of the inequality beyond the usual arguments consider in the context of the kinetic exchange models of markets.
A. Appendix: Derivation of the kinetic exchange models of market

The utility functions describing the preferences of the economic agents in the Kinetic wealth-exchange model of economic growth are defined as:

\[ U_i(x_i, z_i, w_{i}^*) = [Ax_i]^{\theta-q}[Az_i]^{\phi+q}[w_{i}^*]^\lambda, \]
\[ U_j(x_j, z_j, w_{j}^*) = [Ax_j]^{\theta-q}[Az_j]^{\phi+q}[w_{j}^*]^\lambda, \]

where the powers \( \theta \mp q, \phi \pm q \) and \( \lambda \) are normalized to 1 as \( \theta + \phi + \lambda = 1 \). In addition, the restrictions over consumption are defined as follows:

\[ Ap_x x_i + Ap_z z_i + w_{i}^* = w_i + p_x X, \]
\[ Ap_x x_j + Ap_z z_j + w_{j}^* = w_j + p_x Z. \]

The utility functions A-1, A-2 are maximized subject to the constraints A-3, A-4 by defining the following Lagrangian functions, in terms of the multipliers \( \mu_{i,j} \):

\[ L_i(x_i, z_i, w_{i}^*, \mu_i) = [Ax_i]^{\theta-q}[Az_i]^{\phi+q}[w_{i}^*]^\lambda - \mu_i [Ap_x x_i + Ap_z z_i + w_{i}^* - w_i - p_x X], \]
\[ L_j(x_j, z_j, w_{j}^*, \mu_j) = [Ax_j]^{\theta+q}[Az_j]^{\phi-q}[w_{j}^*]^\lambda - \mu_j [Ap_x x_j + Ap_z z_j + w_{j}^* - w_j - p_x Z]. \]

Thus, the maximization process is solved by computing the derivatives equal to zero of \( L_{i,j} \), with respect to \( x_{i,j}, z_{i,j}, \mu_{i,j} \). For the case of the agent \( i \), it is obtained:

\[ \frac{\partial L_i}{\partial x_i} = 0 = a A^a x_i^{a-1}[Az_i][w_{i}^*]^\lambda - \mu_i Ap_x, \]
\[ \frac{\partial L_i}{\partial z_i} = 0 = b [Ax_i]^a A^b z_i^{b-1}[w_{i}^*]^\lambda - \mu_i Ap_z, \]
\[ \frac{\partial L_i}{\partial w_{i}^*} = 0 = \lambda [Ax_i][Az_i][w_{i}^*]^{\lambda-1} - \mu_i, \]
\[ \frac{\partial L_i}{\partial \mu_i} = 0 = Ap_x x_i + Ap_z z_i + w_{i}^* - w_i - p_x X, \]

where the variables \( a = \theta - q, b = \phi + q \) have been introduced in order to simplify the notation.
Replacing the Lagrange multiplier $\mu_i = \lambda [Ax_i]^a [Az_i]^b [w_i^*]^{\lambda-1}$ in equation A-7, the demand function for $x_i$ is computed as:

$$0 = aA^a x_i^{a-1} [Az_i]^b [w_i^*]^{\lambda} - \lambda [Ax_i]^a [Az_i]^b [w_i^*]^{\lambda-1} A p_x$$

$$\Rightarrow 0 = [Ax_i]^a [Az_i]^b [w_i^*]^{\lambda} (ax_i^{a-1} - \lambda[w_i^*]^{-1} A p_x)$$

$$\Rightarrow x_i = \frac{aw_i^*}{\lambda A p_x}. \quad (A-11)$$

Similarly, the demand function for $z_i$ is obtained using equation A-8. The result is:

$$z_i = \frac{bw_i^*}{\lambda A p_z}. \quad (A-12)$$

Replacing $x_i$ and $z_i$ in A-10, the wealth of the agent $i$ at $t + 1$ is obtained as:

$$0 = \frac{aw_i^*}{\lambda} + \frac{bw_i^*}{\lambda} + w_i^* - w_i - p_x X$$

$$\Rightarrow 0 = aw_i^* + bw_i^* + \lambda w_i^* - \lambda w_i - \lambda p_x X$$

$$\Rightarrow w_i^* (a + b + \lambda) = \lambda [w_i + p_x X]$$

$$\Rightarrow w_i^* = \lambda [w_i + p_x X], \quad (A-13)$$

where the normalization condition: $a + b + \lambda = \theta + \phi + \lambda = 1$ was used in the last line.

Applying the same steps over the function A-6, the demand functions obtained for the agent $j$ are:

$$x_j = \frac{a' w_j^*}{\lambda A p_x}, \quad (A-14)$$

$$z_j = \frac{b' w_j^*}{\lambda A p_z}, \quad (A-15)$$

$$w_j^* = \lambda [w_j + p_z Z], \quad (A-16)$$

where the variables $a = \theta + q$, $b = \phi - q$.

The clearing market conditions are defined as $x_i + x_j = X$ and $z_i + z_j = Z$, which implies that the demand perfectly matches the supply. Replacing $x_{i,j}$ the first condition becomes:

$$x_i + x_j = X$$

$$\Leftrightarrow \frac{aw_i^*}{\lambda A p_x} + \frac{a' w_j^*}{\lambda A p_x} = X$$

$$\Leftrightarrow a [w_i + \hat{p}_x X] + a' [w_j + \hat{p}_z Z] = X A \hat{p}_x, \quad (A-17)$$
where $\hat{p}_x$ and $\hat{p}_z$ are the clearing prices.

This equation can be solved by expressing the production of each agent as:

$$-\hat{p}_z Z = \frac{1}{a'} [aw_i + a'w_j + a\hat{p}_x X - A\hat{p}_x X], \quad (A-18)$$

and

$$\hat{p}_x X = \frac{1}{A-a} [aw_i + a'w_j + a'\hat{p}_z Z]. \quad (A-19)$$

Similarly, it the second condition leads to:

$$z_i + z_j = Z$$

$$\iff \frac{bw_i^*}{\lambda A\hat{p}_z} + \frac{b'w_j^*}{\lambda A\hat{p}_z} = Z$$

$$\iff b[w_i + \hat{p}_x X] + b'[w_j + \hat{p}_z Z] = ZA\hat{p}_z, \quad (A-20)$$

which is expressed in the same way of A-19 and A-18 as:

$$\hat{p}_z Z = \frac{1}{A-b'} [bw_i + b'w_j + b\hat{p}_x X], \quad (A-21)$$

and

$$-\hat{p}_x X = \frac{1}{b} [bw_i + b'w_j + b'\hat{p}_z Z - A\hat{p}_z Z]. \quad (A-22)$$

Adding equations A-18 and A-21, the result for the clearing price $\hat{p}_x$ is:

$$\hat{p}_x = \frac{1}{X} \frac{Aaw_i + Aa'w_j + (a'b - ab')w_i}{A(A-a-b') + ab' - a'b} \quad (A-23)$$

On the other hand, the price $\hat{p}_z$ is obtained adding equations A-19 and A-22. The result is:

$$\hat{p}_z = \frac{1}{Z} \frac{Abw_i + Ab'w_j + (a'b - ab')w_j}{A(A-a-b') + ab' - a'b} \quad (A-24)$$

Note that the products between the variables $A, a, a', b, b'$ in the previous expressions can be computed as:

$$ab' - a'b = 2q(\theta + \phi), \quad (A-25)$$

$$A(A-a-b') = A(A + \lambda - 1 + 2q), \quad (A-26)$$
and

\[ A(A - a - b') + ab' - a'b = A(A + \lambda - 1) + 2q(A + \theta + \phi) = A(A + \lambda - 1) + 2q(A + \lambda - 1) = (A + \lambda - 1)(A + 2q). \quad (A-27) \]

Replacing this relations, the clearing prices become:

\[ \hat{p}_x = \frac{A\theta[w_i + w_j] - qA[w_i - w_j] + 2q(\theta + \phi)w_i}{X(\lambda + A - 1)(A + 2q)}, \quad (A-28) \]

\[ \hat{p}_x = \frac{A\phi[w_i + w_j] + qA[w_i - w_j] + 2q(\theta + \phi)w_j}{Z(\lambda + A - 1)(A + 2q)}. \quad (A-29) \]

The stochastic behavior of the model is introduced by defining the variable \( \varepsilon = \frac{\theta}{\theta + \phi} \), such that, for fixed values of \( \lambda \), if \( \theta \) is uniformly distributed over the domain \([0, 1 - \lambda]\), then \( \varepsilon \) is uniformly distributed over the domain \([0, 1]\). The resulting prices obtained from this simplification are:

\[ \hat{p}_x = \frac{A(1 - \lambda)[w_i + w_j] - qA[w_i - w_j] + 2q(1 - \lambda)w_i}{X(\lambda + A - 1)(A + 2q)}, \quad (A-30) \]

\[ \hat{p}_x = \frac{A(1 - \varepsilon)(1 - \lambda)[w_i + w_j] + qA[w_i - w_j] + 2q(1 - \lambda)w_j}{Z(\lambda + A - 1)(A + 2q)}. \quad (A-31) \]

The evolution of wealth in time for each agents is obtained replacing the clearing prices in equations A-13, A-16, as follows:

\[ w_i^* = \lambda w_i + \frac{A(1 - \lambda)\lambda[w_i + w_j] - qA\lambda[w_i - w_j] + 2q\lambda(1 - \lambda)w_i}{(\lambda + A - 1)(A + 2q)}, \quad (A-32) \]

\[ w_j^* = \lambda w_j + \frac{A(1 - \varepsilon)(1 - \lambda)\lambda[w_i + w_j] + qA\lambda[w_i - w_j] + 2q\lambda(1 - \lambda)w_j}{Z(\lambda + A - 1)(A + 2q)}. \quad (A-33) \]

The previous results are written as \( w_i^* = w_i + \Delta w_i \) and \( w_j^* = w_j + \Delta w_j \) by adding and subtracting \( w_{i,j} \), in the corresponding equation. Thus, the result for the first expression is:

\[ w_i^* = w_i - (1 - \lambda)w_i + \frac{A(1 - \lambda)\lambda[w_i + w_j] - qA\lambda[w_i - w_j] + 2q\lambda(1 - \lambda)w_i}{(\lambda + A - 1)(A + 2q)} \]

\[ = w_i - \frac{(1 - \lambda)(\lambda + A - 1)(A + 2q)w_i}{(\lambda + A - 1)(A + 2q)} + \frac{A(1 - \lambda)\lambda[w_i + w_j] - qA\lambda[w_i - w_j] + 2q\lambda(1 - \lambda)w_i}{(\lambda + A - 1)(A + 2q)} \quad (A-34) \]
And for the case of the agent $j$, it is obtained:

$$w_j^* = w_j - (1 - \lambda)w_j + \frac{A(1 - \varepsilon)(1 - \lambda)\lambda[w_i + w_j] + qA\lambda[w_i - w_j] + 2q\lambda(1 - \lambda)w_j}{Z(\lambda + A - 1)(A + 2q)}$$

$$= w_j - \frac{A(1 - \varepsilon)(1 - \lambda)(\lambda + A - 1)(A + 2q)w_j}{(\lambda + A - 1)(A + 2q)}$$

$$+ \frac{A(1 - \varepsilon)(1 - \lambda)\lambda[w_i + w_j] + qA\lambda[w_i - w_j] + 2q\lambda(1 - \lambda)w_j}{Z(\lambda + A - 1)(A + 2q)}. \tag{A-35}$$

Finally, factorizing $A(1 - \lambda)$ and $q$ in both expressions, and replacing $A = 1 - s$, the change of the individual wealth is written as:

$$\Delta w_i = \frac{(1 - \lambda)(1 - s)}{(\lambda - s)(1 - s + 2q)} \left\{ \varepsilon \lambda(w_i + w_j) - \lambda w_i + sw_i \right\}$$

$$+ \frac{q}{(\lambda - s)(1 - s + 2q)} \left\{ \lambda(w_j - w_i) + s[2w_i - \lambda(w_i + w_j)] \right\}, \tag{A-36}$$

$$\Delta w_j = \frac{(1 - \lambda)(1 - s)}{(\lambda - s)(1 - s + 2q)} \left\{ -\varepsilon \lambda(w_i + w_j) + \lambda w_i + sw_j \right\}$$

$$+ \frac{q}{(\lambda - s)(1 - s + 2q)} \left\{ \lambda(w_i - w_j) + s[2w_j - \lambda(w_i + w_j)] \right\}. \tag{A-37}$$

Note that this general expression can be reduced in the particular cases presented in chapter 2 by setting $s = 0$ and/or $q = 0$. In particular, if $s = q = 0$ simultaneously, then expressions A-36 and A-37 correspond to the exchange of money defined in the Chakraborti and Chakrabarti model.
B. Appendix: The gamma distribution

The probability density function for the gamma distribution is defined in terms of the scale and shape parameters, $a$ and $b$, as:

$$f(w)dw = \frac{1}{a\Gamma(b)} \left( \frac{w}{a} \right)^{b-1} \exp \left( -\frac{w}{a} \right) dw,$$

where $\Gamma(b) \equiv \int_0^\infty w^{b-1}e^{-w}dw$ is the gamma function of $b$.

The effect of both parameters can be seen in figures B-1. In particular, for the case $b = 0$, the gamma distribution becomes in the Boltzmann-Gibbs distribution. Using the fact that $\langle w \rangle = ab = a$, it is obtained:

$$f(w)dw = \exp \left( -\frac{w}{\langle w \rangle} \right) dw,$$

where $\langle w \rangle$ is related to temperature. In the case of a closed economy, the average money is defined as the economic temperature.

**Fig. B-1.** Gamma distributions for different values of $a$ and $b$. a-b) The first subplot shows the variation of the distributions for $b$ fixed and different values of $a$. On the other hand, the variation of the distributions in terms of $b$ is shown in the second subplot.
B.1 Moments of the gamma distribution

The moments around zero of this distribution are computed explicitly as:

\[ M_r = \int_0^\infty w^r f(w) \, dw = \int_0^\infty w^r \frac{1}{a \Gamma(b)} \left( \frac{w}{a} \right)^{b-1} \exp \left( -\frac{w}{a} \right) \, dw. \]  \hspace{1cm} (B-3)

Defining the variables \( w' = w/a \), such that \( dw = adw' \), this expression turns into:

\[ M_r = \frac{a^r}{\Gamma(b)} \int_0^{\infty} (w')^{r+b-1} e^{-w'} dw'. \]  \hspace{1cm} (B-4)

The integral in the previous equation corresponds to the gamma function \( \Gamma(r+b) \). Replacing this relation into B-4, the \( r \)th moment reads as:

\[ M_r = \frac{a^r \Gamma(r+b)}{\Gamma(b)}. \]  \hspace{1cm} (B-5)

In particular, the first two moments of the distribution can be computed easily using the recursive relation \( \Gamma(b+1) = b \Gamma(b) \). The result is:

\[ M_1 = ab, \]  \hspace{1cm} (B-6)

\[ M_2 = a^2 b(b+1). \]  \hspace{1cm} (B-7)
C. Gini index

The Gini index is a dispersion metric for distributions, introduced in 1912 by sociologist Corrado Gini, to study the level of inequality in a wealth or income distribution by comparing a set of data with a perfect egalitarian distribution. This comparison is established by means of the Lorentz curves, that are build using the cumulative share of wealth and population from a set of data with \( N \) entrances. In the first case, the cumulative share of wealth is computed by operating the sorted sum over the amount of wealth associated to each entrance, divided by the sum over all the entrances. Similarly, the cumulative share of population is computed as the sorted sum over the number of population units associated to each data point divided by total population. In the case that each entrance of the data corresponds to only one unit of population the cumulative share of population is:

\[
CP_m = \sum_{k=1}^{N} \frac{1}{N} = \frac{m}{N},
\]

where \( m \) is an specific point in the Lorentz curve. In addition, in the case that all the population has the same wealth, the Lorentz curve is a line with slope equal 1, due to the fact that each unity of population contributes with the same amount in the cumulative share of population but also in the cumulative share of wealth. On the other hand, the distance of the curve to the unitary line increases as the distribution becomes less egalitarian. In figures C-1 are shown some examples of Lorentz Curves.

In general the value of the Gini index can be computed by subtracting the areas below the Lorentz curve representing the perfect egalitarian distribution \( A \) and the Lorentz curve for the set of data \( B \) (see figure C-1 a)). The result of this computation is:

\[
G = A - B = 1 - 2 \int_0^1 L(w)dw,
\]

where \( L(w) \) is the Lorentz curve of the distribution.

In the discrete case, this equation transforms into:

\[
G = A - B = \left| \sum_{m=1}^{N} (CP_m - CP_{m-1})(CW_m - CW_{m-1}) \right|,
\]

\(^1\)This appendix focuses on the distribution of wealth. However, all the results can be extrapolated to the case of income.
where the cumulative share of wealth.

In figure C-1, it is shown an example of Lorentz curves for different distributions, built using the simulated data of the KWEMEG. Note that the distances of the Lorentz curves with respect to the line of perfect egalitarianism increases as $\lambda$ and $q$ decreases, but also as $s$ increases. In this case, the Lorentz curves are easily plotted using the fact that every agent contributes with $\frac{1}{N}$ to the cumulative share of population, and that the cumulative share of wealth is computed at every point as:

$$CW_m = \sum_{k=1}^{m} \frac{w_k}{W},$$  \hspace{1cm} (C-4)

where $CW_m$ corresponds to an arbitrary point for an agent $m$, which is computed by summing the wealth of all the agents $k$ such that $w_k < w_m$, divide by total wealth $W$.

**Fig. C-1.** Lorentz curves computed for different cases of the KWEMEG. Note that the distances between the curves and the unitary line representing the perfectly egalitarian distribution increase as $q$ and $\lambda$ decrease and as $s$ increases.

### C.1. Gini index of the gamma distribution

An analytical expression for the Gini index of the gamma distribution can be achieved using the relation for continuous probability density functions:

$$G = \frac{1}{2\langle w \rangle} \int_0^{\infty} \int_0^{\infty} f(w)f(v) |w - v| \, dw \, dv.$$ \hspace{1cm} (C-5)

Replacing the definition B-1, and using the Gaussian Hypergeometric functions it is possible to obtain the analytical relation \cite{55}

$$G = \frac{1}{\sqrt{\pi}} \frac{\Gamma(b + \frac{1}{2})}{\Gamma(b + 1)}.$$ \hspace{1cm} (C-6)
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