Dynkin Functions and Its Applications

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Dynkin Functions and Its Applications.

**Abstract:** Dynkin functions were introduced by Ringel as a tool to investigate combinatorial properties of hereditary artin algebras. According to Ringel, a Dynkin function consists of four sequences associated to $A_n$, $B_n$, $C_n$, $D_n$ and five single values associated to the diagrams $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$. He also proposes to create an On-line Encyclopedia of Dynkin functions (OEDF) with the same purposes as the famous OEIS. Dynkin functions arise from the context of categorification of integer sequences, which according to Ringel and Fahr it means to consider suitable objects in a category instead of numbers of a given integer sequence. They gave a categorification of Fibonacci numbers by using the Gabriel's universal covering theory and the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver. For instance, if $\Lambda$ denotes a hereditary artin algebra associated to a Dynkin diagram $\Delta_n$ then $r(\Delta_n)$ the number of indecomposable modules, $a(\Delta_n)$ the number of antichains in mod $\Lambda$, and $t_n(\Delta_n)$ the number of tilting modules are Dynkin functions. In particular, we are focused on the way that some Dynkin functions act on Dynkin diagrams of type $A_n$.

In this work, we follow the ideas of Ringel regarding Dynkin functions by investigating the number of sections in the Auslander-Reiten quiver of algebras of finite representation type. Dyck paths categories are introduced as a combinatorial model of the category of representations of quivers of Dynkin type $A_n$ and it is shown an algebraic interpretation of frieze patterns as a direct sum of indecomposable objects of the category of Dyck paths. In particular, it is proved that there is a bijection between some Dyck paths and perfect matchings of some snake graphs. The approach allows us to give formulas for cluster variables in cluster algebras of Dynkin type $A_n$ in terms of Dyck paths. At last but not least, it is introduced some Brauer configuration algebras such that the dimension of these algebras and its corresponding centers can be obtained via some combinatorial properties of the Catalan triangle.

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**Keywords:** Auslander-Reiten quiver; categorification; Brauer configuration; Brauer configuration algebra; Catalan triangle; cluster algebras; Dyck paths; Dynkin algebra; Dynkin function; frieze patterns; lattice path; mutation class; perfect matchings; poset; quiver representation; section; triangulations.
Acceptation Note

Thesis Work

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Advisor
Agustín Moreno Cañadas

Bogotá, D.C., October, 2020
Dedicated to

My wife, Estefania.
My son, Andrés.
My parents, Gabriel and Claudia.
My grandmother, Elvia.
My brother, Jorge and his family Andrea and Juan.
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Introduction

Dynkin functions were introduced in 2016 by Ringel with the purpose of giving a systematic study of the relationships between integer sequences and invariants of objects in a category mod $\Lambda$ where $\Lambda$ is a hereditary artin algebra. A Dynkin function $f$ does not depend on orientation and consists of four sequences $f(\mathbb{A}_n)$, $f(\mathbb{B}_n)$, $f(\mathbb{C}_n)$, $f(\mathbb{D}_n)$ and five single values $f(\mathbb{E}_6)$, $f(\mathbb{E}_7)$, $f(\mathbb{E}_8)$ and $f(\mathbb{G}_2)$ [78]. If $\Lambda$ is an algebra of Dynkin type $\Delta_n = \{\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2\}$ then the number $r(\Delta_n)$ of indecomposable modules, the number $a_n(\Delta_n)$ of exceptional antichains in mod $\Lambda$ and $t_n(\Delta_n)$ the number of tilting modules are examples of Dynkin functions. Ringel also proposes to create an On-Line Encyclopedia of Dynkin Functions (ODEF) with the same purposes as the famous On-Line Encyclopedia of Integer Sequences (OEIS) which is the main tool dealing with the research of integer sequences.

Dynkin functions are a way to categorify integer sequences. According to Ringel and Fahr a categorification of an integer sequence means to consider instead of numbers in the sequence suitable invariants of objects in a category. Ringel and Fahr gave a categorification of Fibonacci numbers by using the Gabriel’s universal covering theory and the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver [49, 50]. The categorification of generalized non-crossing partitions (in the sense of Kreweras) of a given finite set has been studied by Hubery, Krausse, Ingalls, Ringel and Thomas amongst others mathematicians [59, 79]. Therefore, researches regarding Dynkin functions not only impact on the theory of representation of algebras if not another fields of the mathematics as combinatorics and number theory, for instance, factorization of numbers associated to invariants of algebras of Dynkin type $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$ seems to be very interesting as Ringel quotes in [78].

Although Ringel’s ideas regarding categorification of integer sequences are so new, they have inspired different researches of many mathematicians, we recall here works on categorification of different integer sequences obtained by the author, Cañadas, and Giraldo et al who have used Kronecker modules, tiled orders and the theory of representation of posets to categorify some integer sequences [27, 31, 35, 41]. In this direction, we use lattice paths connecting points of some suitable posets to investigate the number of sections $S(\Delta_n)$ in the Auslander-Reiten quiver of some algebras as a Dynkin function. Some interesting integer sequences arise from this research, for instance, Fermat numbers (i.e., numbers of the form $2^{2^j} + 1$) is a subsequence of an integer sequence whose some of its elements can be interpreted as the number of some lattice paths via the procedures used in this work. We also give a formula partition for numbers in the sequence A049611 in the OEIS by using sections in the Auslander-Reiten quiver of algebras of Dynkin type $\mathbb{A}_n$. Besides, an explicit formula for sections in the Auslander-Reiten quiver of algebras
of this type is given, in particular, categorifications of the integer sequences A083329 and A000295 in the OEIS are obtained by interpreting each of its elements as the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type $\mathbb{A}_n$ [38].

Another interesting integer sequence with many interpretations in the theory of representation of algebras is the sequence of Catalan numbers, i.e., the sequence whose elements are numbers of the form $C_n = \frac{1}{n+1} \binom{2n}{n}$ [92]. For instance, Gabriel and De la Peña proved that Catalan numbers count the number of discrete subsets contained in the set of representatives of isoclasses of indecomposable finite-dimensional modules over a Dynkin algebra of type $\mathbb{A}_n$ (with $\mathbb{A}_n$ linearly oriented).

In the last few years, researches regarding connections between cluster algebras and different fields of mathematics have been growing. For instance, relationships between cluster algebras, quiver representations, combinatorics and number theory have been reported by Fomin et al., Shiffler et al., K. Baur et al., Assem et al. amongst a great number of mathematicians [4,9,19,22,52,54–56].

Perhaps the Catalan combinatorics (which consists of all the enumeration problems whose solutions are Catalan numbers) is the most appropriate environment for the investigation of cluster algebras of Dynkin type $\mathbb{A}_n$. Among all these kinds of problems, for example, it is possible to prove that the Catalan numbers count [92]:

1. The number of plane binary trees with $n + 1$ endpoints (or $2n + 1$ vertices),
2. The number of ways to parenthesize a string of length $n + 1$ subject to a non associative binary operation,
3. The number of paths $P$ in the $(x, y)$-plane from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ that never pass below the $x$-axis. Such paths are called Dyck paths,
4. The number of triangulations of an $(n + 3)$ polygon,
5. The number of clusters of a cluster algebra of Dynkin type $\mathbb{A}_n$.

Regarding integer friezes, we point out that Propp in [71] reminds that Conway and Coxeter completely classified the frieze patterns whose entries are positive integers, and showed that these frieze patterns constitute a manifestation of the Catalan numbers. Specifically, that there is a natural association between positive integer frieze patterns and triangulations of regular polygons with labelled vertices. According to Baur and Marsh [9], a connection between cluster algebras and frieze patterns was established by Caldero and Chapoton [18], which showed that frieze patterns can be obtained from cluster algebras of Dynkin type $\mathbb{A}_n$.

Another example of the use of the Catalan combinatorics as a tool to describe the structure of cluster algebras, was given by Schiffler et al. [19,22,69], who found out formulas for cluster variables based on its relations with some triangulated surfaces and perfect matchings of snake graphs. They also proved that there is a way of obtaining the number
of perfect matchings of a given snake graph by associating a suitable continued fraction defined by the sign function of the graph.

Given a non-negative integer \( n \) and a triangulation \( T \) of a regular polygon with \((n + 3)\) vertices. Caldero, Chapoton and Schiffler [17] gave a realization of the category \( C_C \) of representations of a quiver \( Q_C \) associated to a cluster \( C \) of a cluster algebra in terms of the diagonals of the \((n + 3)\) polygon. They proved that there is a categorical equivalence between the categories \( C_T \) and \( \text{Mod} \, Q_T \), where \( C_T \) is the category whose objects are positive integral linear combinations of positive roots (i.e., diagonals that does not belong to the triangulation \( T \)), whereas \( \text{Mod} \, Q_T \) denotes the category of modules over the quiver \( Q_T \) with triangular relations induced by the triangulation \( T \).

Following the ideas of Caldero, Chapoton and Schiffler, in this work, a combinatorial model of the category of representations of Dynkin quivers of type \( A_n \) with relations is developed by using Dyck paths. This approach allows us to realize perfect matchings of snake graphs as objects of suitable Dyck paths categories, and with this machinery a formula for cluster variables based on Dyck paths is obtained.

We show that frieze patterns arise from Dyck paths and they can be written in terms of Dyck path categories. We also introduce a family of Brauer configuration algebras associated to Dyck paths. Combinatorial properties of the Catalan triangle are used to find out formulas for the dimension of this type of algebras and its corresponding centers.

Main results, contributions, papers and conferences

This research regards the categorification of integer sequences and some applications of Dynkin functions in representation theory of algebras and combinatorics.

Contributions

The following are the main contributions:

1. It is given a recurrence formula of the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type via lattice paths connecting minimal and maximal points of suitable posets.

2. It is obtained a categorification of integer sequences arising from sections in the Auslander-Reiten quiver of algebras of Dynkin type in the sense of Ringel and Fahr.

3. Dyck paths categories are introduced and it is proved that there exists an equivalence of categories between the category of Dyck paths and the category of representations of Dynkin quivers of type \( A_n \) with relations.

4. It is given a formula of the cluster variables of cluster algebras associated to quivers of type \( A_n \) by using Dyck paths.

5. It is established a bijective correspondence between Dyck paths and frieze patterns, attaining in this way a new algebraic interpretation of frieze patterns as a direct sum of indecomposable objects of Dyck paths categories.
6. It is defined Dyck-Brauer configuration algebras, and it is given an explicit formulas of the dimension of these Brauer configuration algebras and its corresponding centers in terms of the Catalan triangle.

**Papers**

Results of this research allowed us to publish the following papers:

1. On the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type \[^{32}\].

2. Integer sequences arising from Auslander-Reiten quivers of some hereditary artin algebras \[^{38}\].

Results of this research allowed us to submit the following manuscript:

1. Dyck paths categories and its relationships with cluster algebras.

**Conferences**

The main results of this research have been presented in the following conferences:


2. UN Encuentro de Matemáticas. Bogotá-Colombia, 06-2018.


5. Primer Encuentro de Estudiantes de Posgrado en Matemáticas, Medellín -Colombia, 02-2020.


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This thesis is distributed as follows:

Chapter 1 aims to present a theoretical introduction of representation theory of algebras, sections in the Auslander-Reiten quiver, representation theory of posets, Brauer configuration algebras, Dyck paths, cluster algebras, cluster-tilted algebras, friezes, snake graphs, and the category of diagonals, as well as, definitions and notations to be used throughout the work.

In chapter 2 it is described a family of posets that allows us to find a formula of the number of sections in the Auslander-Quiver quiver of algebras of Dynkin type $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$. These formulas establish a categorification of some integer sequences in the sense of Ringel and Fahr.

In chapter 3 it is introduced the category of Dyck paths as a combinatorial model of the category of representations of a quiver of type $A_n$ with relations. It is presented a bijective correspondence between a family of words of Dyck paths and the number of perfect matchings of a snake graph. Besides, it is described a formula of cluster variables arising from the Dyck paths of algebras with an underlying graph of type $A_n$.

In chapter 4 it is defined a basic set called diamond which is used to build frieze patterns, these sets are in bijective correspondence with Dyck paths and triangulations of $(n + 3)$ polygons, and it is presented frieze patterns by using indecomposable objects of Dyck paths categories. Dyck-Brauer configuration algebras are introduced and it is given the dimension of these algebras and its corresponding centers.

Finally, appendix A contains examples of integer sequences arising from the number of sections associated to algebras of type $A_n$ and $D_n$. Besides, it is included examples of family of integer sequences and pairs of matrices associated to Dyck-Brauer configuration algebras.
CHAPTER 1

Preliminaries

In this chapter, we present a brief description and important theorems regarding quiver representations in section 1.1. Sections in the infinite translation quiver and Brauer configuration algebras are described in sections 1.2 and 1.3 respectively. Category of representation of ordinary posets and some classical theorems regarding classification of ordinary posets are introduced in section 1.4. In section 1.5 we recall Dyck paths as a Catalan object, whereas some elementary notions of cluster algebras, category of diagonals of an \((n + 3)\) polygon, cluster-tilted algebras, and friezes are defined in sections 1.6, 1.7 and 1.8. Finally, some definitions and results regarding snake graphs are given in section 1.9. Throughout the thesis, \(k\) denotes an algebraically closed field. \(\mathbb{N}, \mathbb{Z}, \mathbb{R}\) and \(\mathbb{C}\) denote the sets natural, integer, real, and complex numbers, respectively \(\{2, 3, 17, 19, 39, 43, 45, 54, 60, 62, 71, 80, 84, 92, 96, 99\}.

1.1 Representation Theory of Quivers

In this section, we present some concepts regarding representations of a quiver. We recall theorems that describe algebras of finite and tame representation type \(\{3, 60, 80, 84\}.

A quiver \(Q = (Q_0, Q_1, s, t)\) is a quadruple consisting of two sets: \(Q_0\) (whose elements are called points, or vertices) and \(Q_1\) (whose elements are called arrows), and two maps \(s, t : Q_1 \to Q_0\), which associate to each arrow \(\alpha \in Q_1\) its source \(s(\alpha) \in Q_0\) and its target \(t(\alpha) \in Q_0\), respectively. Figure 1.1 shows examples of quivers

\[\begin{align*}
\text{Figure 1.1. Examples of quivers.}
\end{align*}\]
A path of length \( l \geq 1 \) with source \( a \) and target \( b \) is a sequence \((a \mid \alpha_1, \ldots, \alpha_l \mid b)\) where \( \alpha_k \in Q_1 \) for all \( 1 \leq k \leq l \), and we have \( s(\alpha_1) = a, t(\alpha_k) = s(\alpha_{k+1}) \) for each \( 1 \leq k < l \), and finally \( t(\alpha_l) = b \). We denote by \( Q_l \) the set of all paths in \( Q \) of length \( l \). We also agree to associate with each point \( a \in Q_0 \) a path of length \( l = 0 \) (denoted by \( e_a = (a\|a) \)).

The path algebra \( kQ \) of \( Q \) is the \( k \)-algebra whose underlying \( k \)-vector space has as its basis the set of all paths \((a \mid \alpha_1, \ldots, \alpha_l \mid b)\) of length \( l \geq 0 \) in \( Q \) and such that the product of two basis vectors \((a \mid \alpha_1, \ldots, \alpha_l \mid b)\) and \((c \mid \beta_1, \ldots, \beta_k \mid d)\) of \( kQ \) is equal to zero if \( t(\alpha_l) \neq s(\beta_1) \) and is equal to the composed path \((a \mid \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k \mid d)\) if \( t(\alpha_l) = s(\beta_1) \).

Let \( Q \) be a finite and connected quiver. The two-sided ideal of the path algebra \( kQ \) generated (as an ideal) by the arrows of \( Q \) is called the arrow ideal of \( kQ \) and is denoted by \( R_Q \). In particular, for each \( l \geq 1 \),

\[
R_Q^l = \bigoplus_{m \geq l} kQ_m.
\]

\( R_Q^l \) is an ideal of \( kQ \). A two-sided ideal \( I \) of \( kQ \) is said to be admissible ideal if there exists an integer \( m \geq 2 \) such that

\[
R_Q^m \subseteq I \subseteq R_Q^2.
\]

If \( I \) is an admissible ideal of \( kQ \), the pair \((Q, I)\) is said to be a bound quiver. The quotient algebra \( kQ/I \) is said to be the algebra of the bound quiver \((Q, I)\) or, simply, a bound quiver algebra.

A relation in \( Q \) with coefficients in \( k \) is a \( k \)-linear combination of paths of length at least two having the same source and target. Thus, a relation \( \rho \) is an element of \( kQ \) such that

\[
\rho = \sum_{i=1}^{m} \lambda_i w_i,
\]

where the \( \lambda_i \) are scalars and the \( w_i \) are paths in \( Q \) of length at least 2 such that, if \( i \neq j \), then the source (resp. the target) of \( w_i \) coincides with that of \( w_j \). If \((\rho_j)_{j \in J}\) is a set of relations for a quiver \( Q \) such that the ideal they generate \( \langle \rho_j \mid j \in J \rangle \) is admissible, we say that the quiver \( Q \) is bounded by the relations \((\rho_j)_{i \in J}\) or by the relations \( \rho_j = 0 \) for all \( j \in J \).

A representation \( M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1} \) of a quiver \( Q \) is a collection of \( k \)-vector spaces \( M_i \), one for each vertex \( i \in Q_0 \), and a collection of \( k \)-linear maps \( \varphi_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)} \), one for each arrow \( \alpha \in Q_1 \).

Let \( M = (M_i, \varphi_\alpha) \), \( M' = (M'_i, \varphi'_\alpha) \) be two representations of \( Q \). A morphism (or homomorphism) of representations \( f : M \rightarrow M' \) is a collection \((f_i)_{i \in Q_0}\) of linear maps \( f_i : M_i \rightarrow M'_i \), such that for each arrow \( i \xrightarrow{\alpha} j \) in \( Q_1 \) the diagram:
commutes, that is \((f_j \circ \varphi_\alpha)(m) = (\varphi'_\alpha \circ f_i)(m)\) for all \(m \in M_i\). Let \(M = (M_i, \varphi_\alpha)\) and \(M' = (M'_i, \varphi'_\alpha)\) be representations of \(Q\). Then

\[
M \oplus M' = \left( M_i \oplus M'_i, \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{pmatrix} \right)_{i \in Q_0, \alpha \in Q_1},
\]

is a representation of \(Q\) called the \textit{direct sum} of \(M\) and \(M'\).

Rep \(Q\) is the category of representations of a quiver \(Q\), rep \(Q\) is the full subcategory of Rep \(Q\) consisting of the finite dimensional representations. Rep \(Q\) and rep \(Q\) are abelian \(k\)-categories. A representation \(M \in \text{rep} \(Q\)\) is called \textit{indecomposable} if \(M \neq 0\) and \(M\) cannot be written as a direct sum of two nonzero representations, that is, whenever \(M \cong N \oplus L\) with \(N, L \in \text{rep} \(Q\)\), then \(N = 0\) or \(L = 0\). A quiver \(Q\) is said to be of \textit{finite representation type} if the number of isoclasses of indecomposable representations of \(Q\) is finite. A quiver \(Q\) is said to be of \textit{infinite representation type} if \(Q\) is not of finite representation type \cite{80}.

\textbf{Theorem 1.1}. \cite{3}. Let \(A = kQ/I\), where \(Q\) is a finite connected quiver and \(I\) is an admissible ideal of \(kQ\). There exists a \(k\)-linear equivalence of categories

\[
F : \text{Mod} \(A\) \rightarrow \text{Rep} \((Q, I)\),
\]

that restricts to an equivalence of categories \(F : \text{mod} \(A\) \rightarrow \text{rep} \((Q, I)\).

Gabriel \cite{58} and Nazarova \cite{72} proved the following theorems, respectively.

\textbf{Theorem 1.2}. \cite{3}. Let \(Q\) be a finite, connected, and acyclic quiver; \(k\) be an algebraically closed field; and \(A = kQ\) be the path \(k\)-algebra of \(Q\).

(a) The algebra \(A\) is representation-finite if and only if the underlying graph \(\overline{Q}\) of \(Q\) is one of the Dynkin diagrams \(A_n, D_n, E_6, E_7,\) and \(E_8\).

(b) If \(\overline{Q}\) is a Dynkin graph, then the mapping \(\text{dim} : M \rightarrow \text{dim} M\) induces a bijection between the set of isomorphism classes of indecomposable \(A\)-modules and the set \(\{x \in \mathbb{N}^n ; q_Q(x) = 1\}\) of positive roots of the quadratic form \(q_Q\) of \(Q\).

(c) The number of the isomorphism classes of indecomposable \(A\)-modules equals \(\frac{1}{2}n(n+1), n^2 - n, 36, 63,\) and \(120\), if \(\overline{Q}\) is the Dynkin graph \(A_n, D_n, E_6, E_7,\) and \(E_8\), respectively.

\textbf{Theorem 1.3}. \cite{84}. Let \(Q\) be a connected quiver without oriented cycles and \(k\) be an algebraically closed field. Then \(kQ\) is representation-tame if and only if the underlying graph \(\overline{Q}\) of \(Q\) is one of the extended Dynkin diagrams \(\widehat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \) and \(\widehat{E}_8\).
1.2 Sections in the Infinite Translation Quiver

In this section, for the sake of clarity we recall the definitions of section, an orbit in an Auslander-Reiten quiver as Assem et al. described in [3].

Let $\Sigma = (\Sigma_0, \Sigma_1)$ be a connected and acyclic quiver. An infinite translation quiver $(\mathbb{Z}\Sigma, \tau)$ has the set $(\mathbb{Z}\Sigma)_0 = \mathbb{Z} \times \Sigma_0 = \{(n, x) \mid n \in \mathbb{Z}, x \in \Sigma_0\}$ as its set of vertices, and for each arrow $\alpha : x \rightarrow y \in \Sigma_1$ there exist two arrows $(n, \alpha) : (n, x) \rightarrow (n + 1, y)$ and $(n + 1, \alpha') : (n, x) \rightarrow (n + 1, x)$ in $(\mathbb{Z}\Sigma)_1$, 

\begin{equation}
(n, \alpha) : (n, x) \rightarrow (n, y) \quad (n, \alpha') : (n + 1, y) \rightarrow (n, x) \text{ in } (\mathbb{Z}\Sigma)_1, \tag{1.2}
\end{equation}

and these are all the arrows in $(\mathbb{Z}\Sigma)_1$. The translation $\tau$ on $\mathbb{Z}\Sigma$ is given by the formula $\tau(n, x) = (n + 1, x)$, and for every $(n, x) \in (\mathbb{Z}\Sigma)_0$ it is defined a bijection between the set of arrows of target $(n, x)$ and the set of arrows of source $(n + 1, x)$ by the formulas:

\begin{align*}
\sigma(n, \alpha) &= (n, \alpha') & \sigma(n, \alpha') &= (n + 1, \alpha), \tag{1.3}
\end{align*}

Let $\Sigma$ be a quiver described in Figure 1.3.
Then the infinite translation quiver of $\Sigma$ is given by Figure 1.4.

Let $(\Gamma, \tau)$ be a connected translation quiver. A connected full subquiver $\Sigma$ of $\Gamma$ is a section of $\Gamma$ if the following conditions are satisfied:

S(1) $\Sigma$ is acyclic.

S(2) For each $x \in \Gamma_0$, there exists a unique $n \in \mathbb{Z}$ such that $\tau^n x \in \Sigma_0$.

S(3) If $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_t$ is a path in $\Gamma$ with $x_0, x_t \in \Sigma_0$, then $x_i \in \Sigma_0$ for all $i$ such that $0 \leq i \leq t$.

For a translation quiver $(\Gamma, \tau)$, the $\tau$-orbit of a point $x \in \Gamma_0$ is defined to be the set of all points of the form $\tau^n x$ with $n \in \mathbb{Z}$. Thus, any section $\Sigma$ meets each $\tau$-orbit exactly once.

Arrows in a section of a translation quiver $(\Gamma, \tau)$ satisfy the following conditions:

1. If $x \rightarrow y$ is an arrow in $\Gamma$ and $x \in \Sigma_0$, then $y \in \Sigma_0$ or $\tau y \in \Sigma_0$.

2. If $x \rightarrow y$ is an arrow in $\Gamma$ and $y \in \Sigma_0$, then $x \in \Sigma_0$ or $\tau^{-1} x \in \Sigma_0$.

Sections are useful to characterize representation-finite tilted algebras. Regarding this subject, we recall the Happel and Ringel’s criterion which states that a connected representation-finite algebra $B$ is a tilted algebra if and only if the Auslander-Reiten quiver of $B$ contains a section.

Henceforth, we let $O_x$ denote the orbit of a fixed element $x \in \Gamma_0$. In particular, if $\Gamma(\text{Mod } A) = (\Gamma_0, \Gamma_1)$ is the Auslander-Reiten quiver of an algebra of Dynkin type $\Delta_n$ then each element of the $\tau$-orbit of an indecomposable projective module will be denoted $\tau^n_i$, $i \in \mathbb{N}$. We also note that in the case of representation-finite hereditary algebras $A$ the vertices of the Auslander-Reiten quiver $\Gamma_A$ corresponding to the indecomposable
projective modules form in $\Gamma_A$ a section of Dynkin class.

As an example in Figure 1.5 we show an oriented quiver $Q$ of type $A_3$ and the corresponding Auslander-Reiten quiver of the algebra $A = kQ$.

![Figure 1.5. Quiver Q and the Auslander-Reiten quiver of kQ.](image)

In this case sections are $S_1 = \{\tau_1, \tau_2, \tau_3\}$, $S_2 = \{\tau_1, \tau_2, \tau_3^{-1}\}$, $S_3 = \{\tau_1^{-1}, \tau_2, \tau_3\}$, $S_4 = \{\tau_1^{-1}, \tau_2, \tau_3^{-1}\}$ and $S_5 = \{\tau_1^{-1}, \tau_2^{-1}, \tau_3^{-1}\}$ all of them of type $A_3$.

### 1.3 Brauer Configuration Algebras

In 2015 Green and Schroll [61] introduced the concept of Brauer configuration algebra as a generalization of a Brauer graph algebra. In general, these algebras are of wild representation type. They showed that Brauer configuration algebras are finite-dimensional symmetric, multiserial, and others. In this section, we recall definitions of Brauer configuration and its Brauer configuration algebra, we present some properties of these algebras [61,83].

A **Brauer configuration** is a tuple $\Gamma = (\Gamma_0, \Gamma_1, \mu, O)$, where:

- (B1) $\Gamma_0$ is a finite set whose elements are called *vertices*.
- (B2) $\Gamma_1$ is a finite collection of multisets called *polygons*. In this case, if $V \in \Gamma_1$ then the elements of $V$ are vertices possibly with repetitions, $\text{occ}(\alpha, V)$ denotes the frequency of the vertex $\alpha$ in the polygon $V$ and the *valency* of $\alpha$ denoted $\text{val}(\alpha)$ is defined in such a way that:
  \[
  \text{val}(\alpha) = \sum_{V \in \Gamma_1} \text{occ}(\alpha, V). \tag{1.4}
  \]
- (B3) $\mu$ is an integer valued function such that $\mu : \Gamma_0 \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers, it is called the *multiplicity function*.
- (B4) $O$ denotes an orientation defined on $\Gamma_1$ which is a choice, for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which $\alpha$ occurs as a vertex, including repetitions, we denote $S_\alpha$ such collection of polygons. More specifically, if $S_\alpha = \{V_1^{(\alpha_1)}, V_2^{(\alpha_2)}, \ldots, V_t^{(\alpha_t)}\}$ is the collection of polygons where the vertex $\alpha$ occurs with $\alpha_i = \text{occ}(\alpha, V_i)$ and $V_i^{(\alpha_i)}$ meaning that $S_\alpha$ has $\alpha_i$ copies of $V_i$ then an orientation
\[ \mathcal{O} \] is obtained by endowing a linear order \(<\) to \(S_\alpha\) and adding a relation \(V_t < V_1\), if \(V_1 = \min S_\alpha\) and \(V_t = \max S_\alpha\). According to this order the \(\alpha_i\) copies of \(V_t\) can be ordered as \(V_{1,i} < V_{2,i} < \cdots < V_{(\alpha_i-1),i} < V_{\alpha_i,i}\) and \(S_\alpha\) can be ordered in the form \(V_1^{(\alpha_1)} < V_2^{(\alpha_2)} < \cdots < V_{(t-1)}^{(\alpha_{t-1})} < V_t^{(\alpha_t)}\).

(B5) Every vertex in \(\Gamma_0\) is a vertex in at least one polygon in \(\Gamma_1\).

(B6) Every polygon has at least two vertices.

(B7) Every polygon in \(\Gamma_1\) has at least one vertex \(\alpha\) such that \(\mu(\alpha)\text{val}(\alpha) > 1\).

The set \((S_\alpha, <)\) is called the \textit{successor sequence} at the vertex \(\alpha\).

A vertex \(\alpha \in \Gamma_0\) is said to be \textit{truncated} if \(\text{val}(\alpha)\mu(\alpha) = 1\), that is, \(\alpha\) is truncated if it occurs exactly once in exactly one \(V \in \Gamma_1\) and \(\mu(\alpha) = 1\). A vertex is \textit{nontruncated} if it is not truncated.

Given a Brauer configuration \(\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})\) we say that the polygon \(V \in \Gamma_1\) is a \(d\)-gon if the number of vertices appearing in \(V\) is \(d\). We say that the configuration \(\Gamma\) is \textit{reduced} if and only if every polygon \(V \in \Gamma_1\) satisfies one of the following conditions:

(i) \(V \cap S_\Gamma = \emptyset\),

(ii) if \(V \cap S_\Gamma \neq \emptyset\), then \(V\) is a 2-gon with only one truncated vertex,

where \(S_\Gamma = \{\alpha \in \Gamma_0 \mid \mu(\alpha)\text{val}(\alpha) = 1\}\).

\textbf{The Quiver of a Brauer Configuration Algebra}

The quiver \(Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1)\) of a Brauer configuration algebra is defined in such a way that the vertex set \((Q_\Gamma)_0 = \{v_1, v_2, \ldots, v_m\}\) of \(Q_\Gamma\) is in correspondence with the set of polygons \(\{V_1, V_2, \ldots, V_m\}\) in \(\Gamma_1\), noting that there is one vertex in \((Q_\Gamma)_0\) for every polygon in \(\Gamma_1\).

Arrows in \(Q_\Gamma\) are defined by the successor sequences. That is, there is an arrow \(v_i \xrightarrow{s_1} v_{i+1}\) provided that \(V_i < V_{i+1}\) in \((S_\alpha, <)\cup\{V_i < V_1\}\) for some nontruncated vertex \(\alpha \in \Gamma_0\). In other words, for each nontruncated vertex \(\alpha \in \Gamma_0\) and each successor \(V'\) of \(V\) at \(\alpha\), there is an arrow from \(v\) to \(v'\) in \(Q_\Gamma\) where \(v\) and \(v'\) are the vertices in \(Q_\Gamma\) associated to the polygons \(V\) and \(V'\) in \(\Gamma_1\), respectively.

\textbf{Ideal of Relations and Definition of a Brauer Configuration Algebra}

Fix a polygon \(V \in \Gamma_1\) and suppose that \(\text{occ}(\alpha, V) = t \geq 1\) then there are \(t\) indices \(i_1, \ldots, i_t\) such that \(V = V_{i_j}\). Then the \textit{special \(\alpha\)-cycles} at \(v\) are the cycles \(C_{i_1}, C_{i_2}, \ldots, C_{i_t}\) where \(v\) is the vertex in the quiver of \(Q_\Gamma\) associated to the polygon \(V\). If \(\alpha\) occurs only once in \(V\) and \(\mu(\alpha) = 1\) then there is only one special \(\alpha\)-cycle at \(v\).
Let $k$ be a field and $\Gamma$ a Brauer configuration. The Brauer configuration algebra associated to $\Gamma$ is defined to be the bounded path algebra $\Lambda_{\Gamma} = kQ_{\Gamma}/I_{\Gamma}$, where $Q_{\Gamma}$ is the quiver associated to $\Gamma$ and $I_{\Gamma}$ is the ideal in $kQ_{\Gamma}$ generated by the following set of relations $\rho_{\Gamma}$ of type I, II and III.

- **Relations of type I.** For each polygon $V = \{\alpha_1, \ldots, \alpha_m\} \in \Gamma_1$ and each pair of nontruncated vertices $\alpha_i$ and $\alpha_j$ in $V$, $\rho_{\Gamma}$ contains all relations of the form $C^{\mu(\alpha_i)} - (C')^{\mu(\alpha_i)}$ or $(C')^{\mu(\alpha_i)} - C^{\mu(\alpha_i)}$ where $C$ is a special $\alpha_i$-cycle at $v$ and $C'$ is a special $\alpha_j$-cycle at $v$.

- **Relations of type II.** The type two relations are all paths of the form $C^{\mu(\alpha)}a$ where $C$ is a special $\alpha$-cycle and $a$ is the first arrow in $C$.

- **Relations of type III.** These relations are quadratic monomial relations of the form $ab$ in $kQ_{\Gamma}$ where $ab$ is not a subpath of any special cycle.

For example, let $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ be a Brauer configuration, where $\Gamma_0 = \{1, 2, 3\}$, $\Gamma_1 = \{V_1 = \{1, 1, 3\}, V_2 = \{1, 2\}, V_3 = \{2, 3, 3\}\}$, $\mu(1) = \mu(3) = 1$ and $\mu(2) = 2$. The successor sequence of vertex 1 is $V_1 < V_1 < V_2$, the successor sequence of vertex 2 is $V_2 < V_3$, and the successor sequence of vertex 3 is $V_1 < V_3 < V_3$. There are two special 1-cycles at $v_1$, $a_1a_2a_3$ and $a_2a_3a_1$. There is only one special 3-cycle at $v_1$, $c_1c_2c_3$. There is one special 1-cycle at $v_2$, $a_3a_1a_2$. The special 2-cycle at $v_2$ is $b_1b_2$. The special 2-cycle at $v_3$ is $b_2b_1$. There are two special 3-cycles at $v_3$, $c_2c_3c_1$ and $c_3c_1c_2$. The ideal $I_{\Gamma}$ is generated by following relations in $\rho_{\Gamma}$:

$$
\begin{align*}
    a_1a_2a_3 &= a_2a_3a_1 = c_1c_2c_3; \\
    a_3a_1a_2 &= (b_1b_2)^2; \\
    (b_2b_1)^2 &= c_2c_3c_1 = c_3c_1c_2; \\
    a_1a_2a_3a_1 &= a_2a_3a_1a_2; \\
    a_3a_1a_2a_3 &= (b_1b_2)^2; \\
    c_1c_2c_3c_1 &= c_2c_3c_1c_2; \\
    c_3c_1c_2c_3 &= c_1c_3a_1; \\
    a_2b_1 &= c_3a_1; \\
    b_1 &= a_2c_3; \\
    b_1 &= a_3c_1; \\
    b_2 &= b_1c_3; \\
    b_2 &= c_2b_2; \\
    c_1 &= b_1b_2.
\end{align*}
$$

(Figure 1.6) Quiver $Q_{\Gamma}$ associated to the Brauer configuration $\Gamma$.

Figures 1.6 and 1.7 show the quiver associated to $\Gamma$ and the indecomposable projective modules of $\Lambda_{\Gamma}$. 
The following results show some properties of Brauer configuration algebras \[61\].

**Theorem 1.4.** Let \( \Lambda \) be a Brauer configuration algebra with Brauer configuration \( \Gamma \).

(i) A Brauer configuration algebra is a finite dimensional symmetric algebra.

(ii) Suppose \( \Gamma = \Gamma_1 \cup \Gamma_2 \) is a decomposition of \( \Gamma \) into two disconnected Brauer configurations \( \Gamma_1 \) and \( \Gamma_2 \). Then there is an algebra isomorphism \( \Lambda_\Gamma \simeq \Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2} \) between the associated Brauer configuration algebra.

(iii) The Brauer configuration algebra associated to a connected Brauer configuration is an indecomposable algebra.

(iv) A Brauer graph algebra is a Brauer configuration algebra.

(v) There is a bijective correspondence between the set of indecomposable projective \( \Lambda \)-modules and the polygons in \( \Gamma \).

(vi) If \( P \) is an indecomposable projective \( \Lambda \)-module corresponding to a polygon \( V \) in \( \Gamma \). Then \( \text{rad}(P) \) is a sum of \( r \) indecomposable uniserial modules, where \( r \) is the number of (nontruncated) vertices of \( V \) and where the intersection of any two of the uniserial modules is a simple \( \Lambda \)-module.

(vii) A Brauer configuration algebra is a multiserial algebra.

**Proposition 1.1.** Let \( \Lambda \) be a Brauer configuration algebra associated to the Brauer configuration \( \Gamma \) and let \( C = \{C_1, \ldots, C_t\} \) be a full set of equivalence class representatives of special cycles. Assume that, for \( i = 1, \ldots, t \), \( C_i \) is a special \( \alpha_i \)-cycle where \( \alpha_i \) is a nontruncated vertex in \( \Gamma \). Then

\[
\text{dim}_k \Lambda = 2|Q_0| + \sum_{C_i \in C} |C_i| (n_i |C_i| - 1),
\]

where \( |Q_0| \) denotes the number of vertices of \( Q \), \( |C_i| \) denotes the number of arrows in the \( \alpha_i \)-cycle \( C_i \) and \( n_i = \mu(\alpha_i) \).

Sierra proved the following result \[83\].
Theorem 1.5. Let $\Lambda = kQ_\Gamma / I_\Gamma$ be the Brauer configuration algebra associated to the connected and reduced Brauer configuration $\Gamma$. Then

$$\dim_k Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#\text{Loops}(Q_\Gamma) - |\mathcal{C}_\Gamma|,$$

where $\mathcal{C}_\Gamma = \{ \gamma \in \Gamma_0 | \text{val}(\gamma) = 1 \text{ and } \mu(\gamma) > 1 \}$.

For the case of $\Lambda_\Gamma$ in the previous example, the dimension of $\Lambda_\Gamma$ is equal to 24, and the dimension of its center is 7.

1.4 Representation Theory of Ordinary Posets

The theory of representation of posets was introduced and developed by Nazarova, Roiter and their students in Kiev at the 1970s, one of their ideas was to use it as a way of giving a solution of the second Brauer-Thrall conjecture regarding classification of algebras \cite{74, 75, 84}. The main tool to classify posets both ordinary and with additional structures have been the algorithms of differentiation which are functors defined to reduce dimension of the objects of the categories involved in the procedure. The first of these algorithms of differentiation known as the algorithm with respect to a maximal point was introduced by Nazarova and Roiter in 1972, it was used by Kleiner to obtain a criterion to classify posets of finite representation type and by Nazarova in order to classify posets of tame representation type in 1977 \cite{63, 76}. In 1977 as well Zavadskij introduced the algorithm of differentiation with respect to a suitable pair of points which was used by him and Nazarova in 1981 to classify posets of finite growth \cite{77, 84, 97}. We recall that in 1991 Zavadskij introduced an apparatus of differentiation for posets consisting of the algorithms of differentiation DI, DII, DIII, DIV and DV this apparatus was used by him and Bondarenko to classify posets of tame and finite growth with an involution \cite{11, 98} (see in \cite{39}). Particularly in Colombia, Cañadas et al. have studied applications of the theory of representation of posets and its generalizations \cite{24, 26, 28, 30, 36, 37, 39, 40, 42, 43}. In this section, we introduce some elementary notions of the matrix problems, ordinary posets, and classical theorems regarding classification of ordinary posets \cite{2, 39, 60, 84, 99}.

Let $\text{Mat}$ be a set of finite matrices with coefficients in $k$ which is closed under direct sums and direct summands, where for matrices $A, B$ we set

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Suppose that $\mathcal{G}$ is a set of elementary transformations on rows and columns of matrices in $\text{Mat}$. We say that $A$ is $\mathcal{G}$–equivalent to $B$ ($A \sim_\mathcal{G} B$) if $B$ can be obtained from $A$ by applying a sequence of transformations from $\mathcal{G} \cup \mathcal{G}^{-1}$. A class $A/\sim_\mathcal{G}$ represented by $A$ in $\text{Mat}$ is said to be decomposable if there is a matrix $B$ in $\text{Mat}$ of the form $B = C \oplus D$ such that $A \sim_\mathcal{G} B$. A class $A/\sim_\mathcal{G}$ is indecomposable if $A$ is not $\mathcal{G}$–equivalent to zero matrix and $A/\sim_\mathcal{G}$ it is not decomposable. The problem of classifying the indecomposables in the residue class set $\text{Mat}/\sim_\mathcal{G}$ is called a matrix problem and we denote it by $(\text{Mat}, \mathcal{G})$. This is equivalent to a reduction of any matrix in $\text{Mat}$ to a canonical form by applying the transformations in $\mathcal{G} \cup \mathcal{G}^{-1}$. The problem $(\text{Mat}, \mathcal{G})$ is of finite representation type if
the set of indecomposable $\mathcal{G}$–equivalence classes of matrices in $\text{Mat}$ is finite [84].

A poset is an ordered pair of the form $(\mathcal{P}, \leq)$ of a set $\mathcal{P}$ and a binary relation $\leq$ contained in $\mathcal{P} \times \mathcal{P}$, called the order on $\mathcal{P}$ such that $\leq$ is reflexive, antisymmetric and transitive [46].

![Figure 1.8. Hasse diagrams of $\mathcal{P}$ and $\Omega$.](image)

A representation of $\mathcal{P}$ over a field $k$ is a collection $U = (U_x | x \in \mathcal{P})$, where $U_0$ is a finite-dimensional $k$–space and $U_x$ is a subspace in $U_0$ for each $x \in \mathcal{P}$, such that $U_x \subseteq U_y$ if the relation $x \leq y$ holds in $\mathcal{P}$. The dimension vector of a representation $U$ is the vector $d = (\dim U_x | x \in \mathcal{P}) \in \mathbb{Z}^\mathcal{P}$ where $d_0 = \dim U_0$ and $d_x = \dim U_x/U_x$ with $\text{rad } U_x = U_x = \sum_{y \prec x} U_y$.

A morphism $\varphi : U \rightarrow V$ from a representation $U$ to a representation $V$ is any $k$–linear map $\varphi : U_0 \rightarrow V_0$ with the condition $\varphi(U_x) \subseteq V_x$ for all $x \in \mathcal{P}$. The category of representations of $\mathcal{P}$ over $k$ is denoted by $\text{rep } (\mathcal{P}, k) = \text{rep } \mathcal{P}$. Two objects $U, V$ are isomorphic in $\text{rep } \mathcal{P}$ if and only if there exists an isomorphism of $k$–spaces $\varphi : U_0 \rightarrow V_0$ such that $\varphi(U_x) = V_x$ for all $x \in \mathcal{P}$. Denote by $\text{Ind } \mathcal{P}$ a complete set of pairwise non-isomorphic indecomposable representations of $\mathcal{P}$ over $k$.

The direct sum $U \oplus V$ of two representations $U, V \in \mathcal{P}$ is the representation $U \oplus V = (U_0 \oplus V_0, U_x \oplus V_x | x \in \mathcal{P})$. A representation $U$ is said to be decomposable if there exist two representations $U' \neq 0, U'' \neq 0$ such that $U \cong U' \oplus U''$. Otherwise, $U$ is an indecomposable representation (Krull-Schmidt category). We say that a representation $U$ is trivial if $\dim U_0 = 1$, i.e., $U_0 = k$.

An ordered set $C$ is called a chain (or a totally ordered set or a linearly ordered set) if and only if for all $p, q \in C$ we have $p \leq q$ or $q \leq p$ (i.e., $p$ and $q$ are comparable). On the other hand, an ordered set $\mathcal{P}$ is called an antichain if $x \leq y$ in $\mathcal{P}$ only if $x = y$. An antichain consisting exactly of two (resp. three) points is called a dyad (resp. triad). If some subsets $X_1, \ldots, X_n \subseteq \mathcal{P}$ do not intersect mutually (but may have comparable points), then their union $X_1 \cup \cdots \cup X_n$ is called a sum and is denoted by $X_1 + \cdots + X_n$. We denote by $w(\mathcal{P})$ the width of a poset $\mathcal{P}$, i.e., the maximal cardinality of its antichains. Accordingly to the known Dilworth’s theorem [82], each poset of finite width $n$ is a sum of $n$ chains.

For a point $a \in \mathcal{P}$ and a subset $A \subseteq \mathcal{P}$, we define their up- and down-cones

$$a^\triangledown = \{x \in \mathcal{P} | a \leq x\}, \quad a_\triangle = \{x \in \mathcal{P} | x \leq a\}, \quad A^\triangledown = \bigcup_{a \in A} a^\triangledown, \quad A_\triangle = \bigcup_{a \in A} a_\triangle.$$
For any subset $A \subseteq P$, we define a trivial representation $k(A) = k(A^c) = (k; U_x \mid x \in P)$ of $P$ where

$$U_x = \begin{cases} k, & \text{if } x \in A^c, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $k(\emptyset) = (k, 0, \ldots, 0)$. We write often $k(X_1, \ldots, X_n)$ instead of $k(X_1 \cup \cdots \cup X_n)$. For example, let $P_1 = \{a, b, c\}$ be the triad, i.e., three incomparable points, the elements of $\text{Ind} (P)$ are $k(\emptyset), k(a), k(b), k(c), k(a, b), k(b, c), k(a, c), k(a, b, c)$ and $U = (k \oplus k, k \oplus 0, 0 \oplus k, (1, 1)k)$ (see [2]).

Attached to each representation $U$ there exists its matrix representation $M = M_U$ choosing some basis $B_0$ in $U_0$ and for each $x \in P$, some system $B_x$ of linearly independent generators of $U_x$ modulo the radical subspace $\text{rad} U_x$. Then

$$M = \begin{bmatrix} M_{x_1} & \cdots & M_{x_n} \end{bmatrix}$$

with entries in $k$, partitioned horizontally into $n = |P|$ blocks (strips). The set of all matrix representations of $P$ is denoted by $\text{Mat}_P$.

If $M$ and $M'$ are matrix representations of a poset $P = \{x_i \mid 1 \leq i \leq n\}$ given by

$$M = \begin{bmatrix} M_{x_1} & \cdots & M_{x_n} \end{bmatrix}, \quad M' = \begin{bmatrix} M'_{x_1} & \cdots & M'_{x_n} \end{bmatrix},$$

then its direct sum $M \oplus M'$ is given by the formula

$$M \oplus M' = \begin{bmatrix} M_{x_1} & 0 & \cdots & M_{x_n} & 0 \\ 0 & M'_{x_1} & \cdots & 0 & M'_{x_n} \end{bmatrix}$$

Two representations $M$ and $N$ of a poset $P$ are isomorphic if and only if their matrix representations can be turned into each other with help of the following admissible transformations (denoted by $\mathcal{G}_P$):

(i) Elementary transformations of rows of the whole matrix $M$.

(ii) Elementary transformations of columns within each vertical strip.

(iii) Additions of columns of a strip $M_i$ to columns of a strip $M_j$ if $i \leq j$ in $P$.

Then we have defined a matrix problem $(\text{Mat}_P, \mathcal{G}_P)$. A poset $P$ is said to be of representation-finite if $(\text{Mat}_P, \mathcal{G}_P)$ is of finite representation type.

**Remark 1.1.** $(\text{Mat}_P, \mathcal{G}_P)$ is a category whose objects are the matrices $M$ in $\text{Mat}_P$ and morphisms are pairs of matrices $(C, D)$, where $C \in \text{Gl}(|B_0|, k)$ and $D$ is a matrix in $\text{Gl}(|B_0| + \cdots + |B_n|, k)$ which is a composition of elementary matrices corresponding to admissible transformations $\mathcal{G}_P$ ($|B_i|$ denote the number of independent generators of $U_i$, for $0 \leq i \leq n$) [84].
Figure 1.9 is an example of a matrix representation of the triad [60].

$$\begin{array}{ccc}
a & b & c \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.
\end{array}$$

Figure 1.9. Example of a matrix representation of the triad.

Kleiner presented the finite representation type criterion [63].

**Theorem 1.6.** [2]. Let $\mathcal{P}$ be a finite poset. Then $\mathcal{P}$ does not contain $\mathcal{K}_1 = (1, 1, 1, 1)$, $\mathcal{K}_2 = (2, 2, 2)$, $\mathcal{K}_3 = (1, 3, 3)$, $\mathcal{K}_4 = (N, 4)$ or $\mathcal{K}_5 = (1, 2, 5)$ as a subposet if and only if $\text{rep } \mathcal{P}$ has finite representation type.

![Diagram of K1 - K5 and N1 - N6](image)

Figure 1.10. Kleiner's critical $\mathcal{K}_1 - \mathcal{K}_5$ and Nazarova's critical $\mathcal{N}_1 - \mathcal{N}_6$.

Nazarova extended the result of Kleiner and showed the tame representation type criterion [73].

**Theorem 1.7.** [2]. Let $\mathcal{P}$ be a finite poset and $k$ a field. Then $\text{rep } \mathcal{P}$ has wild representation type if and only if $\mathcal{P}$ contains $\mathcal{N}_1 = (1, 1, 1, 1)$, $\mathcal{N}_2 = (1, 1, 1, 2)$, $\mathcal{N}_3 = (2, 2, 3)$, $\mathcal{N}_4 = (1, 3, 4)$, $\mathcal{N}_5 = (N, 5)$ or $\mathcal{N}_6 = (1, 2, 6)$ as a subposet.
1.5 Dyck Paths

Dyck paths is an important tool in combinatorics which is in relationship with Catalan objects as permutations, binary trees, non-decreasing parking functions, triangulations of a regular polygon, etc [10, 92]. Dyck paths can be defined as lattice paths connecting points in a square lattice that satisfies some conditions in the $xy$ plane. Such Dyck paths are also described by using some Dyck words. In this section, we present the concept of a square lattice, lattice path, Dyck words. A connection between Dyck words and Dyck paths is given as well [6,12,48,92,96].

A lattice $\Lambda = (V,E)$ is a mathematical model of a discrete space. It consists of two sets, a set $V \subset \mathbb{R}^n$ of vertices and a set $E \subset \mathbb{R}^n \times \mathbb{R}^n$ of edges, with no more than two edges between any two vertices. If two vectors are connected via an edge, we call them nearest neighbors.

Let $\Lambda = (V,E)$, an $n$-step lattice path or lattice walk or walk from $s \in V$ to $x \in V$ is a sequence $w = (w_0, \ldots, w_n)$ of elements in $V$, such that

1. $w_0 = s, w_n = x$,
2. $(w_i, w_{i+1}) \in E$.

The length $|w|$ of a lattice path is the number $n$ of steps (edges) in the sequence $w$.

The Euclidean lattice is a lattice where $V = \mathbb{Z}^d$. The edges are mostly defined through a so called step set. On this lattice an alternative definition via the step set can be used. A step set $S \subset \mathbb{Z}^d$ is the fixed and finite set of possible steps. The elements of $S$ are called steps. If the step set $S$ is a subset of $\{-1, 0, 1\}^2 \setminus \{(0,0)\}$, then we say $S$ is a set of small steps.

![Figure 1.11. Square lattice (left) and triangular lattice (right).](image)

For the square lattice and triangular lattice in Figure 1.11, the sets of small steps are $S_1 = \{(1,0), (0,1), (-1,0), (0,-1)\}$ and $S_2 = \{(1,0), (0,1), (-1,0), (0,-1), (1,1), (-1,-1)\}$, respectively.

An $n$-step lattice path or lattice walk or walk from $s \in \mathbb{Z}^d$ to $x \in \mathbb{Z}^d$ relative to $S$ is a sequence $w = (w_0, \ldots, w_n)$ of elements in $\mathbb{Z}^d$, such that
1. \( w_0 = s, \ w_n = x, \)
2. \( w_{i+1} - w_i \in S \)

Let \( A \) be the diagonal square lattice where \( V_A = \{(x,y) \in \mathbb{Z}^2 \mid x \geq 0, y \geq 0 \} \) and \( S_A = \{(1,1), (1,-1)\} \), walks on the diagonal square lattice \( A \) are equivalent to walks on the square lattice \( B \) with \( V_B = \{(x,y) \in \mathbb{Z}^2 \mid x \geq 0, y = x \} \) and \( S_B = \{(1,0), (0,1)\} \).

A Dyck path is a lattice path in \( \mathbb{Z}^2 \) with steps \((1,1)\) and \((1,-1)\), such that the path starts at \((0,0)\) and ends at \((2n,0)\) and it does not pass below the \(x\)-axis. The number of Dyck paths of length \(2n\) is equal to the \(n\)–th Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n} \).

Figure 1.12 shows the set of all lattice paths of length 6 in the square lattice \( B \) as described above.

Let \( X \) be an alphabet. We define the free monoid generated by \( X \), denoted by \( X^* \), as the set of the finite words written with \( X \)'s letters. The product of \( u = u_1 \ldots u_p \in X^* \) and \( v = v_1 \ldots v_q \in X^* \) is defined as the concatenation of these words: \( uv = u_1 \ldots u_p v_1 \ldots v_q \). The word \( u \) is called a left factor of the word \( w = uv \). The empty word is denoted by \( e \). The number of occurrences of the letter \( a \in X \) in the word \( w \) is denoted by \(|w|_a\), and the length of \( w \) by

\[
|w| = \sum_{a \in X} |w|_a, \quad (1.6)
\]

The set of Dyck words is the set of words \( w \in X^* = \{U, D\}^* \) characterized by the following two conditions [6]:

- for any left factor \( u \) of \( w \), \( |u|_U \geq |u|_D \),
- \( |w|_U = |w|_D \).

For example, the set of Dyck words of length 6 is

\[
\{UDUDUD, UDUDUD, UDUDUD, UDUDUD, UDUDUD, UDUDUD\}. \quad (1.7)
\]

The number of Dyck words of length \(2n\) is equal to the \(n\)–th Catalan number [47].

There is a bijective correspondence between the set of Dyck paths and the set of Dyck words [6].
1.6 Cluster Algebras

In 2002, Fomin and Zelevinsky introduced the term of the cluster algebra \cite{54} as a subalgebra of a field of rational functions generated by the set of cluster variables \cite{52,55,56}. The cluster algebras are in connection with different topics as algebraic combinatorics, Lie theory, discrete dynamical systems, tropical geometry, and others. Afterwards, Fomin, Schiffler et al introduced cluster algebras associated to surfaces \cite{19,52–56,69}.

The definition of a cluster algebra $\mathcal{A}$ starts by introducing its ground ring. Let $(\mathbb{P}, \oplus, \cdot)$ be a semifield, i.e., an abelian multiplicative group endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$. The group ring $\mathbb{Z}\mathbb{P}$ will be used as a field of scalars (ground ring) for $\mathcal{A}$.

Let $J$ be a finite set of labels, and let $\text{Trop}(u_j : j \in J)$ be an abelian group (written multiplicatively) freely generated by the elements $u_j$. We define the addition $\oplus$ in $\text{Trop}(u_j; i \in J)$ by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j,b_j)},$$

and call $(\text{Trop}(u_j : j \in J), \oplus, \cdot)$ a tropical semifield. To illustrate, $u_2 \oplus u_2^2 u_1^{-1} = u_2^{-1}$ in $\text{Trop}(u_1,u_2)$. The group ring of $\text{Trop}(u_j : j \in J)$ is the ring of Laurent polynomials in the variables $u_j$. If $J$ is empty, we obtain the trivial semifield consisting of a single element 1.

As an ambient field for a cluster algebra $\mathcal{A}$, we take a field $\mathcal{F}$ isomorphic to the field of rational functions in $n$ independent variables (here $n$ is the rank of $\mathcal{A}$), with coefficients in $\mathbb{Q}\mathbb{P}$. Note that the definition of $\mathcal{F}$ ignores the auxiliary addition in $\mathbb{P}$.

A labeled $Y$–seed in $\mathbb{P}$ is a pair $(y, B)$, where:

- $y = (y_1, \ldots, y_n)$ is an $n$–tuple of elements of $\mathbb{P}$,
- $B = (b_{ij})$ is an $n \times n$ integer matrix which is skew-symmetrizable.

That is, $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \ldots, d_n$. A labeled seed in $\mathcal{F}$ is a triple $(x, y, B)$, where;

- $(y, B)$ is a labeled $Y$–seed,
- $x = (x_1, \ldots, x_n)$ is an $n$–tuple of elements of $\mathcal{F}$ forming a free generating set.

That is, $x_1, \ldots, x_n$ are algebraically independent over $\mathbb{Q}\mathbb{P}$, and $\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \ldots, x_n)$. We refer to $x$ as the (labeled) cluster of a labeled seed $(x, y, B)$, to the tuple $y$ as the coefficient tuple, and to the matrix $B$ as the exchange matrix.
The (unlabeled) seeds are obtained by identifying labeled seeds that differ from each other by simultaneous permutations of the components in \( x \) and \( y \), and of the rows and columns of \( B \).

We use the notation \([x]_+ = \max\ (x, 0)\), \([1, n] = \{1, \ldots, n\}\), and

\[
\text{sgn}(x) = \begin{cases} 
1, & \text{if } x > 0, \\
0, & \text{if } x = 0, \\
-1, & \text{if } x < 0.
\end{cases}
\]

Let \((x, y, B)\) be a labeled seed in \( F \), and let \( k \in [1, n]\). The seed mutation \( \mu_k \) in direction \( k \) transforms \((x, y, B)\) into the labeled seed \( \mu_k(x, y, B) = (x', y', B) \) defined as follows:

- The entries of \( B' = (b'_{ij}) \) are given by
  \[
b'_{ij} = \begin{cases} 
-b_{ij}, & \text{if } i = k \text{ or } j = k, \\
b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise}.
\end{cases}
\]

- The coefficient tuple \( y' = (y'_1, \ldots, y'_n) \) is given by
  \[
y'_j = \begin{cases} 
y_k^{-1}, & \text{if } j = k, \\
y_jy_k[b_{kj}]_+((y_k \oplus 1)b_{kj})_-, & \text{if } j \neq k.
\end{cases}
\]

- The cluster \( x' = (x'_1, \ldots, x'_n) \) is given by \( x'_j = x_j \) for \( j \neq k \), whereas \( x'_k \in F \) is determined by the exchange relation
  \[
x'_k = \frac{y_k \prod x_i[b_{ik}]_+ + \prod x_i[-b_{ik}]_+}{(y_k \oplus 1)x_k}.
\]

We consider the \( n \)-regular tree \( T_n \) whose edges are labeled by the numbers \( 1, \ldots, n \), so that the \( n \) edges emanating from each vertex receive different labels. We write \( t \xrightarrow{k} t' \) to indicate that vertices \( t, t' \in T_n \) are joined by an edge labeled by \( k \). A cluster pattern is an assignment of a labeled seed \( \Sigma_t = (x_t, y_t, B_t) \) to every vertex \( t \in T_n \), such that the seeds assigned to the endpoints of any edge \( t \xrightarrow{k} t' \) are obtained from each other by the seed mutation in direction \( k \). The elements of \( \Sigma_t \) are written as follows:

\[
x_t = (x_{1,t}, \ldots, x_{n,t}), \quad y_t = (y_{1,t}, \ldots, y_{n,t}), \quad B_t = (b'_{ij}).
\]

A cluster pattern is uniquely determined by each of its seeds, which can be chosen arbitrarily.

For example (case \( A_2 \), see [56]), let \( n = 2 \), then the tree \( T_2 \) is an infinite chain. We denote its vertices by \( \ldots, t_{-1}, t_0, t_1, \ldots \), and label its edges as follows:

\[
\ldots \xrightarrow{2} t_{-1} \xrightarrow{1} t_0 \xrightarrow{2} t_1 \xrightarrow{1} t_2 \xrightarrow{2} \ldots
\]
We denote the corresponding seeds by $\Sigma_m = \Sigma_{t_m} = (x_m, y_m, B_m)$, for $m \in \mathbb{Z}$. Let the initial seed $\Sigma_0$ be

$$\begin{align*} x_0 = (x_1, x_2), \quad y_0 = (y_1, y_2), \quad B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{align*}$$

(1.12)

We then recursively compute the seeds $\Sigma_1, \ldots, \Sigma_5$ as shown in Table 1.1 with $B_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$B_t$</th>
<th>$y_t$</th>
<th>$x_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$B_0$</td>
<td>$y_1$</td>
<td>$y_2$</td>
</tr>
<tr>
<td>1</td>
<td>$B_1$</td>
<td>$y_1(y_2 \oplus 1)$</td>
<td>$\frac{1}{y_2}$</td>
</tr>
<tr>
<td>2</td>
<td>$B_0$</td>
<td>$\frac{1}{y_1(y_2 \oplus 1)}$</td>
<td>$\frac{1}{y_2}$</td>
</tr>
<tr>
<td>3</td>
<td>$B_1$</td>
<td>$y_1y_2$</td>
<td>$y_1y_2 \oplus y_1 + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$B_0$</td>
<td>$y_1 \oplus 1$</td>
<td>$\frac{1}{y_1}$</td>
</tr>
<tr>
<td>5</td>
<td>$B_1$</td>
<td>$y_2$</td>
<td>$y_1$</td>
</tr>
</tbody>
</table>

Table 1.1. Seeds for the case $A_2$.

**Cluster Algebra**

Given a cluster pattern, we denote by

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} x_t = \{x_{i,t} : t \in \mathbb{T}_n, 1 \leq i \leq n\},$$

the union of clusters of all of the seeds in the pattern. We refer to the elements $x_{i,t} \in \mathcal{X}$ as *cluster variables*. The *cluster algebra* $\mathcal{A}$ associated with a given cluster pattern is the $\mathbb{Z}\mathbb{P}$–subalgebra of the ambient field $\mathcal{F}$ generated by all cluster variables: $\mathcal{A} = \mathbb{Z}\mathbb{P}[\mathcal{X}]$. We denote $\mathcal{A} = \mathcal{A}(x, y, B)$, where $(x, y, B) = (x_t, y_t, B_t)$ is any labeled seed in the underlying cluster pattern. A cluster algebra is of *geometric type* if the coefficient semifield $\mathbb{P}$ is a tropical semifield.

We say that a cluster algebra is of finite type if it has finitely many seeds. More specifically, we define the diagram $\Gamma(B)$ associated to an $n \times n$ exchange matrix $B$ to be a weighted directed graph on nodes $v_1, \ldots, v_n$, with $v_i$ directed towards $v_j$ if and only if $b_{ij} > 0$. In that case, we label this edge by $|b_{ij}b_{ji}|$. Then $\mathcal{A} = \mathcal{A}(x, y, B)$ is of finite type if and only if $\Gamma(B)$ is mutation-equivalent to an orientation of a *finite type* Dynkin diagram [55]. In this case, we say that $B$ and $\Gamma(B)$ are of finite type. We say that a matrix $B$ (and the corresponding cluster algebra) has finite mutation type if its mutation equivalence class is finite, i.e. only finitely many matrices can be obtained from $B$ by repeated matrix mutations. A classification of all cluster algebras of finite mutation type with skew-symmetric exchange matrices was given by Felikson, Shapiro, and Tumarkin [51] (see [69]).
Cluster Algebras From Quivers

For quivers, cluster algebras are defined as follows:

Fix an integer $n \geq 1$. In this case, a seed $(Q, u)$ consists of a finite quiver $Q$ without loops or 2-cycles with vertex set $\{1, \ldots, n\}$, whereas $u$ is a free-generating set $\{u_1, \ldots, u_n\}$ of the field $\mathbb{Q}(x_1, \ldots, x_n)$.

Let $(Q, u)$ be a seed and $k$ a vertex of $Q$. The mutation $\mu_k(Q, u)$ of $(Q, u)$ at $k$ is the seed $(Q', u')$, where;

(a) $Q'$ is obtained all arrows incident with $k$,

1. reverse all arrows incident with $k$,
2. for all vertices $i \neq j$ distinct from $k$, modify the number of arrows between $i$ and $j$, in such a way that a system of arrows of the form $(i \xrightarrow{r} j, i \xrightarrow{s} k, k \xrightarrow{t} j)$ is transformed into the system $(i \xrightarrow{r+s} j, k \xrightarrow{s} i, j \xrightarrow{t} k)$. And the system $(i \xrightarrow{r} j, j \xrightarrow{t} k, k \xrightarrow{s} i)$ is transformed into the system $(i \xrightarrow{r-st} j, i \xrightarrow{s} k, k \xrightarrow{t} j)$. Where, $r, s$ and $t$ are non-negative integers, an arrow $i \xrightarrow{l} j$, with $l \geq 0$ means that $l$ arrows go from $i$ to $j$ and an arrow $i \xrightarrow{-l} j$, with $l \leq 0$ means that $-l$ arrows go from $j$ to $i$.

(b) $u'$ is obtained form $u$ by replacing the element $u_k$ with

$$u_k = \frac{1}{u_k} \prod_{\text{arrows } i \rightarrow k} u_i + \prod_{\text{arrows } k \rightarrow j} u_j. \quad (1.13)$$

If there are no arrows from $i$ with target $k$, the product is taken over the empty set and equals 1. It is not hard to see that $\mu_k(\mu_k(Q, u)) = (Q, u)$. Thus, if $Q$ is a finite quiver without loops or 2-cycles with vertex set $\{1, \ldots, n\}$, the following interpretations have place:

1. the clusters with respect to $Q$ are the sets $u$ appearing in seeds, $(Q, u)$ obtained from a initial seed $(Q, x)$ by iterated mutation,
2. the cluster variables for $Q$ are the elements of all clusters,
3. the cluster algebra $\mathcal{A}(Q)$ is the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all the cluster variables.

As example, the cluster variables associated to the quiver $Q = 1 \rightarrow 2$ are:

$$\left\{x_1, x_2, \frac{1 + x_2}{x_1}, \frac{1 + x_1 + x_2}{x_1 x_2}, \frac{1 + x_1}{x_2}\right\}.$$
Cluster Algebra Arising from Unpunctured Surface

Let $S$ be a connected oriented 2-dimensional Riemann surface with nonempty boundary, and let $M$ be a nonempty finite subset of the boundary of $S$, such that each boundary component of $S$ contains at least one point of $M$. The elements of $M$ are called marked points. The pairs $(S, M)$ is called a bordered surface with marked points. Marked points in the interior of $S$ are called punctures (For technical reasons, we require that $(S, M)$ is not a disk with 1, 2 or 3 marked points) \[22\].

An arc $\gamma$ in $(S, M)$ is a curve in $S$, considered up to isotopy, such that:

(i) the endpoints of $\gamma$ are in $M$,

(ii) $\gamma$ does not cross itself, except that its endpoints, may coincide,

(iii) except for the endpoints, $\gamma$ is disjoint from the boundary of $S$,

(iv) $\gamma$ does not cut out a monogon or a bigon.

Curves that connect two marked points and lie entirely on the boundary of $S$ without passing through a third marked point are boundary segments. Note that boundary segments are not arcs. For any two arcs $\gamma, \gamma'$ in $S$, let $e(\gamma, \gamma')$ be the minimal number of crossings of arcs $\alpha$ and $\alpha'$, where $\alpha$ and $\alpha'$ range over all arcs isotopic to $\gamma$ and $\gamma'$, respectively. We say that arcs $\gamma$ and $\gamma'$ are compatible if $e(\gamma, \gamma') = 0$.

A triangulation is a maximal collection of pairwise compatible arcs (together with all boundary segments). Triangulations are connected to each other by sequences of flips. Each flip replaces a single arc $\gamma$ in a triangulation $T$ by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in $T$, forms a new triangulation.

Choose any triangulation $T$ of $(S, M)$, and let $\tau_1, \ldots, \tau_n$ be the $n$ arcs of $T$. For any triangle $\Delta$ in $T$, we define a matrix $B^\Delta = (b^\Delta_{ij})_{1 \leq i, j \leq n}$ as follows.

- $b^\Delta_{ij} = 1$ and $b^\Delta_{ji} = -1$ if $\tau_i$ and $\tau_j$ are sides of $\Delta$ with $\tau_j$ following $\tau_i$ in the clockwise order,
- $b^\Delta_{ij} = 0$ otherwise.

Then define the matrix $B_T = (b_{ij})_{1 \leq i, j \leq n}$ by $b_{ij} = \sum_\Delta b^\Delta_{ij}$, where the sum is taken over all triangles in $T$. Note that $B_T$ is skew-symmetric and each entry $b_{ij}$ is either 0, ±1, or ±2, since every arc $\tau$ is in at most two triangles.

According to Schiffler and Canakci \[22\], Fomin, Shapiro and Thurston \[54\] associated a cluster algebra $A(S, M)$ to any bordered surface with marked points $(S, M)$, and the cluster variables of $A(S, M)$ are in bijection with the (tagged) arcs of $(S, M)$.

The following theorem regarding relationships between cluster algebras and surface triangulations was obtained Fomin, Shapiro, and Thurston \[52, 53\].
Theorem 1.8. \[\text{Fix a bordered surface } (S, M) \text{ and let } \mathcal{A} \text{ be the cluster algebra associated to the signed adjacency matrix of a tagged triangulation. Then the (unlabeled) seed } \Sigma_T \text{ of } \mathcal{A} \text{ are in bijection with tagged triangulations } T \text{ of } (S, M), \text{ and the cluster variables are in bijection with the tagged arcs of } (S, M) \text{ (so we can denote each by } x_\gamma, \text{ where } \gamma \text{ is a tagged arc). Moreover, each seed in } \mathcal{A} \text{ is uniquely determined by its cluster. Furthermore, if a tagged triangulation } T' \text{ is obtained from another tagged triangulation } T \text{ by flipping a tagged arc } \gamma \in T \text{ and obtaining } \gamma', \text{ then } \Sigma_{T'} \text{ is obtained from } \Sigma_T \text{ by the seed mutation replacing } x_\gamma \text{ by } x_{\gamma'}\].

1.7 Category of Diagonals and Cluster-tilted Algebras

In 2006 [17], Caldero, Chapoton, and Schiffler introduced the category of diagonals of a polygon with \(n + 3\) sides associated to a triangulation \(T\), in this case, the diagonals are called roots which can be classified as negative or positive, negative roots are those roots belonging to the triangulation \(T\) [17, 80].

The combinatorial \(\mathbb{C}\)-linear additive category \(C_T\) is described as follows. The objects are positive integral linear combinations of positive roots, and the space of morphisms from a positive root \(\alpha\) to a positive root \(\alpha'\) is a quotient of the vector space over \(\mathbb{C}\) spanned by pivoting paths from \(\alpha\) to \(\alpha'\). The subspace which defines the quotient is spanned by the so-called mesh relations. For any couple \(\alpha, \alpha'\) of positive roots such that \(\alpha\) is related to \(\alpha'\) by two consecutive pivoting elementary moves with distinct pivots, the mesh relations are given by the identity \(P_{v_2'}P_{v_1'} = P_{v_2}P_{v_1}\), where \(v_1, v_2\) (resp. \(v_1', v_2'\)) are the vertices of \(\alpha\) (resp. \(\alpha'\)) such that \(P_{v_1'}P_{v_2} = \alpha'\).

Let \(T\) be a triangulation, then one can define a planar tree \(t_T\) as follows. Its vertices are the triangles of \(T\) and the edges connect adjacent triangles. In the same way, we can define a graph \(Q_T\) whose vertices are the inner edges of \(T\) and are related to each other by an edge, if they bound the same triangle. An orientation can be defined by using graph \(Q_T\), in such a way that a vertex \(i\) connects a vertex \(j\) (denoted \(i \rightarrow j\)), if \(-\alpha_j\) can be obtained from the diagonal \(-\alpha_i\) by rotating anticlockwise about their common vertex.

According to Caldero, Chapoton, and Schiffler [17], one can define a \(\mathbb{C}\)-linear abelian category \(\text{Mod } Q_T\) as follows. This is the category of modules over the quiver \(Q_T\) with the following relations, called triangle relations:

In any triangle, the composition of two successive maps is zero.

These relations are exactly the relations prescribed by [17, Definition 1].

Figure 1.13 shows an example of the tree and the quiver associated to a triangulation.
The following results regarding the category of diagonals were given by Caldero, Chapoton, and Schiffler in [17].

**Theorem 1.9.** There is an equivalence of categories between $C_T$ and $\text{Mod} \ Q_T$.

**Corollary 1.1.** There exists a bijection $\varphi$ between $\text{Ind} \ Q_T$ and the diagonals of the polygon not in $T$. Moreover, for $M$ in $\text{Ind} \ Q_T$ and any vertex $i$ of $Q_T$, the multiplicity of the simple module $S_i$ in the module $M$ is 1 if $\varphi(M)$ crosses the $i^{th}$ diagonal of $T$ and 0 if not. In particular, for two isoclasses $M, M'$ in $\text{Ind} \ Q_T$, we have $M = M'$ if and only if $n_i(M) = n_i(M')$ for all $i$.

**Theorem 1.10.** Let $T$ be a triangulation of the $n + 3$ polygon, and let $C_T$ be the corresponding category, then:

(i) The irreducible morphisms of $C_T$ are direct sums of the generating morphisms given by pivoting elementary moves.

(ii) The mesh relation of $C_T$ are the mesh relations [5] of the Auslander-Reiten quiver of $C_T$.

(iii) The Auslander-Reiten translate is given on diagonals by $r^-$.

(iv) The indecomposable projective objects of $C_T$ are diagonals in $r^+(T)$.

(v) The indecomposable injective objects of $C_T$ are diagonals in $r^-(T)$.

with $r^+$ (resp. $r^-$) the elementary rotation of the polygon in the positive (resp. negative) direction.

**Theorem 1.11.** Let $C = \{u_1, \ldots, u_n\}$ be a cluster of a cluster algebra of type $A_n$ and let $V$ be the set of all cluster variables of the algebra. Let $Q_C$ be the quiver with relations associated to $C$ and $\text{Ind} \ Q_C$ the set of isoclasses of indecomposable modules. Then there is a bijection

$$\text{Ind} \ Q_C \rightarrow V \setminus C, \ \alpha \mapsto w_\alpha,$$

such that

$$w_\alpha = \frac{P(u_1, \ldots, u_n)}{\prod_{i=1}^{n} u_i^{n_i(\alpha)}},$$

where $P$ is a polynomial such that none of the $u_i$ divides $P$ ($i = 1, \ldots, n$) and $n_i(\alpha)$ is the multiplicity of the simple module $\alpha_i$ in the module $\alpha$. 
The Auslander-Reiten quiver of the quiver shown in Figure 1.13 is given by Figure 1.14.

![Figure 1.14. Auslander-Reiten quiver of $Q_T$.](image)

**Cluster-tilted Algebras of Type $A_n$**

In this section, we recall some results regarding cluster-tilted algebras [95].

The cluster category was introduced independently in [17] for type $A_n$ and in [13] for the general case. Let $\mathcal{D}^b(\text{mod } H)$ be the bounded derived category of the finitely generated modules over a finite dimensional hereditary algebra $H$ over a field $k$. In [13] the cluster category was defined as the orbit category $\mathcal{C} = \mathcal{D}^b(\text{mod } H)/\tau^{-1}[1]$, where $\tau$ is the Auslander-Reiten translation and $[1]$ the suspension functor. The cluster-tilted algebras are the algebras of the form $\Gamma = \text{End}_\mathcal{C}(B)^{\text{op}}$, where $B$ is a cluster-tilting object in $\mathcal{C}$ [14].

Let $Q$ be a quiver with no multiple arrows, no loops and no oriented cycles of length two and let $Q'$ be a quiver obtained from $Q$ via mutations. We say that a quiver $Q$ is mutation equivalent to $Q'$, if $Q'$ can be obtained from $Q$ by a finite number of mutations. The mutation class of $Q$ is all quivers mutation equivalent to $Q$. The mutation class of a Dynkin quiver $Q$ is finite [55].

If $\Gamma$ is a cluster-tilted algebra, then we say that $\Gamma$ is of type $A_n$ if it arises from the cluster category of a path algebra of Dynkin type $A_n$. Let $Q$ be a quiver of a cluster-tilted algebra $\Gamma$, if $Q'$ is obtained from $Q$ by a finite number of mutations, then there is a cluster-tilted algebra $\Gamma'$ with quiver $Q'$. Moreover, $\Gamma$ is of finite representation type if and only if $\Gamma'$ is of finite representation type. We also have that $\Gamma$ is of type $A_n$ if and only if $\Gamma'$ is of type $A_n$. We know that a cluster-tilted algebra is up to isomorphism uniquely determined by its quiver [13] [17] [95]. It follows from this that to count the number of cluster-tilted algebras of type $A_n$, it is enough to count the mutation class of any quiver with underlying graph $A_n$. 
We define mutation of a triangulation at a given diagonal, by replacing this diagonal with another one. This can be done in one and only one way. Let $Q_T$ be a quiver corresponding to a triangulation $T$. Then mutation of $Q_T$ at the vertex $i$ corresponds to mutation of $T$ at the diagonal corresponding to $i$.

Let $\mathcal{M}_n$ be the mutation class of $A_n$, i.e. all quivers obtained by repeated mutation from $A_n$, up to isomorphisms of quivers. Let $\mathcal{T}_n$ be the set of all triangulations of an $n + 3$ polygon. We can define a function $\gamma : \mathcal{T}_n \rightarrow \mathcal{M}_n$ where we set $\gamma(T) = Q_T$ for any triangulation $T$ in $\mathcal{T}_n$. Note that $\gamma$ is surjective.

For a triangulation $T$ of an $n + 3$ polygon, let us denote by $T^i$ the triangulation obtained from $T$ by rotating $T$ $i$ steps in the clockwise direction. We define an equivalence relation on $\mathcal{T}_n$, where we let $T \sim T^i$ for all $i$. We define a new function $\overline{\gamma} : (\mathcal{T}_n \setminus \sim) \rightarrow \mathcal{M}_n$ induced from $\gamma$.

The following results regarding cluster-tilted algebras of type $A_n$ were obtained by Torkildsen in [95].

**Theorem 1.12.** The function $\overline{\gamma} : (\mathcal{T}_n \setminus \sim) \rightarrow \mathcal{M}_n$ is bijective for all $n \geq 2$.

**Corollary 1.2.** The number $a(n)$ of non-isomorphic basic cluster-tilted algebras of type $A_n$ is the number of triangulations of the disk with $n$ diagonals, i.e.

$$a(n) = C_{n+1}/(n+3) + C_{(n+1)/2}/2 + (2/3)C_{n/3},$$

where $C_i$ is the $i$-th Catalan number and the second term is omitted if $(n + 1)/2$ is not an integer and the third term is omitted if $n/3$ is not an integer.

### 1.8 Friezes

In this section, we recall the concepts of frieze patterns, a generalization associated to Cartan matrix, vector friezes and its connection with cluster algebras [4,7,43–45,62,71].

Coxeter introduced frieze patterns in [45] in the early 1970s, inspired by Gauss’s penta-gramma mirificum. A frieze pattern is a grid of positive integers, with a finite number of infinite rows, where the top and bottom rows are bi-infinite repetition of 0s and the second to top and the second to bottom row are bi-infinite repetitions of 1s, and every four adjacent numbers of the following square

\[
\begin{array}{ccc}
  & b & \\
  a & & c \\
  & d &
\end{array}
\]

satisfy the identity $ac - bd = 1$. The sequence of integers in the first non-trivial row, $(m_i)_{i \in \mathbb{Z}}$, is called **quiddity sequence**. This sequence completely determines the frieze pattern. Each frieze pattern is also periodic, since it is invariant under glide reflection. The order of the frieze pattern is defined to be the number of rows minus one. It follows
that each frieze pattern of order $n$ is $n$-periodic \cite{7,8}. Conway and Coxeter classified completely the frieze patterns whose entries are positive integers, and show that these frieze patterns constitute a manifestation of the Catalan numbers \cite{43,44}. Specifically, there is a natural association between positive integer frieze patterns and triangulations of regular polygons with labeled vertices. From every triangulation $T$ of a regular $n$-gon with vertices cyclically labeled 1 through $n$, Conway and Coxeter build an $n$-rowed frieze pattern determined by the numbers $a_1, a_2, \ldots, a_n$ where $a_k$ is the number of triangles in $T$ incident with vertex $k$. Specifically \cite{71}:

1. the top row of the array is $\ldots, 0, 0, 0, \ldots$;
2. the second row (offset from the first) is $\ldots, 1, 1, 1, \ldots$;
3. the third row is $\ldots, a_1, \ldots, a_n, a_1, \ldots$ (with period $n$);
4. each succeeding row (offset from the one before) is determined by the frieze recurrence of the four adjacent numbers given as above.

For instance, given a frieze pattern

\[
\begin{array}{cccccc}
\ldots & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 2 & 3 & 1 & 2 & 3 & \ldots \\
\ldots & 5 & 2 & 1 & 5 & \ldots \\
\ldots & 2 & 3 & 1 & 2 & 3 & \ldots \\
\ldots & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & \ldots \\
\end{array}
\]

this is in relationship with a triangulation of the form

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure1.15.png}
\caption{Example of triangulation associated to a frieze pattern.}
\end{figure}

In 2010 Assem, Reutenauer and Smith \cite{4} introduced a generalization of friezes associated to Cartan matrix (see \cite{3} 226p.), in the following way, let $C = (C_{i,j})_{n \times n}$ be a Cartan matrix of a connected Quiver $Q$, then a frieze is a collection of positive integers $a(j, m)$, with $j \in \{1, \ldots, n\}$ and $m \in \mathbb{Z}$, such that

\[
a(j, m)a(j, m + 1) = 1 + \left( \prod_{i=1}^{m} a(i, m)^{C_{i,j}} \right) \left( \prod_{i=j+1}^{m+1} a(i, m+1)^{|C_{i,j}|} \right). \tag{1.15}
\]
For instance, if $Q$ is a Dynkin diagram of type $D_6$ with any orientation, a frieze associated to $D_6$ is

\[
\begin{array}{cccccc}
\ldots & 1 & 3 & 3 & 1 & \ldots \\
\ldots & 2 & 2 & 8 & 2 & 2 & \ldots \\
\ldots & 3 & 5 & 5 & 3 & \ldots \\
\ldots & 7 & 4 & 7 & 2 & 3 & 2 & 7 & 4 & 7 & \ldots \\
\ldots & 4 & 2 & 2 & 4 & \ldots 
\end{array}
\]

Many authors have studied properties of the friezes, and they have found connections with different topics (see examples in [4,7–9,57,62,66,67,71]). In particular, Fontaine and Plamondon [57] obtained the following results.

**Theorem 1.13.** The number of friezes of type $D_n$ is $\sum_{m=1}^{n} d(m) (2^{n-m-1})$ where $d(m)$ denotes the number of divisors of $m$.

**Corollary 1.3.** The number of friezes in type, $B_n$, $C_n$, and $G_2$ is $\sum_{m=\sqrt{n}+1}^{\infty} (2^{n-m^2+1}), (\binom{2n}{n})$, and 9, respectively.

Fontaine, Plamondon and Propp (in type $E_6$) conjectured that the number of friezes in type $E_6$, $E_7$, $E_8$, and $F_4$ is 868, 4400, 26592, and 112, respectively [57,71].

One way to define friezes is to say that they are ring homomorphisms from a cluster algebra to the ring of integers such that all cluster variables are sent to positive integers [57]. Let $Q$ be a quiver without loops and 2–cycles and let $\mathcal{A}(Q)$ be the corresponding cluster algebra with trivial coefficients (see [54]).

(i) A frieze of type $Q$ is a ring homomorphism $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{R}$ from the cluster algebra to an integral domain $\mathbb{R}$. The frieze is called integral if $\mathbb{R} = \mathbb{Z}$.

(ii) A frieze $\mathcal{F}$ is said to be unitary if there exists a cluster $x$ in $\mathcal{A}(Q)$ such that every cluster variable $x \in x$ is mapped by $\mathcal{F}$ to a unit in $\mathbb{R}$.

(iii) A frieze is said to be non-zero if every cluster variable in $\mathcal{A}(Q)$ is mapped by $\mathcal{F}$ to a non-zero element of $\mathbb{R}$.

(iv) An integral frieze is said to be positive if every cluster variable in $\mathcal{A}(Q)$ is mapped by $\mathcal{F}$ to a positive integer.

Let $x = (x_1, \ldots, x_n)$ be a cluster of $\mathcal{A}(Q)$.

(i) A vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$ is called a frieze vector relative to $x$ if the frieze $\mathcal{F}$ defined by $\mathcal{F}(x_i) = a_i$ has values in $\mathbb{R}$. If the frieze $\mathcal{F}$ is unitary we say that the frieze vector $(a_1, \ldots, a_n)$ is unitary.
(ii) A vector \((a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n\) is called a positive frieze vector relative to \(x\) if the frieze \(F\) defined by \(F(x_i) = a_i\) is positive integral.

Gunawan and Schiffler proved the following result \([62]\).

**Theorem 1.14.** Let \(Q\) be a quiver without loops and 2-cycles and let \(x = (x_1, \ldots, x_n)\) be an arbitrary cluster of \(A(Q)\). Then there is a bijection

\[
\phi : \{\text{unordered clusters in } A(Q)\} \rightarrow \{\text{positive unitary frieze vectors relative to } x\}
\]

\[
x' = (x'_1, \ldots, x'_n) \mapsto \phi(x') = (a_1, \ldots, a_n).
\]

### 1.9 Snake Graphs

Snake graph is a combinatorial tool that has appeared in cluster algebras. According to Propp, given a triangulation \(T\), we can define a graph whose \(n\) vertices correspond to the vertices in \(T\) and \(n - 2\) vertices corresponded to the triangular faces of \(T\) \([71]\). Canakci and Schiffler have studied relationships between snake graphs and continued fractions, introducing a calculus for cluster algebras \([19–23]\) (see other works \([68, 81]\)). In particular, Musiker, Schiffler, and Williams introduced a combinatorial formula for the cluster variables of cluster algebras from surfaces by using snake graphs and its perfect matchings \([69]\). In this section, we recall the definition of a snake graph, the number of perfect matchings associated to these graphs, and the way that these concepts can be used to find out a formula for the cluster variables of a cluster algebra associated to a surface \([19, 21, 22, 69]\).

A *tile* \(G\) is a square of fixed side-length in the plane whose sides are parallel or orthogonal to the fixed basis.

\[
\begin{array}{c}
\text{North} \\
\text{West} & G & \text{East.} \\
\text{South} \\
\end{array}
\]

We consider a tile \(G\) as a graph with four vertices and four edges in the obvious way. A *snake graph* \(G\) is a connected graph consisting of a finite sequence of tiles \(G_1, \ldots, G_d\) with \(d \geq 1\), such that for each \(i = 1, \ldots, d - 1\)

(i) \(G_i\) and \(G_{i+1}\) share exactly one edge \(e_i\) and this edge is either the north edge of \(G_i\) and the south edge of \(G_{i+1}\) or the east edge of \(G_i\) and the west edge of \(G_{i+1}\).

(ii) \(G_i\) and \(G_j\) have no edge in common whenever \(|i - j| \geq 2\).

(iii) \(G_i\) and \(G_j\) are disjoint whenever \(|i - j| \geq 3\).

For notation, \(G[i, i+t] = (G_i, \ldots, G_{i+t})\) is a subgraph of \(G = (G_1, \ldots, G_n)\), the \(d - 1\) edges \(e_1, \ldots, e_{d-1}\) which are contained in two tiles are called *interior edges* of \(G\) and the other edges are called *boundary edges*. A *perfect matching* \(P\) of a graph \(G\) is a subset of the
set of edges of $G$ such that each vertex of $G$ is incident to exactly one edge in $P$. Let $\text{Match}(G)$ denote the set of all perfect matchings of the graph $G$. The following figure presents some perfect matchings of a snake graph.

**Figure 1.16.** Perfect matchings of a snake graph.

### Snake Graphs and Cluster Algebras

Let $T$ be a triangulation of a surface $(S, M)$ and let $\gamma$ be an arc in $(S, M)$ which is not in $T$. Choose an orientation on $\gamma$, let $s \in M$ be its starting point, and let $t \in M$ be its endpoint. Denote by $s = p_0, p_1, \ldots, p_{d+1} = t$ the ordered points of intersection of $\gamma$ and $T$. For $j = 1, 2, \ldots, d$, let $\tau_{ij}$ be the arc of $T$ containing $p_j$, and let $\Delta_{j-1}$ and $\Delta_j$ be the two triangles in $T$ on either side of $\tau_{ij}$. Then, for $j = 1, \ldots, d - 1$, the arcs $\tau_{ij}$ and $\tau_{ij+1}$ form two sides of the triangle $\Delta_j$ in $T$ and we define $e_j$ to be the third arc in this triangle.

Let $G_j$ be the quadrilateral in $T$ that contains $\tau_{ij}$ as a diagonal (a tile) whose edges are arcs in $T$, thus, they are labeled edges. Define a sign function $f$ of the edges $e_1, \ldots, e_d$ by

$$f(e_j) = \begin{cases} +1, & \text{if } e_j \text{ lies on the right of } \gamma \text{ when passing through } \Delta_j, \\ -1, & \text{otherwise.} \end{cases}$$

(1.16)

The labeled snake graph $\mathcal{G}_\gamma = (G_1, \ldots, G_d)$ with tiles $G_i$ and sign function $f$ is called the snake graph associated to the arc $\gamma$. Each edge $e$ of $\mathcal{G}_\gamma$ is labeled by an arc $\tau(e)$ of the triangulation $T$. Such an arc defines the weight $x(e)$ of the edge $e$ to be the cluster variable associated to the arc $\tau(e)$. Thus $x(e) = x_{\tau(e)}$.

In [69] Musiker, Schiffler, and Williams showed a combinatorial formula for cluster variables of a cluster algebra of surface type $\mathcal{A}(S, M)$ with principal coefficients $\Sigma_T = (x_T, y_T, B_T)$. In such a case, if $\gamma$ is an arc, $\mathcal{G}_\gamma$ is its snake graph, and the triangulation $T$ has no self-folded triangles. Then the corresponding cluster variable $x_\gamma$ is given by the identity

$$x_\gamma = \frac{1}{\text{cross}(\gamma, T)} \sum_{P \in \text{Match}(\mathcal{G}_\gamma)} x(P),$$

(1.17)

where the sum runs over all perfect matchings of $\mathcal{G}_\gamma$, the summand $x(P) = \prod_{e \in P} x(e)$ is the weight of the perfect matching $P$, and $\text{cross}(T, \gamma) = \prod_{j=1}^d x_{\tau_{ij}}$ is the product of all initial cluster variables whose arcs cross $\gamma$.

A relationship between cluster variables and continued fractions is described by Schiffler and Canakci in [22], who claimed that, the numerator of a continued fraction is equal
to the number of perfect matchings of the corresponding abstract snake graph, and that it can therefore be interpreted as the number of terms in the numerator of the Laurent expansion of an associated cluster variable. Thus, the Laurent polynomials of the cluster variable can be recovered from the continued fraction.

For example, let $T$ be a triangulation, and let $\gamma$ be a diagonal which is not in $T$.

We can build the snake graph $G_\gamma$ associated to $\gamma$ (see Figure 1.17). The set of all perfect matchings of $G_\gamma$ are shown in Figure 1.18, and the cluster variable associated to $x_\gamma$ is given by the identity

$$x_\gamma = \frac{x_4 + x_1x_3x_4 + x_2}{x_2x_3}.$$

We can build the snake graph $G_\gamma$ associated to $\gamma$ (see Figure 1.17). The set of all perfect matchings of $G_\gamma$ are shown in Figure 1.18, and the cluster variable associated to $x_\gamma$ is given by the identity

$$x_\gamma = \frac{x_4 + x_1x_3x_4 + x_2}{x_2x_3}.$$
Integer Sequences Arising From Auslander-Reiten Quivers

Ringel and Fahr called categorification of an integer sequence the process for which numbers in the sequence can been seen as suitable invariants of objects in a category and proposed a categorification of Fibonacci numbers by using the Gabriel’s universal covering theory and the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver [49, 50]. In this chapter, we study sections in the Auslander-Reiten quiver of algebras of Dynkin type as a tool that provides categorifications of some integer sequences in the Online Encyclopedia of Integer Sequences (OEIS) [85–89]. Posets of type $b$, $d$ and $h$ and some properties of its lattice paths are introduced in section 2.1. In section 2.2 lattice paths connecting minimal and maximal points in posets of type $b$, $d$ and $h$ are used to enumerate sections in the Auslander-Reiten quiver of algebras of Dynkin type $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$ [32,38]. As a consequence of this chapter, we conclude that the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type is not a Dynkin function.

2.1 Posets of Type $b$, $d$, and $h$

In this section, we build families of posets (almost all of wild representation type), and we present integer sequences associated to the lattices paths over these posets.

2.1.1 Posets and Lattice Paths

If $\mathcal{P} = (\mathbb{N}^2, \preceq)$ is a poset where $\mathbb{N}$ denotes the set of natural numbers endowed with the usual order and $(x, y) \preceq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Then, a lattice path $P \subseteq \mathcal{P}$ is a sequence of points $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \subseteq \mathcal{P}$ where $(x_i, y_i) \preceq (x_{i+1}, y_{i+1})$ for any $1 \leq i \leq n-1$, and either $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$ or $y_{i+1} = y_i + 1$ and $x_{i+1} = x_i$.

An order ideal of a poset $(\mathcal{P}, \preceq)$ is a subset $I$ of $\mathcal{P}$ such that if $x \in I$ and $y \preceq x$, then $y \in I$ (i.e., $x$ covers $y$). We let $J(\mathcal{P})$ denote the set of all order ideals of $\mathcal{P}$, ordered by inclusion. Note that, $m$-element antichains in $\mathcal{P}$ correspond to elements of $J(\mathcal{P})$ that cover exactly
Given a finite poset $P$ with $|P| = n$ in [93] it is defined an extension of $P$ to a total order or linear extension of $P$ as an order-preserving bijection $\sigma : P \rightarrow n$. The number of extensions of $P$ to a total order is denoted $e(P)$. Actually, $e(P)$ is also equal to the number of maximal chains of $J(P)$.

According to Stanley [91,93] the enumeration of lattice paths is an extensively developed subject, the point in this chapter is that certain lattice path problems are equivalent to determining $e(P)$ for a given poset $P$, or equivalent to the problem of finding the number of sections in the Auslander-Reiten quiver of some finite-representation algebras. In this fashion, it is possible to establish connections between the theory of partitions, the theory of partially ordered sets and the theory of representation of algebras.

If $M = 2 \times n$ then it can be shown that the number of lattice paths from $(0,0)$ to $(n,n)$ with steps $(1,0)$ and $(0,1)$, which never rise above the main diagonal $x = y$ of the plane $(x,y)$-plane equals the number of linear extensions $e(M)$ of the poset $M$ and that $e(2 \times n) = \frac{1}{n+1} \binom{2n}{n} = C_n$ [91]. Figure 2.1 shows the number of lattice paths from each point $(x,y) \in M_3 = 2 \times 3$ to the maximal point, note that the number of lattice paths from the minimal to the maximal point is $C_3 = 5$. We will make the same computations for other types of posets in order to enumerate sections in the Auslander-Reiten quiver of some hereditary algebras of finite-representation type.

![Figure 2.1. Number of lattice paths from each $C_t$ to the maximal points $C_0$ is a Catalan number.](image)

More connections between the theory of partitions and the theory of partially ordered sets via lattice paths have been quoted by Andrews and Stanley in [1,93]. Firstly by establishing an identity between inversions and $p(m_1, m_2; n)$ the number of partitions of an integer number $n$ into at most $m_2$ parts no greater than a given integer $m_1$. And secondly by using $\mathcal{P}$-partitions, i.e., order-preserving maps from a partially ordered set $\mathcal{P}$ to a chain with special rules specifying where equal values may occur. For instance, if $\mathcal{P}$
is a $p$-element chain, then a $P$-partition of a positive integer $n$ is equivalent to an ordinary partition of $n$ into at most $p$ parts. Some relationships between $P$-partitions and the counting of chains in the set of order ideals of $P$ ordered by inclusion are well described by Stanley in [91, 93]. Actually, he describes in [93] the following relation between the number of some $P$-partitions of a positive integer $n$, denoted $m_n$, and the number $e(P)$ of extensions of $P$ to a total order. In this case, we have considered that $|P| = p$:

$$m_n = \frac{e(P)n^{p-1}(1+o(\frac{1}{n}))}{p!(1-p)!} \quad \text{as} \quad n \to \infty.$$  

The theory of $P$-partitions has been used by Petersen in [70] and Stembridge in [94] to investigate peak algebras and descent algebras.

### 2.1.2 Some Integer Sequences

We will see that the following sequences $\{a_n\}_{n \geq 0}$ and $\{C_m^n\}_{m \geq 0}$ are useful to enumerate the number of sections in the Auslander-Reiten quiver of some algebras of Dynkin type. Sequence $\{a_n\}$ is defined as follows:

$$a_0 = 1,$$

$$a_n = a_{n-2^x-1} + a_{n-2^x-1+y}, \quad (2.1)$$

where $x$ stands for the length of the binary expansion of $n$ and $y$ denotes the largest power of 2 associated to a zero occurring in such expansion bearing in mind that $y = 0$ if the binary expansion of $n$ has no 0 's. The following are the first 20 terms of $\{a_n\}$.

$$\{1, 2, 3, 4, 4, 6, 7, 8, 5, 8, 10, 12, 11, 14, 15, 16, 6, 10, 13, 16\}.$$

Note that,

$$a_{2^k+j} = a_{2^k-1+j} + a_j,$$ for each $k \geq 2$ and $0 \leq j \leq 2^{k-1} - 1. \quad (2.2)$$

In particular,

$$a_{2^k(2^k)-1} = 2^{2k} + 1 \text{ (a Fermat number).} \quad (2.3)$$

Sequence $C_m^n$ is defined in such a way for $n \geq 3$ fixed it holds that:

$$C_m^n = \begin{cases} 0, & \text{if } m = 0, \\ C_m^{n-1} + C_m^{n-1}, & \text{if } 0 < m < 2^{n-3}, \\ 2C_m^{n-1} + a_{2^{n-2}-(m-1)}, & \text{if } 2^{n-3} \leq m < 2^{n-2}. \end{cases} \quad (2.4)$$

In this case, for $n > 1$ and $m \geq 0$, $p_1$ denotes the number of digits in the binary expansion of the number $m$ and $a_{2^{n-2}-(m-1)} \in \{a_n\}$. Besides, for $m \geq 0$, $C_m^0 = 0$, further $C_2^1 = 1$ (see Appendix, Table A.1).
Remark 2.1. If \( n = \{1, 2, \ldots, n\} \) is an \( n \)-point chain then \( \mathcal{C}_{(1,n)} \) stands for all admissible subchains \( \mathcal{C} \) of \( n \) with \( \min \mathcal{C} = 1 \) and \( \max \mathcal{C} = n \). For instance, \( \{1, 3, 5, 7\} \) and \( \{1, 4, 6, 7\} \) are four-point subchains contained in \( \mathcal{C}_{(1,7)} \). Note that, the number of admissible chains in \( n \) equals \( 2^{n-2} \). To enumerate admissible subchains is a particular case of another interesting problem in combinatorics which consists of finding the number of chains contained in a poset \( (L, \preceq) \) where \( \preceq \) is the dominance order defined on the lattice of integer points \( (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d \). And for fixed nonnegative integers \( n_1, n_2, \ldots, n_d \), points \( (a_1, a_2, \ldots, a_d) \in L \) are defined in such a way that \( 0 \leq a_i \leq n_i \) for \( 1 \leq i \leq d \). Stanley proved that in the case \( d = 2 \) and \( n_1, n_2 \) share common value \( n \) then the total number of chains in \( L \) equals \( 2^{n+1}d_n \) where \( d_n \) denotes the \( n \)-th Delannoy number \( [92] \).

2.1.3 Posets of Type \( b_{j_0:j_1: \ldots: j_m}^{i_0:i_1: \ldots: i_k} \)

In this section we define the first type of poset we are interested in. Henceforth, we assume that \( i_0 = j_0 = 0 \), and the set \( \{j_1, j_2, \ldots, j_m, i_1, i_2, \ldots, i_k\} \) is an admissible subchain where either \( i_1 = 1 \) or \( j_1 = 1 \).

Given the partially ordered set \( \mathcal{P} = (\mathbb{Z}^2, \preceq) \) where \( (x, y) \preceq (x', y') \) if and only if \( x \leq x' \) and \( y \leq y' \), it is denoted as \( b_{j_0:j_1: \ldots: j_m}^{i_0:i_1: \ldots: i_k} \) a subposet of \( \mathcal{P} \) whose points \( i_r = (x_r, y_r), j_s = (w_s, z_s), i_r = (j_r^t)^t = (z_r, w_r) \) and \( j_s^t = (j_s^t)^t = (y_s, x_s) \) satisfy the following conditions:

1. If \( i_1 = 1 \) and \( k = m \),
   \[ i_1 = (x_1, y_1) = (n - 1, 0) \text{ and } j_1 = (w_1, z_1) = (0, 0). \]

2. If \( i_1 = 1 \) and \( k > 1 \),
   \[ i_1 = (x_1, y_1) = \left( \sum_{t=1}^{m} |i_t - j_t|, \sum_{t=1}^{\max\{k,m\}-1} |j_t - i_{t+1}| \right), \]
   \[ j_1 = (w_1, z_1) = (x_1 - |i_1 - j_1|, y_1), \]
   \[ i_r = (x_r, y_r) = \left( x_1 - \sum_{t=1}^{r-1} |i_t - j_t|, y_1 - \sum_{t=1}^{r-1} |j_t - i_{t+1}| \right), \]
   \[ j_s = (w_s, z_s) = \left( x_1 - \sum_{t=1}^{s-1} |i_t - j_t|, y_1 - \sum_{t=1}^{s-1} |j_t - i_{t+1}| \right), \]

for \( 1 < r \leq k \) and \( 1 < s \leq m \).

The admissible subchain \( \mathcal{C} = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq n \) must satisfy the following constraints for \( 1 \leq r \leq k \) and \( 1 \leq s \leq m \):

- If \( i_1 = 1 \) and \( k = m \) then \( i_1 < j_1 < \cdots < i_k < j_m = n \).

- If \( i_1 = 1 \) and \( k = m + 1 \) then \( i_1 < j_1 < \cdots < i_k < j_m < i_k = n \).

- If \( j_1 = 1 \) and \( k = m \) then \( j_1 < i_1 < \cdots < j_m < i_k = n \).
• If \( j_1 = 1 \) and \( m = k + 1 \) then \( j_1 < i_1 < \cdots < j_m < i_k < j_m = n \).

If for \( s \) fixed, \( 1 \leq s \leq m \), it is defined

\[
H_{zs} = \left\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq y \leq z_s, \; x \geq w_s, \; x + y \leq \max \{i_k, j_m\} - 1 \right\}.
\] (2.5)

Then

\[
b_{i_0i_1...i_kj_0j_1...j_m}^{i_0i_1...i_k} = \bigcup_{s=1}^{m} H_{zs}.
\] (2.6)

The following algorithm summarizes the construction of posets of type \( b_{i_0i_1...i_kj_0j_1...j_m}^{i_0i_1...i_k} \):

**Algorithm 2.1.**

1. If \( i_0 = j_0 = 0 \), then either \( i_1 = 1 \) or \( j_1 = 1 \),
2. Fix \( k \in \{m, m + 1\} \) and \( n \),
3. For \( 1 \leq r \leq k \) and \( 1 \leq s \leq m \) do;
   - If \( i_1 = 1 \) and either \( k = m \) or \( k = m + 1 \) then the subchain \( C = \{i_1 < j_1 < \cdots < i_k < j_m = n\} \) or \( C = \{i_1 < j_1 < \cdots < i_k < j_m = n\} \), respectively,
   - Else
     - If \( j_1 = 1 \) and either \( m = k \) or \( m = k + 1 \) then the subchain \( C = \{j_1 < i_1 < \cdots < j_m < i_k = n\} \) or \( C = \{j_1 < i_1 < \cdots < j_m < i_k = n\} \), respectively.
4. For \( 1 \leq s \leq m \) do;
5. \( i_r^*, j_s^*, (i_r^*)^t, (j_s^*)^t \), and \( H_{zs} \),
6. Do \( \bigcup_{s=1}^{m} H_{zs} \).

**Remark 2.2.** The main problem regarding posets of type \( b_{i_0i_1...i_kj_0j_1...j_m}^{i_0i_1...i_k} \) consists of finding the number of lattice paths from \((0,0)\) to each point \( p \in M_b \) where \( M_b \) denotes the set of maximal points contained in \( b_{i_0i_1...i_kj_0j_1...j_m}^{i_0i_1...i_k} \). Actually, points \( p \in M_b \) are solutions of the following linear programming problem:

**Problem 1**

Maximize \( x + y \);
Subject to the constraints:

\[
x \geq w_s,
\]
\[
y \leq z_s,
\]
\[
y \leq n - x - 1,
\]
\[
x \geq 0, \quad y \geq 0.
\] (2.7)

Henceforth, we let \([b_{i_0i_1...i_kj_0j_1...j_m}^{i_0i_1...i_k}]\) denote such a number and \(|P_{(x,y)}^{(x',y')}|\) the number of lattice paths from a point \((x,y)\) to a point \((x',y')\) in a given poset \(\mathcal{P}\). \([A]\) will denote the number of lattice paths from the set of minimal points to the set of maximal points of a subset \(A \subseteq \mathcal{P}\).
The following example shows the procedure described above to construct the poset $b_{025}^{14}$. 

(1) Firstly we make, $i_0 = j_0 = 0$. 

(2) Choose $m = k = 2$ and $n = 5$, thus $i_r, j_s \in \{1, 2\}$. 

(3) Select the admissible subchain $i_1 = 1 < j_1 = 2 < i_2 = 4 < j_2 = 5$ among all possible admissible subchains in $C_{(1,5)}$ satisfying the constraints. 

(4) Points $i_r, j_s$ are given by the following identities: 
\[
\begin{align*}
i_1 &= (x_1, y_1) = (2, 2), \\
j_1 &= (w_1, z_1) = (1, 2), \\
i_2 &= (x_2, y_2) = (1, 0), \\
j_2 &= (w_2, z_2) = (0, 0).
\end{align*}
\]

(5) Subsets $H_{z_1}$ and $H_{z_2}$ are given by the identities: 
\[
\begin{align*}
H_{z_1} &= \{(1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (2, 1), (3, 1), (1, 2), (2, 2)\}, \\
H_{z_2} &= \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)\}.
\end{align*}
\]

(6) We conclude finally that: 
\[
b_{025}^{14} = H_{z_1} \cup H_{z_2} = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (1, 1), (2, 1), (3, 1), (1, 2), (2, 2)\}.
\]

Figure 2.2 shows the way points in $b_{025}^{14}$ are connected by lattice paths. 

**Figure 2.2.** Lattice paths in poset $b_{025}^{14}$. 

Figure 2.3 shows other examples of lattice paths in posets of type $b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k}$. 

**Remark 2.3.** Let $P_b$ be a poset of type $b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k}$ then the derivatives $\frac{\partial P_b}{\partial x}$ and $\frac{\partial P_b}{\partial y}$ are defined in such a way that: 
\[
\begin{align*}
\frac{\partial P_b}{\partial x} &= P_b \backslash \{(n - 1, 0)\}, \\
\frac{\partial P_b}{\partial y} &= P_b \backslash \{(0, n - 1)\}.
\end{align*}
\]
Formulas (2.10) allow to establish the following equalities regarding the number of lattice paths from $(0, 0)$ to each point $p \in M_b$ as follows:

$$[p_b] = \begin{cases} \left( \frac{\partial p_b}{\partial x} + \frac{\partial p_b}{\partial y} \right), & \text{if } i_k = n, \\ \left( \frac{\partial p_b}{\partial y} \right), & \text{otherwise.} \end{cases}$$

In posets of type $b_{i_0i_1...i_k}^{j_0j_1...j_m}$ these identities have the following interpretations:
Lemma 2.1. Posets \( b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \) satisfy the following identities:

\[
\begin{aligned}
[ b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} ] &= \begin{cases} 
  b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} & \text{if } 1 \leq i_k < n-1, \\
  b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} & \text{if } i_k = n-1, \\
  b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} + b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k - 1} & \text{if } i_k = n, j_m = n-1, |j_m - i_k - 1| > 1, \\
  b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} + b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k - 1} & \text{if } i_k = n, j_m = n-1, |j_m - i_k - 1| = 1, \\
  b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} + b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k - 1} & \text{if } i_k = n, 1 < j_m < n-1, |j_m - i_k - 1| > 1, \\
  b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} + b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k - 1} & \text{if } i_k = n, 1 < j_m < n-1, |j_m - i_k - 1| = 1, \\
  b_{j_0j_1}^{i_0i_1} + b_{j_0j_1}^{i_0i_1 - 1} & \text{if } i_1 = n, j_m = 1.
\end{cases}
\end{aligned}
\]

The following theorem shows a connection between sequence (2.1) and the number of lattice paths in posets of type \( b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \) from \((0,0)\) to points \( p \in \mathbb{M}_b \).

Theorem 2.1. For a given poset \( b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \) (of type b) associated to an admissible subchain \( \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \) it holds the identity

\[
[ b_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} ] = a_r,
\]

where for \( n > 1 \)

\[
r = \begin{cases} 
  \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1}, & \text{if } 1 < i_k < n, \\
  \sum_{t=k-m+1}^{k} 2^{i_t-1} - \sum_{t=1}^{m} 2^{j_t-1}, & \text{if } i_k = n, \\
  0, & \text{if } i_k = 1.
\end{cases}
\]

Proof. By induction. For \( n = 2 \) we have two cases. If \( i_k = 1 \) the associated poset is given by \( b_{j_0j_1}^{i_0i_1} = \{(0,0), (1,0)\} \) and the only lattice path is \((0,0) \rightarrow (1,0)\). Thus, \( [ b_{j_0j_1}^{i_0i_1} ] = 1 = a_0 \).

On the other hand, if \( i_k = 2 \) the associated poset is \( b_{j_0j_1}^{i_0i_2} = \{(0,0), (0,1), (1,0)\} \) with two lattice paths. Since \( r = 1 \) and \( a_1 = 1 \) it holds that \( [ b_{j_0j_1}^{i_0i_2} ] = 2 = a_1 \). Suppose now that the case holds for \( n \leq p \) with \( 1 \leq i_k \leq p \).

If \( n = p + 1 \) and \( i_k = 1 \) then we have that \( [ b_{j_0j_1\ldots j_p+1}^{i_0i_1\ldots i_k} ] = [ b_{j_0j_1\ldots j_p}^{i_0i_1\ldots i_k} ] = a_r \) with \( r = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} \), if \( i_k = p \) we have that \( j_{m-1} < p \), thus \( [ b_{j_0j_1\ldots j_p+1}^{i_0i_1\ldots i_p} ] = [ b_{j_0j_1\ldots j_{m-1}}^{i_0i_1\ldots i_k} ] = a_r \) with \( r = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} \).

Now if \( i_k = p + 1 \), then the following cases hold:

Case 1. If \( j_m = p, |j_m - k_{k-1}| > 1 \) and \( i_{k-1} > 1 \), we have that \( [ b_{j_0j_1\ldots j_p}^{i_0i_1\ldots i_{k-1}} ] = [ b_{j_0j_1\ldots j_{p-1}}^{i_0i_1\ldots i_{k-1}} ] + a_r + a_s \) where \( r = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} \) and \( s = \sum_{t=k-m+1}^{k-1} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} \).
\[ \sum_{t=1}^{m} 2^{i_t - 1} + 2^{p-2}, \text{ actually, } r = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1} \text{ and } s = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1} + 2^{p-2}. \text{ Note that, if } w = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} \text{ then the number of digits of } w \text{ is } x = p \text{ and the largest power of } 2 \text{ associated to a zero in the binary expansion of } y \text{ is } 2^{p-2} \text{ (see formula (2.1)), thus}
\]
\[ \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}+1}} \right] = a_w. \]

Now, if \( i_{k-1} = 1 \) then \( \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}+1}} \right] = \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}} + 1} \right] = a_r + a_s \) where \( r = 0 \) and \( s = 2^{p-1} - 2^{p-2} \). Since \( w = 2^p - 2^{p-1} \) then the number of digits of \( w \) is \( x = p \) and the largest power of \( 2 \) associated to a zero in the binary expansion of \( y \) is again \( 2^{p-2} \), therefore,
\[ \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}} + 1} \right] = a_w. \]

**Case 2.** If \( i_k = p + 1, j_m = p \) and \( |j_m - i_{k-1}| = 1 \), then \( \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}+1}} \right] = \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}}} \right] + \left[ b_{j_{0j1\ldots j_m-1}}^{i_{j_{0j1\ldots j_m}}} \right] \), \( a_r + a_s \) where \( r = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1} \) and \( s = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1} + 2^{p-2}, x = p \) and \( y = 2^{p-2} \), therefore
\[ \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}+1}} \right] = a_w. \]

**Case 3.** If \( i_k = p + 1, 1 < j_m < p \) and \( |j_m - i_{k-1}| > 1 \) then \( \left[ b_{j_{0j1\ldots j_m-1}}^{i_{j_{0j1\ldots j_m}}} \right] = a_r + a_w \) where \( r = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1}, s = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1} + 2^{j_m-2}, x = p \) and \( y = 2^{j_m-2} \), thus
\[ \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}+1}} \right] = a_w. \]

**Case 4.** If \( i_k = p + 1, 1 < j_m < p \) and \( |j_m - i_{k-1}| = 1 \) then \( \left[ b_{j_{0j1\ldots j_m-1}}^{i_{j_{0j1\ldots j_m}}} \right] = a_r + a_w \) where \( r = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1}, s = \sum_{t=k-m+1}^{k} 2^{i_t - 1} - \sum_{t=1}^{m} 2^{i_t - 1} - 2^{p-1} + 2^{j_m-2}, \) in this case, \( x = p \) and \( y = 2^{j_m-2} \), therefore
\[ \left[ b_{j_{0j1\ldots p}}^{i_{j_{0j1\ldots p}+1}} \right] = a_w. \]
Case 5. If \( i_k = p + 1 \) and \( j_1 = 1 \), \( b_{j_0 p+1}^{i_0} \) = \( b_{j_0 p}^{i_0} \) + \( b_{j_0}^{i_0+1} \) + \( a_r \) where \( r = 2^{p-1} - 1 \), the number of digits of \( w = 2^p - 1 \) is \( x = p \) and \( w \) has no zeroes in its binary expansion, that is, \( y = 0 \). Therefore, \( r = w - 2^{x-1} \) and \( b_{j_0}^{i_0+1} \) = \( a_w \). \( \square \)

### 2.1.4 Posets of Type \( d_{j_0 i_1 \ldots i_k} \)

In this section, we define another type of posets whose lattice paths will allow to enumerate sections in Auslander-Reiten quivers of algebras of Dynkin type \( A_n \).

Points \( i_r^* = (x_r, y_r) \), \( j_s^* = (x_s, y_s) \), \( i_{rs} = (x_{rs}, y_{rs}) \) and \( j_{ss} = (x_{ss}, y_{ss}) \) in posets of type \( d_{j_0 i_1 \ldots i_k} \subset \mathcal{P} \) satisfy the following conditions:

- If \( i_1 = 1 \)

\[
\begin{align*}
i_r^* &= (x_r, y_r) = \left( - \sum_{w=0}^{r-1} l_w, u - \sum_{w=0}^{r-1} d_w \right), \\
j_s^* &= (w_s, z_s) = \left( - \sum_{w=0}^{s-1} l_w, u - \sum_{w=0}^{s-1} d_w \right), \\
i_{rs} &= (x_{rs}, y_{rs}) = \left( \sum_{w=0}^{r-1} d_w, -n + 1 + u + \sum_{w=0}^{r-1} l_w \right), \\
j_{ss} &= (x_{ss}, y_{ss}) = \left( \sum_{w=0}^{s-1} d_w, -n + 1 + u + \sum_{w=0}^{s-1} l_w \right).
\end{align*}
\]

- If \( j_1 = 1 \)

\[
\begin{align*}
i_r^* &= (x_r, y_r) = \left( - \sum_{w=0}^{s-1} l_w, u - \sum_{w=0}^{s-1} d_w \right), \\
j_s^* &= (w_s, z_s) = \left( - \sum_{w=0}^{r-1} l_w, u - \sum_{w=0}^{r-1} d_w \right), \\
i_{rs} &= (x_{rs}, y_{rs}) = \left( \sum_{w=0}^{s-1} d_w, -n + 1 + u + \sum_{w=0}^{s-1} l_w \right), \\
j_{ss} &= (x_{ss}, y_{ss}) = \left( \sum_{w=0}^{r-1} d_w, -n + 1 + u + \sum_{w=0}^{r-1} l_w \right),
\end{align*}
\]

where \( 1 \leq r \leq k \) and \( 1 \leq s \leq m \).

In these cases,

\[
u = \begin{cases} 
\sum_{l=0}^{t-1} |i_{k-l} - j_{m-l}|, & \text{if } 1 \leq i_k < n \text{ and } t = \min \{k, m\}, \\
n - 1 - \sum_{l=0}^{t-1} |i_{k-l} - j_{m-l}|, & \text{if } i_k = n.
\end{cases}
\]
Numbers $d_w$ and $l_w$ are given by the following relations:

(1) If $i_1 = 1$ and
   - $k = m = 1$ then $d_1 = |j_1 - i_1|$ and $l_0 = 0$.
   - $k = m > 1$ then $d_r = |j_r - i_r|$ and $l_s = |i_{s+1} - j_s|$ for $1 \leq r \leq k$ and $1 \leq s \leq k - 1$.
   - $1 < k = m + 1$ then $d_r = |j_r - i_r|$ and $l_s = |i_{s+1} - j_s|$ for $1 \leq r \leq k - 1$ and $1 \leq s \leq k - 1$.

If $j_1 = 1$ and
   - $k = m = 1$ then $d_0 = 0$ and $l_1 = |j_1 - i_1|$.
   - $k = m > 1$ then $d_s = |j_{s+1} - i_s|$ and $l_r = |j_r - i_r|$ for $1 \leq r \leq k - 1$ and $1 \leq s \leq k$.
   - $1 < k = m + 1$ then $d_s = |j_{s+1} - i_s|$ and $l_r = |j_r - i_r|$ for $1 \leq r \leq m - 1$ and $1 \leq s \leq m - 1$.

$l_0 = 0$ and $d_0 = 0$.

(2) $d_{j_0j_1...j_m}^{i_0i_1...i_k} = A_1 \cup A_2$ where

\[ A_1 = \bigcup_{r=1}^{k} H_{y_r} \quad \text{and} \quad A_2 = \bigcup_{s=1}^{m-1} H_{\mathbb{Z}_s}, \]

with

\[ H_{y_r} = \left\{ (x, y) \in \mathbb{Z}^2 \mid 0 \leq y \leq y_r, x \geq x_r, x + y \leq u \right\} \]

\[ H_{\mathbb{Z}_s} = \left\{ (x, y) \in \mathbb{Z}^2 \mid \mathbb{Z}_s \leq y \leq 0, x \leq \mathbb{Z}_s, x + y \geq -n + 1 + u \right\} \]

The following algorithm summarizes the construction of posets of type $d_{j_0j_1...j_m}^{i_0i_1...i_k}$.

**Algorithm 2.2.** (1) Fix $i_0 = 0 = j_0$,

(2) Define $k \in \{m, m+1\}$, $n$ and either $i_1 = 1$ or $j_1 = 1$,

(3) If $i_1 = 1$ and $k = m$ or $k = m + 1$ then the subchain $C = \{i_1 < j_1 < i_2 < j_2 < \cdots < j_{k-1} < i_k < j_k\}$ or $C = \{i_1 < j_1 < i_2 < j_2 < \cdots < j_{k-2} < i_{k-1} < j_{k-1} < i_k\}$,

(4) Else if $j_1 = 1$ and $m = k$ or $m = k + 1$ then the subchain $C = \{j_1 < i_1 < j_2 < i_2 < \cdots < i_{k-1} < j_k < i_k\}$ or $C = \{j_1 < i_1 < j_2 < i_2 < \cdots < i_{k-1} < j_k < i_k < j_{k+1}\}$,

(5) If $i_1 = 1$ then for $1 \leq r \leq k$ and $1 \leq s \leq k - 1$ or $1 \leq r \leq k - 1$ and $1 \leq s \leq k - 1$ do $d_r$ and $l_s$,

(6) Else if $j_1 = 1$ and for $1 \leq r \leq k - 1$ and $1 \leq s \leq k$ or $1 \leq r \leq k - 1$ and $1 \leq s \leq k - 1$ do $d_s$ and $l_r$,

(7) Compute $u$,

(8) For $1 \leq r \leq k$ and $1 \leq s \leq m$ do $i_{r^*}$, $j_{s^*}$, $i_{r^*}$, $j_{s^*}$,

(9) For $1 \leq r \leq k$ and $1 \leq s \leq m - 1$ do $H_{y_r}$, $H_{\mathbb{Z}_s}$.
(10) Compute $A_1$, $A_2$ and $A_1 \cup A_2$.

As an example, we build the poset $d_{025}^{013}$ by using the Algorithm 2.2.

1. $i_0 = j_0 = 0$.
2. $k = m = 2$, $n = 5$, $i_1 = 1$.
3. It is constructed the admissible subchain $1 < 2 < 3 < 5$ with $i_1 = 1$, $j_1 = 2$, $i_2 = 3$ and $j_2 = 5$.
4. $d_1 = |j_1 - i_1| = 1$, $d_2 = |j_2 - i_2| = 2$ and $l_1 = |i_2 - j_1| = 1$,
5. $3 = i_2 < 5$ and $t = \min \{k, m\} = 2$
   \[ u = |i_2 - j_2| + |i_1 - j_1| = 3, \]
6. $1 \leq r, s \leq 2$
   \[ i_1^* = (x_1, y_1) = (-l_0, 3 - d_0) = (0, 3), \]
   \[ i_2^* = (x_2, y_2) = (-l_1, 3 - d_1) = (-1, 2), \]
   \[ i_{1*} = (\overline{x}_1, \overline{y}_1) = (d_0, -1 + l_0) = (0, -1), \]
   \[ i_{2*} = (\overline{x}_2, \overline{y}_2) = (d_1, -1 + l_1) = (1, 0), \]
   \[ j_1^* = (w_1, z_1) = (l_0, 3 - d_1) = (0, 2), \]
   \[ j_2^* = (w_2, z_2) = (-l_1, 3 - d_1 - d_2) = (-1, 0), \]
   \[ j_{1*} = (\overline{w}_1, \overline{z}_1) = (d_1, -1 + l_0) = (1, -1), \]
   \[ j_{2*} = (\overline{w}_2, \overline{z}_2) = (d_1 + d_2, -1 + l_1) = (3, 0). \]
7. $H_{y_1} = \{(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (0, 3)\}$,
   $H_{y_2} = \{(-1, 0), (0, 0), (1, 0), (2, 0), (3, 0), (-1, 1), (0, 1), (1, 1), (2, 1), (-1, 2), (0, 2), (1, 2)\}$,
   $H_{\overline{y}_1} = \{(1, -1), (0, -1), (1, 0), (0, 0), (-1, 0)\}$.
8. $A_1 = H_{y_1} \cup H_{y_2}$, $A_2 = H_{\overline{y}_1}$, \[ d_{025}^{013} = A_1 \cup A_2. \]

The following is the linear programming problem associated to posets of type $d$.

Problem 2
Maximize (Minimize) \( x + y \);
Subject to the constraints:

\[
\begin{align*}
x_r & \leq x \leq w_s, \\
z_s & \leq y \leq y_r, \\
-n + 1 & \leq x + y - u \leq 0.
\end{align*}
\]

(2.11)

As for posets of type \( b \), if \( M^d \) denotes the set of minimal points in a poset of the form \( d_{i_0j_1...j_m}^{i_1...i_k} \) then the main problem for this kind of posets (of type \( d \)) consists of finding the number \([\alpha_{i_0j_1...j_m}^{i_1...i_k}]\) of lattice paths from points \( p \in M^d \) to points \( q \in M_d \) satisfying the following conditions:

\[
\begin{align*}
x_i & \leq w_{i-1}, \ y = y_i \text{ for any } 2 \leq i \leq k, \text{ if } i_1 = 1, \\
x_i & \leq w_i, \ y = y_i \text{ for any } 1 \leq i \leq k, \text{ if } j_1 = 1.
\end{align*}
\]

(2.12)

Figure 2.4 shows lattice paths linking points in \( d_{025}^{013} \).

![Figure 2.4. Lattice paths in poset of type \( d_{025}^{013} \).](image)

Figure 2.5 below shows other examples of lattice paths in posets of type \( d_{0247}^{0135} \) and \( d_{01357}^{0246} \).
Lemma 2.2. Numbers \( [\alpha_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k}] \) with \( j_m = n \) satisfy the following identities:

\[
\begin{align*}
[\alpha_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k}] &= \begin{cases} 
0, & \text{if } i_k = 1, \\
1, & \text{if } i_k = 2, \\
\alpha_{j_0j_1\ldots j_m-1}^{i_0i_1\ldots i_k} + \alpha_{j_0j_1\ldots j_m-1}^{i_0i_1\ldots i_{k-1}}, & \text{if } 2 < i_k \leq n - 2, i_k - j_{m-1} > 1, \\
\alpha_{j_0j_1\ldots j_m-1}^{i_0i_1\ldots i_k} + \alpha_{j_0j_1\ldots j_m-2}^{i_0i_1\ldots i_{k-1}}, & \text{if } 2 < i_k \leq n - 2, i_k - j_{m-1} = 1, \\
2[\alpha_{j_0j_1\ldots j_m-2}^{i_0i_1\ldots i_{k-1}}] + [b_{i_0i_1\ldots i_k}], & \text{if } 2 < i_k = n - 1, j_{m-1} = n - 2, \\
2[\alpha_{j_0j_1\ldots j_m-1}^{i_0i_1\ldots i_{k-1}}] + [b_{i_0i_1\ldots i_k}], & \text{if } 2 < i_k = n - 1, j_{m-1} < n - 2.
\end{cases}
\]

Proof. We have the following cases:

Case 1. If \( i_k = 1 \) then \( \alpha_{j_0n}^{i_01} = 0 \).

Case 2. If \( i_k = 2 \) then \( \alpha_{j_01n}^{i_02} = |P_{(1,0)}^{0,n-2}| = 1 \).

Case 3. If \( 2 < i_k \leq n - 2, i_k - j_{m-1} > 1 \), \( A = A_1 \cup A_2 \) and \( B = B_1 \cup B_2 \cup B_3 \) with

\[
A_1 = \bigcup_{r=1}^{k} \{(x, y) \in d_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \mid 1 \leq y \leq y_r, x \geq x_r, x + y \leq u \},
\]

\[
A_2 = \bigcup_{s=1}^{m-1} \{(x, y) \in d_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \mid \overline{z}_s + 1 \leq y \leq 1, x \leq \overline{w}_s, x + y \geq -n + 2 + u \},
\]

\[
B_1 = \bigcup_{r=1}^{k-1} \{(x, y) \in d_{j_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \mid 0 \leq y \leq y_r, x \geq x_r, x + y \leq u \},
\]
\[ B_2 = \{(x, y) \in d_{\alpha}^{i_1 \ldots i_k} | 0 \leq y \leq y_k, x \geq x_k + 1, x + y \leq u\} \] and
\[ B_3 = \bigcup_{s=1}^{m-1} \{(x, y) \in d_{\alpha}^{i_1 \ldots i_k} | \overline{s} + 1 \leq y \leq 0, x \leq \overline{w}, x + y \geq -n + 2 + u\}, \]
then the maps \(g_1 : d_{\alpha}^{i_1 \ldots i_k} \rightarrow A\) and \(g_2 : d_{\alpha}^{i_1 \ldots i_k - 1} \rightarrow B\) such that:
\[
g_1(x, y) = (x, y + 1),
g_2(x, y) = (x, y), \tag{2.13}
\]
are isomorphisms.

If \(C = \{(x, y) \in b_{\alpha}^{i_1 \ldots i_k}, |x + y = -n + 1 + u\}\) then the union \(C \cup A\) is a poset with relations of type \((x, y) \leq (x + 1, y)\) for \((x, y) \in C\) and \((x, y + 1) \in A\). Since \(P^{(e, f)}(x, y) = P^{(e, f)}(x, y - 1)\) where all of paths \(P^{(e, f)}\) contain at least one point satisfying conditions (2.12) and points \((x, y), (e, f)\) are chosen in such a way that, \(x + y - 1 = -n + 1 + u\) and \(e + f = u\) then \(\left[C \cup A\right] = \left[a_{\alpha}^{i_1 \ldots i_k}\right].\)

Now, we define the poset \(C \cup B\) with relations of the form \((x, y) \leq (x + 1, y)\) with \((x, y) \in C\) and \((x + 1, y) \in B\). Since \(P^{(e, f)}(x, y) = P^{(e, f)}(x, y)\) then \(\left[C \cup B\right] = \left[a_{\alpha}^{i_1 \ldots i_k - 1}\right].\) Thus,
\[
\left[a_{\alpha}^{i_1 \ldots i_k}\right] = \left[C \cup A\right] + \left[C \cup B\right] = \left[a_{\alpha}^{i_1 \ldots i_k - 1}\right].
\]

**Case 4.** If \(2 < i_k \leq n - 2, i_k - j_{m-1} = 1, (A, C\) described as before\) and \(D = B_1 UC_1\) where
\[ C_1 = \bigcup_{s=1}^{m-2} \{(x, y) \in d_{\alpha}^{i_1 \ldots i_k} | \overline{s} + 1 \leq y \leq 0, x \leq \overline{w}, x + y \geq -n + 2 + u\}, \]
then the maps \(g_3 : d_{\alpha}^{i_1 \ldots i_k - 1} \rightarrow A\) and \(g_4 : d_{\alpha}^{i_1 \ldots i_k - 1} \rightarrow D\) defined as \(g_1\) and \(g_2\), respectively are isomorphisms, sets \(C \cup A\) and \(C \cup D\) are posets with the same relations as in Case 3 for points \((x, y) \in C \cup A\). Whereas, relations between points \((x, y) \in C \cup D\) are of the form \((x, y) \leq (x + 1, y)\) for \((x, y) \in C\) and \((x + 1, y) \in D\). Thus \(\left[C \cup A\right] = \left[a_{\alpha}^{i_1 \ldots i_k}\right]\) and \(\left[C \cup D\right] = \left[a_{\alpha}^{i_1 \ldots i_k - 1}\right].\) Therefore, \(\left[a_{\alpha}^{i_1 \ldots i_k - 1}\right] = \left[C \cup A\right] + \left[C \cup D\right] = \left[a_{\alpha}^{i_1 \ldots i_k}\right] + \left[a_{\alpha}^{i_1 \ldots i_k - 1}\right].\)

**Case 5.** If \(2 < i_k = n - 1, i_{m-1} = n - 2, D\) is described as before and
\[ E = \{(x, y) \in d_{\alpha}^{i_1 \ldots i_k} | y \geq y_k, x_k \leq x\}, \]
then maps \(g_5\) and
\[
g_5 : b_{\alpha}^{i_{j_1 \ldots j_m}} \rightarrow E \quad (x, y) \mapsto (x - \sum_{l=1}^{k} l, y + 1)\]
where for each edge \((x, y), (w, z)\), it holds that
\[
g_5((x, y), (w, z)) = \left( \left( x - \sum_{i=1}^{k} l_i, y + 1 \right), \left( w - \sum_{i=1}^{k} l_i, z + 1 \right) \right),
\]
Eq. (2.14)
are isomorphisms.

Now, the set \(C \cup D\) is a poset with lattice paths induced by \(d_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k}}\) and \(P_{(x,y)}^{(e,f)}\) where paths \(P_{(x,y)}^{(e,f)}\) satisfy conditions as described in Case 3. Therefore \(\big[ C \cup D \big] = 2 E_{j_{0}j_{1}...j_{m-2}j_{m-1}}^{i_{1}...i_{k}}\).

The set \(\{(n + 1 + u, 0)\} \cup E\) is a poset whose lattice paths are induced by \(d_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k}}\) and \(P_{(-n+1+u,0)}^{(e,f)}\) then \(\{(n + 1 + u, 0)\} \cup E\) = \([a_{j_{0}j_{1}...j_{m-1}}^{i_{1}...i_{k-1}}] + [b_{j_{0}j_{1}...j_{m-1}}^{i_{1}...i_{k-1}}]\).

**Case 6.** If \(2 < i_k = n - 1, i_{m-1} < n - 2, B, E, C, g_2, g_5\) and \(\{(n + 1 + u, 0)\} \cup E\) are described as before then the set \(C \cup B\) is a poset with lattice paths induced by \(d_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k}}\). Then \([ C \cup B \big] = 2 E_{j_{0}j_{1}...j_{m-1}}^{i_{1}...i_{k-1}}\). Thus, \(\big[ a_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k-1}} \big] = [ C \cup D \big] + \{(n + 1 + u, 0)\} \cup E\) = \(2 E_{j_{0}j_{1}...j_{m-1}}^{i_{1}...i_{k-1}} + b_{j_{0}j_{1}...j_{m-1}}^{i_{1}...i_{k-1}}\).

If \(r = 0\) and \(r = 1\) are associated to numbers \([a_{j_{0}j_{1}}^{i_{1}}]\) and \([a_{j_{0}j_{1}}^{i_{2}j_{1}}]\) respectively. Then the following result shows a relationship between numbers \([a_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k}}]\) and elements in the integer sequence \(C_{m}^{n}\) (see formula 2.4).

**Theorem 2.2.** Let \(d_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k}}\) be a poset of type \(d\) with \(2 < i_k < n\) and \(j_{m} = n\) then \([a_{j_{0}j_{1}...j_{m}}^{i_{1}...i_{k}}] = C_{m}^{n}\).

where \(r = \sum_{t=k-m+2}^{k} 2^{i_{t}-1} - \sum_{t=1}^{m-1} 2^{i_{t}-1}\).

**Proof.** (Induction). If \(n = 4\) and \(2 \leq i_k < 4\) then \(j_{m-1} = 1\), and \([a_{j_{0}j_{1}}^{i_{3}j_{1}}] = 2 E_{j_{0}j_{1}j_{2}j_{3}}^{i_{1}j_{3}j_{2}}\) with \(r = 2^{3} - 1\). If \(j_{m-1} = 2\) \([a_{j_{0}j_{1}}^{i_{2}j_{1}j_{2}}] = 2 E_{j_{0}j_{1}j_{2}j_{3}}^{i_{1}j_{2}j_{1}}\) with \(r = 2^{2} - 2\).

Suppose that the hypothesis holds for \(n \leq p\) with \(2 \leq i_k < p\). Then if \(n = p + 1\) the following cases have place:
Case 1. If $2 < i_k \leq p - 1$ and $|i_k - j_{m-1}| > 1$ we have that $\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] + \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_w^p + C_s^p$ where $w = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1}$ and $s = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} + 2^k$. Since, it is easy to see that the number of digits in the binary expansion of $w = s + 2^k < 2^{p-2}$ is $x = i_k - 1$ then we conclude that,

$$\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_w^{p+1}.$$ 

Case 2. If $2 < i_k \leq p - 1$ and $|i_k - j_{m-1}| = 1$ then we have that $\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] + \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_w^p + C_s^p$, in this case, $w = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1}$ and $s = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-2} 2^{j_t-1} + 2^k$ then $s = w - 2^k$ and the number of digits in the binary expansion of $w$ is $x = i_k - 1$, thus,

$$\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_w^{p+1}.$$ 

Case 3. If $2 < i_k = p$ and $i_{m-1} = p - 1$, it follows that $\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = 2 \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] + \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_w^p + a_s$, in this case, if $k = m$ then $w = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} = r - 2^{p-2}$ and $s = 2^{p-1} - (r + 1)$ with $r = \sum_{t=1}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1}$. On the other hand, if $k = m + 1$ then $w = r - 2^{p-2}$ and $s = 2^{p-1} - (r + 1)$ with $r = \sum_{t=1}^{m} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1}$, therefore,

$$\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_r^{p+1}.$$ 

Case 4. If $2 < i_k = p$ and $i_{m-1} < p - 1$, it holds that $\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = 2 \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] + \left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = 2C_w^p + a_s$ where $w = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1} + 2^{p-2} = r - 2^{p-2}$ and $s = \sum_{t=k-m+1}^{m-1} 2^{j_t-1} - \sum_{t=1}^{k} 2^{i_t-1} = 2^{p-1} - (r + 1)$, with $r = \sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{j_t-1}$. In this case, we also have that $\left[ \alpha_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k} \right] = C_r^{p+1}$. \[\square\]

### 2.1.5 Posets of Type $h_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k}$

We let $h_{i_0j_1\ldots j_m}^{i_0i_1\ldots i_k}$ denote a poset which can be defined by following the steps:
1. Follow steps 1-5 of Algorithm 2.1.

2. Define

\[ G = \bigcup_{s=1}^{m} \left\{ (x, y) \in \mathbb{Z}^2 \mid a \leq y \leq z_s, x = w_s \right\} \cup \left\{ (x, y) \in \mathbb{Z}^2 \mid y = z_s, w_s \leq x \leq b \right\}, \]

where

\[ a = \begin{cases} z_{s+1}, & \text{if } s < m, \\ 0, & \text{if } s = m \end{cases}, \quad b = \begin{cases} w_{s-1}, & \text{if } s > 1, \\ x_1, & \text{if } s = 1. \end{cases} \]

3. \( h_{i_0j_1...i_k}^{j_0j_1...j_m} = \left\{ (x, y) \in \mathbb{Z}^2 \mid x = w - i, \ y = z + i, \ (w, z) \in G \right\} \) for \( 0 \leq i \leq \max \{i_k, j_m\}. \)

The following are examples of lattice paths in posets of type \( h_{i_0j_1...i_k}^{j_0j_1...j_m} \):

![Examples of oriented lattice paths of posets of type \( h_{i_0j_1...i_k}^{j_0j_1...j_m} \).](image)

Now, we intend to enumerate the number of lattice paths \( |P_{(x,y)}^{(e,f)}| \) connecting points \((x, y)\) and \((e, f)\) with \( y = -x \) and \( e + f = \max \{i_k, j_m\} \), \((x, y), (e, f) \in h_{i_0j_1...i_k}^{j_0j_1...j_m} \), and conclude that

\[ |P_{(x,y)}^{(e,f)}| = \sum |P_{(x,y)}^{(e,f)}|. \]

Numbers \( b_{R_t} \) and \( b_{L_r} \) are defined as follows for cases \( i_1 = 1, \ j_1 = 1 \):

- If \( j_1 = 1 \) then

\[ b_{R_t} = \begin{cases} \left[ b_{j_0...j_{t-1}(j_1+p_1)} \right], & \text{for } 1 \leq t \leq k, \ 1 \leq p_1 \leq |R_t|, \\ \left[ b_{j_0} \right], & \text{for } t = 0, \end{cases} \]

\[ b_{L_r} = \begin{cases} \left[ b_{j_0...j_{r-1}j_m} \right], & \text{for } 1 \leq r \leq m, \ 1 \leq p_1 \leq |L_r| \ y \ p_2 \neq |L_m| \\ \left[ b_{j_0j_1...j_k} \right], & \text{for } p_2 = |L_m| \ \text{and} \ r = m, \end{cases} \]

with \( |R_t| = |i_t - j_t|, \ |R_0| = 1, \ |L_r| = |j_r - i_{r-1}| \ \text{and} \ |L_1| = 1 \) for \( 1 \leq t \leq k \) and \( 1 < r \leq m. \)

- If \( j_1 = 1 \)
Theorem 2.3.

\[ b_{R_t} = \left[ b_{i_0, \ldots, i_t-1}^{j_0, \ldots, j_{t-1}, j_m} \right] \text{ para } 1 \leq t \leq k, 1 \leq p_1 \leq |R_t|, \]

\[ b_{L_r} = \left\{ \begin{array}{ll}
& \left[ b_{i_0, \ldots, i_{r-1}}^{j_0, \ldots, j_{r-1}, i_k} \right], \quad \text{for } 1 \leq r \leq m, 1 \leq p_1 \leq |L_m| \text{ and } p_2 \neq |L_m|, \\
& \left[ b_{i_0, \ldots, i_k}^{j_0, \ldots, j_m} \right], \quad \text{for } p_2 = |L_m| \text{ and } r = m, \\
& \left[ b_{i_0}^{j_0, \ldots, j_m} \right], \quad \text{for } r = 0,
\end{array} \right. \]

with \( |R_t| = |i_t - j_t - 1|, \) \( |R_1| = 1, \) \( |L_r| = |j_r - i_r| \) and \( |L_1| = 1 \) for \( 1 \leq t \leq k \) and \( 1 \leq r \leq m. \)

**Proof.** Let \( b_{i_0, \ldots, i_k}^{j_0, \ldots, j_m} \) be a poset with \( j_m = n \) and \( m \geq k, \) and sets \( b_u \) defined in such a way that:

\[ b_u = \{(x,y) \in b_{i_0, \ldots, i_k}^{j_0, \ldots, j_m} | x \geq x_m - u, y \geq y_m + u, x + y \leq n - 1\}, \]

with \( 0 \leq u \leq n. \) If \( j_1 = 1, \) it is possible to build the sets \( R_t = \{ y \in \mathbb{Z} | z_t \leq y < y_t \}, \)
\( R_0 = \{ y_1 \}, \)
\( L_r = \{ x \in \mathbb{Z} | x_r \leq x \leq w_{r-1} \} \)
and \( L_1 = \{ x_1 \} \) for \( 1 \leq t \leq k \) \( y \leq r \leq m \)
where \( |R_t| = |z_t - y_t| = |i_t - j_t|, |R_0| = 1, |L_r| = |x_r - w_{r-1}| = |j_r - i_{r-1}| \) and \( |L_1| = 1. \) If
\[ \sum_{a=0}^{k-t} |R_{k+1-a}| \leq u < \sum_{a=0}^{k+1-t} |R_{k+1-a}| (|R_{k+1}| = 0), \]
then there exists an isomorphism

\[ f_u : b_{i_0, \ldots, i_t-1}^{j_0, \ldots, j_{t-1}, j_m} \rightarrow b_u \]
\[ (x, y) \rightarrow (y - u, x + u) \]

with

\[ f_u((x, y), (z, w)) = (((y - u, x + u), (w - u, z + u))) \text{ for any edge } ((x, y), (z, w)). \quad (2.15) \]

For \( 1 \leq p_1 \leq |R_t|, \) where \( u = \sum_{a=0}^{k+1} |R_{k+1-a}| \) then there exists an isomorphism \( f_u : b_{i_0}^{j_0, \ldots, j_m} \rightarrow b_u \) defined as in \((2.15).\) Similarly, if \( \sum_{a=0}^{k-t} |R_{k+1-a}| + \sum_{a=0}^{r-1} |L_a| \leq u < \sum_{a=0}^{k+1} |R_{k+1-a}| + \sum_{a=0}^{r} |L_a| \)
\((|L_0| = 0) \) or \( u = |L_m|, \) it is possible to define an isomorphism defined as in \((2.15)\) with

\[ g_u : b_{i_0, \ldots, i_{r-1}, i_t}^{j_0, \ldots, j_{r-1}, j_{t+1}+p_2} \rightarrow b_u, \text{ for } 1 \leq p_2 \leq |L_r|. \]

The same can be done for a homomorphism

\[ g_u : b_{i_0, \ldots, i_k}^{j_0, \ldots, j_m} \rightarrow b_n, \text{ if } r = m \text{ and } |L_m| = p_2. \]
Thus,

\[
\left[ h_{i_0j_1...j_m}^{i_1...i_k} \right] = \sum_{t,p_1} b_{R_t} + \sum_{r,p_1,p_2} b_{L_r}. \tag{2.16}
\]

If \( i_1 = 1 \) then we can construct sets \( R_t = \{ y \in \mathbb{Z} | z_t \leq y < y_{t-1} \} \), \( R_1 = \{ z_1 \} \), \( L_r = \{ x \in \mathbb{Z} | x_j \leq x < w_r \} \) and \( L_0 = \{ w_1 \} \) for \( 1 < t \leq k \) and \( 1 \leq r \leq m \), where \( |R_t| = |z_t - y_{t-1}| = |i_t - j_{t-1}|, |R_1| = |L_r| = |x_j - w_r| = |j_t - i_r| \) and \( |L_0| = 1 \).

We conclude that the theorem holds provided that isomorphisms of the following types:

\[
\begin{align*}
    f_u & : b_{i_{t-1}+p_1}^{j_{t-1}...j_m} \rightarrow b_u, \\
    g_u & : b_{i_0}^{j_{t-1}...j_m} \rightarrow b_u, \\
    h_u & : b_{i_0...i_r+1}^{j_{t-1}...j_m} \rightarrow b_u, \\
    i_n & : b_{i_0...i_k}^{j_{t-1}...j_m} \rightarrow b_n
\end{align*}
\]  

(2.17)

can be defined respectively according to the following cases for \( u, p_1 \) and \( p_2 \):

\[
\begin{align*}
    \sum_{a=0}^{k-t} |R_{k+1-a}| \leq u < \sum_{a=0}^{k+1-t} |R_{k+1-a}| \text{ and } 1 \leq p_1 \leq |R_t|, \tag{2.18} \\
    \sum_{a=0}^{k+1} |R_{k+1-a}| + |L_0| \leq u < \sum_{a=0}^{k+1} |R_{k+1-a}| + |L_0| + |L_1|, \tag{2.19} \\
    \sum_{a=0}^{k+1} |R_{k+1-a}| + \sum_{a=0}^{r-1} |L_a| \leq u < \sum_{a=0}^{k+1} |R_{k+1-a}| + \sum_{a=0}^{r} |L_a|, \text{ and } 1 \leq p_2 \leq |L_r|, p_2 \neq |L_m|, \tag{2.20}
\end{align*}
\]

\[
r = m \text{ and } p_2 = |L_m|. \tag{2.21}
\]

We are done. \( \square \)

**Remark 2.4.** On sets \( \{ b_{R_t} \} \) (resp. \( \{ b_{L_m} \} \)) it is defined a partial order such that \( b_{R_t} < b_{R_s} \) (resp. \( b_{L_t} < b_{L_s} \)) if and only if \( i_{k_s} < i_{k_t} \) (resp. \( i_{k_t} < i_{k_s} \)) with < the relation induced by the usual order of natural numbers, thus elements in the set \( \{ \{ b_{R_t} \}, \{ b_{L_m} \} \} \) can be written as a vector

\[
\left[ h_{i_0j_1...j_m}^{i_1...i_k} \right] = (v_0, \ldots, v_n),
\]

where if \( b_{R_t} < b_{R_s} \) then \( v_u = b_{R_t} \) and \( v_{u+1} = b_{R_s} \) for \( 0 \leq u < |\{ b_{R_t} \}| - 1 \), and if \( b_{L_t} < b_{L_s} \) then \( v_r = b_{L_s} \) and \( v_{r+1} = b_{L_s} \) for \( |\{ b_{R_t} \}| \leq r < |\{ b_{R_t} \}| + |\{ b_{L_m} \}| \).

As an example, we have the following identities:
2.2 Sections in the Auslander-Reiten Quiver of Algebras of Dynkin Type

In this section we use quivers of type \( b, d \) and \( h \) in order to enumerate the number of sections in the Auslander-Reiten quiver of algebras of type \( A_n \) and \( D_n \). We present the same description for the case of \( E_6, E_7 \) and \( E_8 \).

2.2.1 Sections in the Auslander-Reiten Quivers of Algebras of Type \( A_n \)

Let \( \mathcal{A} = kA_n \) be a path algebra induced by an oriented Dynkin diagram of type \( A_n \) with \( k \) sinks and \( m \) sources, \( \Gamma(\text{mod } \mathcal{A}) \) be the corresponding Auslander-Reiten quiver and \( S^{(k_n)}_{i_1 \ldots i_k} \) the number of sections in \( \Gamma(\text{mod } \mathcal{A}) \) of these kind of algebras, where \( i_t \) represents the location of a sink for \( 1 \leq t \leq k \) and \( j_s \) represents the location of a source for \( 1 \leq s \leq m \), points \( i_t \) and \( j_s \) follow conditions defined in steps 1-3 of Algorithm 2.1. If we have an algebra \( \mathcal{B} \) with only one point then the number of sections in the Auslander-Reiten quiver will be denoted \( S^{(k_1)}_{i_1} \). For vertices in Dynkin diagrams of type \( A_n \), we assume the numbering described in Figure 1.2 (Section 1.1).

If \( \mathcal{A} \) is an algebra as described above then \( \Gamma(\text{mod } \mathcal{A}) \) is isomorphic to the quiver \( d_{j_0 j_1 \ldots j_m}^{i_0 i_1 \ldots i_k} \) obtained from \( d_{j_0 j_1 \ldots j_m}^{i_0 i_1 \ldots i_k} \) by orienting each edge \( ((x, y), (x', y')) \) as \( (x, y) \rightarrow (x', y') \). Such an isomorphism can be defined by associating to each \( \tau \)-orbit of a given vertex \( x_t \in \Gamma_0 \) points \( (x, y) \in d_{j_0 j_1 \ldots j_m}^{i_0 i_1 \ldots i_k} \) such that \( x + y = u + 1 - t \) for \( 1 \leq t \leq n \). If \( \mathcal{A} \) is an algebra of type \( A_n \) with sinks located at points \( \{i_1, \ldots, i_k\} \) and sources located at points \( \{j_1, \ldots, j_m\} \) and \( \mathcal{B} \) is an algebra of type \( A_n \) with sinks at \( \{j_1, \ldots, j_m\} \) and sources at \( \{i_1, \ldots, i_k\} \) then there exists an isomorphism:
Proof. Suppose that the algebra $\mathcal{B}$ satisfies hypothesis of the theorem then since $\Gamma \mod A$ is isomorphic to the quiver $\varphi : \frac{d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}}{\mathcal{B}} \to \frac{d^{j_0j_1\cdots j_{m-1}}_{i_1\cdots i_k}}{\mathcal{B}}$ with vertices $B_0$ can be written in the form $B_0 = B_1 \cup B_2$ where

$$B_1 = \bigcup_{r=1}^{k} \{ (x, y) \in \frac{d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}} {\mathcal{B}} \mid 0 \leq y \leq y_r - 1, x > x_r, x + y \leq u - 1 \},$$

$$B_2 = \bigcup_{s=1}^{m-1} \{ (x, y) \in \frac{d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}} {\mathcal{B}} \mid \underline{x} \leq y \leq 0, x \leq \underline{w}, x + y \geq -n + u \}.$$

If $|y_k - 1 - z_m| > 1$ ($i_k < n - 1$) then $i'_t = (x, y - 1) = j'_s = (x, y - 1)$ and $j'_m = j^*_m$ for $1 \leq t \leq k$ and $1 < s < m$; therefore $B_0 = d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}$ and $B = \frac{d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}} {\mathcal{B}}$.

If $|y_k - 1 - z_m| = 1$ ($i_k = n - 1$) it holds that $i'_t = (x, y - 1) = j'_s$ and $i'_k = j^*_m$ for $1 \leq t < k$ and $1 < s < m$, thus as before $B_0 = d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}$ and $B = \frac{d^{i_1\cdots i_k}_{j_0j_1\cdots j_{m-1}}} {\mathcal{B}}$.

If $C = \{ (x, y) \in \frac{d^{i_1\cdots i_k}_{j_0j_1\cdots j_m}} {\mathcal{B}} \mid x + y = u \}$ and it is defined the quiver $C \cup B$ with arrows induced by $d^{i_1\cdots i_k}_{j_0j_1\cdots j_m}$ then since $|P_{(c,d)}^{(x,y)}| = 2$ we have

$$|C \cup B| = \sum_{a,b,x,y} |P_{(a,b)}^{(x,y)}| = 2 \sum_{a,b,c,d} |P_{(a,b)}^{(c,d)}| = 2 \left[ B \right],$$
where \( a + b = u - n \) and \( c + d = u - 1 \).

Since \( \sum_{a,b,h,p} |P_{(a,b)}^{(h,p)}| \) contains at least one point of \((C \cup B)^c\) for \( a + b = u - n + 1 \), \( h + p = u \) and \((a, b), (h, p) \in a_{ij1}^{i_1} \ldots i_k \) then \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = 2 [B] + [a_{ij1}^{i_1} \ldots i_k] \). \( \square \)

**Remark 2.5.** Numbers \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} \) with \( i_k < n \) are shown in Table A.2 (see Appendix), where rows represents values \( j_m = n \) and columns are given by the identities:

\[
r = \begin{cases} 
\sum_{t=k-m+2}^{k} 2^{i_t-1} - \sum_{t=1}^{m-1} 2^{i_t-1}, & \text{if } i_k > 1, \\
0, & \text{if } i_k = 1.
\end{cases} \tag{2.22}
\]

For instance, if \( \mathbb{A} \) is an algebra of type \( \mathbb{A}_7 \) with sinks and sources at points of the sets \( \{1, 4, 7\} \) and \( \{3, 5\} \) respectively then it holds that \( S_{(\mathbb{A}_7)^{0147}} = S_{(\mathbb{A}_7)^{0147}}^\alpha = \sum_{t=0}^{3} \binom{n-2}{i} \) for \( n \geq 3 \), \( 1 \leq m < n \) with \( S_{(\mathbb{A}_7)^{0147}} = 2^{2n-1} \).

\[
S_{(\mathbb{A}_7)^{0147}} = S_{(\mathbb{A}_7)^{0147}}^\alpha = 2S_{(\mathbb{A}_7)^{0147}}^\alpha + S_{(\mathbb{A}_7)^{0147}}^\alpha = 2(64) + 17 = 145.
\]

We recall that sections in algebras of type \( \mathbb{A}_n \) with sinks and sources at sets \( \{1, 3\} \) and \( \{2, n\} \) respectively categorifies the sequence A176448 = \{5, 12, 26, 54, 110, \ldots \} in the OEIS A88.

The following corollaries dealing with oriented Dynkin diagram of type \( \mathbb{A}_n \) with only sink were reported by the author, Cañadas and Giraldo in \[32\]. Henceforth, we assume the notation \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} \) for \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} \).

**Corollary 2.1.** \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = 2 \left( S_{(\mathbb{A}_n-1)^{011 \ldots j_1 \ldots j_m}} \right) + \sum_{i=0}^{m-2} \binom{n-2}{i} \) for \( n \geq 3 \), \( 1 \leq m < n \) with \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = 2^{n-1} \).

**Corollary 2.2.**

\[
S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = S_{(\mathbb{A}_n-1)^{011 \ldots j_1 \ldots j_m}} + S_{(\mathbb{A}_n-1)^{011 \ldots j_1 \ldots j_m}} + S_{(\mathbb{A}_n-2)^{011 \ldots j_1 \ldots j_m}} \tag{2.23}
\]

for \( n \geq 3 \), \( 1 \leq m < n \).

**Proof.** We proceed by induction on \( n \) taking into account that \( S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = S_{(\mathbb{A}_n)^{011 \ldots j_1 \ldots j_m}} = 2^{n-1} \). If \( n = 3 \) and \( m = 2 \) we have that

\[
S_{(\mathbb{A}_3)^{011 \ldots j_1 \ldots j_m}} = 2(S_{(\mathbb{A}_2)^{011 \ldots j_1 \ldots j_m}} + 1) = S_{(\mathbb{A}_3)^{011 \ldots j_1 \ldots j_m}} + S_{(\mathbb{A}_2)^{011 \ldots j_1 \ldots j_m}} + S_{(\mathbb{A}_1)^{011 \ldots j_1 \ldots j_m}}.
\]

Suppose that the assertion is true for \( 3 \leq k \leq n \) and \( 2 \leq m \leq n - 1 \). Thus
\[ S(A_k)^m = 2(S(A_k)^m) + \sum_{i=0}^{m-2} \binom{k-1}{i} = 2(S(A_k)^m) + \sum_{i=0}^{m-2} \binom{k-2}{i} + \sum_{i=0}^{m-3} \binom{k-3}{i}. \]

Therefore

\[ S(A_{k+1})^m = 2(S(A_{k+1})^m - S(A_{k-1})^m + 2^{k-3} + \sum_{i=0}^{m-2} \binom{k-2}{i} + \sum_{i=0}^{m-3} \binom{k-3}{i}. \]

Identity \[2.23\] allows to give a partition-formula for numbers in the sequence A049611 or A084851 \[87\].

**Corollary 2.3.** \( c_n = \sum_{h=1}^{n} S(A_n)^h = 2^{n-4}(n^2 + 5n + 2), \ n \geq 4. \)

**Proof.** Rows in the next table show the number of sections in the Auslander-Reiten quiver of an algebra \( A \) associated to a Dynkin graph of type \( A_n \) with an unique sink allocated at the \( h \)th position (from the left to the right), \( 1 \leq h \leq n \) and \( 1 \leq n \leq 7. \)

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>2</td>
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<td>4</td>
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<td>8</td>
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<tr>
<td>16</td>
</tr>
<tr>
<td>32</td>
</tr>
<tr>
<td>64</td>
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For instance, according to the Corollary \[2.22\] we see that,

\begin{align*}
95 &= 16 + 32 + 47, \\
120 &= 16 + 47 + 57, \\
130 &= 57 + 57 + 16, \tag{2.24}
\end{align*}

and

\begin{align*}
688 &= 2(64) + 2(95) + 2(120) + 130 \\
&= 2(2^6) + 2(2^5) + 9(2^4) + 16(2^3) + 28(2^2) + 48(2^1) + 16(1). \tag{2.25}
\end{align*}

Actually, it is easy to see that in the case \( n = k \geq 4 \), it holds that

\[ \sum_{h=1}^{k} S(A_h)^h = 2(2^k) + 2^{k-3} + \sum_{j=1}^{k} (k+2) - (k-j)2^{k-j} - 2^{j-3} = 2(2^k) + 2^{k-3} + \sum_{i=6}^{n+2} i2^{k-3}. \]
Thus
\[ \sum_{h=1}^{k} S_{(h)} h^2 = \left(\frac{(k+2)(k+3)}{2} - 14\right) 2^{k-3} + 2^k + 2^{k-1} = \left(\frac{(k+2)(k+3)}{2} - 14\right) 2^{k-3} + 3(2^{k-1}) = 2^{k-4}[k^2 + 5k - 22 + 24] = 2^{k-4}[k^2 + 5k + 2]. \] Since \( k \geq 4 \) is arbitrary we are done. \( \square \)

**Corollary 2.4.** \( S_{(h)} h^2 = 3(2^{n-2}) - 1. \)

**Proof.** \( S_{(h)} h^2 = 5 = 2 + 2 + 1 \) and \( S_{(h)} h^2 = S_{(h)} h_2 + (2^2) + (2^1) = 5 + (2^2) + (2^1) = (2^2) + (2^0) + (2^2) + (2^1) = (2^3) + (2^1) + (2^0). \) Thus for any \( k \geq 3 \) it holds that

\[ S_{(h)} h^2 = S_{(h-1)} h^2 + (2^{k-2}). \] (2.26)

Therefore
\[ S_{(h)} h^2 = 2(2^{k-2}) + 2(2^{k-4}) + 2(2^{k-5}) + \cdots + 2(2^2) + 3(2^1) = (2^{k-1}) + \sum_{j=0}^{k-3} 2^j \] (2.27)
\[ = 2^{k-1} + 2^{k-2} - 1 = 3(2^{k-2}) - 1. \] \( \square \)

Corollaries 2.2 and 2.4 allow us to establish the following result.

**Corollary 2.5.** \( S_{(h)} h^2 = (h+1)2^{k-2} - \sum_{j=1}^{\left\lfloor \frac{h}{2} \right\rfloor} j(k+1). \)

**Proof.** (Induction on \( h \)) Firstly, we recall the following identities:
\[ h \sum_{j=h+1}^{k-1} 2^{j-2} = h(2^{k-2}) - h2^{h-2}, \]
\[ \sum_{j=h-2}^{k-3} 2^j = (2^{k-2}) - 2^{h-2}, \] (2.28)
\[ \sum_{j=h+1}^{k-1} \binom{j+1}{i} = \binom{k+1}{i+1} - \binom{h+1}{i+1}. \]

Corollaries 2.2 and 2.4 induce the following identity where \( S_{(h)} h^2 = 2^h: \)
\[ S_{(h)} h^2 = \sum_{j=h+1}^{k-1} S_{(h)} h^2 + \sum_{j=h-2}^{k-3} 2^j + S_{(h+1)} h^2. \] (2.29)

Now if we assume that the theorem is true for any fixed \( k, k \geq 1 \) and \( 1 \leq s \leq h \) then the theorem holds for \( s = h+1 \) if identities (2.28) are applied to the summands in (2.29). \( \square \)

**Remark 2.6.** We note that
1. For \( n \geq 2 \), the sequence \( a_n = S_{(h)} h^2 \) appears in the OEIS as A083329. \( \square \)
2. For \( n \geq 3 \), the sequence \( b_n = S_{(\mathcal{A}_n)3} \) appears in the OEIS as A000295 [86].

2.2.2 Sections in the Auslander-Reiten Quiver of Algebras of Type \( \mathbb{D}_n \)

Let \( \mathcal{A} \) be an algebra with underlying diagram of type \( \mathbb{D}_n \), with \( k \) sinks and \( m \) sources. We assume the numbering for Dynkin diagrams of type \( \mathbb{D}_n \) described in Figure 1.2 (section 1.1).

Let \( \mathcal{B} \) be an algebra of type \( \mathbb{D}_n (n \geq 4) \) whose sinks and sources are located at points of the sets \( \{j_1, \ldots, j_m\} \) and \( \{i_1, \ldots, i_k\} \) respectively. Then, if there exists an irreducible morphism in \( \Gamma(\text{mod } \mathcal{A}) \) of the form \( \tau_a^{-s} \rightarrow \tau_b^{-r} \) then there exists a morphism in \( \Gamma(\text{mod } B) \) of the form \( \tau_b^{-s} \rightarrow \tau_a^{-r} \) for some \( s, r \in \mathbb{Z} \). \( \mathcal{B} \) denotes the conjugate quiver of \( \mathcal{A} \) and the following identity has place:

\[
S_{(\mathcal{B}_n)\{i_1, \ldots, i_k\}}^{\{j_1, \ldots, j_m\}} = S_{(\mathcal{B}_n)\{j_1, \ldots, j_m\}}^{\{i_1, \ldots, i_k\}},
\]

when \( \tau_a^{-s} \rightarrow \tau_b^{-r} \) with \( r \leq k \) and \( w \leq n \) is a subquiver of \( \Gamma(\text{mod } \mathcal{A}) \), where each \( \tau^{-}\)-orbit of a point \( x_t \in \Gamma_0 \) has associated points \( (x, y) \in h_{i_1, \ldots, i_r}^{j_1, \ldots, j_w} \) with \( x + y = n - 1 - t \) for \( 1 \leq t \leq n - 2 \). According to these arguments it suffices to consider the subquiver \( \mathcal{A}'_{n-2} \) with vertices \( 1 \ldots n-2 \) a sink at the vertex \( n-2 \). Thus, we can enumerate sections in \( \Gamma(\mathcal{A}) \) via the following three cases described in Theorem 2.5.

**Theorem 2.5.** Let \( \mathcal{A} \) be an algebra of type \( \mathbb{D}_n \) with sink and sources located at the sets \( \{i_1, \ldots, i_k\} \) and \( \{j_1, \ldots, j_m\} \). If the subquiver \( \mathcal{A}'_{n-2} \) has a sink at the vertex \( n-2 \) then:

\[
S_{(\mathcal{D}_n)\{i_1, \ldots, i_k\}}^{\{j_1, \ldots, j_m\}} = \begin{cases} 
4 \sum_{s=0}^{n-3} v_s + v_{n-2}, & \text{if } n, n-1 \text{ are sources}, \\
4 \sum_{s=1}^{n-2} v_s + 2(v_{n-2} + v_0), & \text{if } n \text{ is a source and } n-1 \text{ is a sink}, (viceversa),
4 \sum_{s=1}^{n-2} v_s + v_0, & \text{if } n, n-1 \text{ are sinks},
\end{cases}
\]

where \( v_s \in h_{i_1, \ldots, i_r}^{j_1, \ldots, j_w} \) with \( r \leq k \) and \( w \leq n \) (see Theorem 2.3 and Remark 2.4).

**Proof.** If the algebra \( \mathcal{A} \) satisfies the hypothesis and the subquiver \( \mathcal{A}'_{n-2} \) has a sink at the vertex \( n-2 \). Then, we can take a subquiver \( \mathcal{B} \) of \( \Gamma(\text{mod } \mathcal{A}) \) which is isomorphic to \( h_{i_1, \ldots, i_r}^{j_1, \ldots, j_w} \) with \( r \leq k \) and \( w \leq m \). Let us observe that each orbit \( \tau_{n-2}^{-s} \in \Gamma(\text{mod } \mathcal{A}) \) has associated point \( (-n-2+s, n-2-s) \in h_{i_1, \ldots, i_r}^{j_1, \ldots, j_w} \) with \( 0 \leq s \leq n-2 \), thus \( \sum_{s=0}^{n-2} \left| P_{(-n-2+s, n-2-s)}^{(a,b)} \right| = v_{n-2-s} \) for \( a + b = n - 3 \). Now, we have three cases:

- If \( n \) and \( n-1 \) are sources for each \( \tau_{n-2}^{-s} \) with \( 1 \leq s \leq n-2 \) then there exist the irreducible morphisms \( \tau_{n-1}^{-(s-1)} \rightarrow \tau_{n-2}^{-s}, \tau_{n}^{-(s-1)} \rightarrow \tau_{n-2}^{-s}, \tau_{n-2}^{-s} \rightarrow \tau_{n-1}^{-s} \), and
For example, let $Q_1$ be a quiver of type $\mathbb{D}_n$ whose sinks and sources are located at points of the sets $\{1, 4, 6, 7\}$ and $\{3, 5\}$ respectively, since the vertex 5 is not a sink, we take the conjugate quiver of $Q_1$, Formula A.1 and Table A.3 (see Appendix) establishes that $r = 4$ and

$$S_{(\mathbb{D}_n)}^{0,1, \ldots i_m} = 4 \sum_{s=1}^{n-3} v_s + v_{n-2}. \quad \Box$$

In the same way, the author, Cañadas and Giraldo showed the next recurrence formula for algebras of Dynkin type $\mathbb{D}_n$ and $\mathbb{D}_n$ with just one sink (by notation $S_{(\mathbb{D}_n)}^{m} = S_{(\mathbb{D}_n)}^{0,1, \ldots i_m}$) \textbf{32}.

**Corollary 2.6.** $S_{(\mathbb{D}_n)}^{m} = S_{(\mathbb{D}_n-1)}^{m-1} + S_{(\mathbb{D}_n-1)}^{m} + 3 \left(2^{n-3}\right)$ for $n \geq 5$, $1 \leq m < n-2$ with $S_{(\mathbb{D}_n)}^{1} = 2^{n-3} (2n - 1)$ and $S_{(\mathbb{D}_n)}^{n-2} = 2^{n-2} (n+1) - 3.$
Corollary 2.6 allows to build the following triangular table where the rows give the number of sections in the Auslander-Reiten quiver of an algebra $\mathcal{A}$ associated to a Dynkin graph $D_n$ with a unique sink allocated at the $h$th position, $1 \leq h \leq n - 2$:

\[
\begin{array}{cccccc}
14 & 17 & & & & \\
36 & 43 & 45 & & & \\
88 & 103 & 112 & 109 & & \\
208 & 239 & 263 & 269 & 253 & \\
480 & 543 & 598 & 628 & 618 & 573 \\
1088 & 1215 & 133 & 1418 & 1438 & 1383 & 1277 \\
\end{array}
\]

Remark 2.7. Sequence $c_n = S_{(D_n)}$ for $n \geq 4$ appears in the OEIS as A052951.

2.2.3 Sections in the Auslander-Reiten Quiver of Algebras of Type $E_6$, $E_7$ and $E_8$

In order to give the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type $E_6, E_7$ and $E_8$. Let $\mathcal{A}$ be a path algebra induced by an oriented Dynkin diagram of type $E_l$ ($l = 6, 7, 8$) with $k$ sinks and $m$ sources. Let $\Gamma(\text{mod } \mathcal{A})$ be the Auslander-Reiten quiver of $\mathcal{A}$ and $S_{(E_l)}^{i_0i_1...i_k}$ is the number of section in $\Gamma(\text{mod } \mathcal{A})$. We assume the numbering described in Figure 1.2 (Section 1.1).

Let $\mathcal{A}_{i-1}'$ be the subgraph of the vertices $\{1, \ldots, l-1\}$ of $E_l$ and suppose that $i_k < i - 1$, we define the vectors $S_{E_{l}^{\uparrow}} = (v_0, \ldots, v_{2^{l-3}-1})$, and $S_{E_{l}^{\downarrow}} = (w_0, \ldots, w_{2^{l-3}-1})$ in the same way:

\[
\begin{align*}
S_{E_6^{\uparrow}} &= (124, 134, 136, 132, 146, 150, 146, 134), \\
S_{E_6^{\downarrow}} &= (124, 139, 146, 147, 136, 145, 146, 139), \\
S_{E_7^{\uparrow}} &= (408, 430, 436, 434, 460, 472, 468, 450, 454, 470, 466, 456, 478, 476, 438), \\
S_{E_7^{\downarrow}} &= (412, 443, 458, 465, 448, 468, 472, 462, 452, 473, 478, 477, 452, 461, 458, 439), \\
S_{E_8^{\uparrow}} &= (1520, 1566, 1580, 1584, 1632, 1662, 1660, 1636, 1628, 1664, 1668, 1650, 1692, 1698, 1680, 1632, 1614, 1654, 1662, 1650, 1698, 1712, 1698, 1694, 1676, 1698, 1692, 1656, 1690, 1678, 1650, 1590), \\
S_{E_8^{\downarrow}} &= (1532, 1595, 1626, 1647, 1620, 1663, 1674, 1663, 1624, 1676, 1696, 1694, 1652, 1673, 1670, 1637, 1616, 1673, 1698, 1703, 1664, 1694, 1696, 1748, 1653, 1693, 1702, 1681, 1632, 1637, 1626, 1583).
\end{align*}
\]
therefore

\[
S_{(E_l)_{j_1 \cdots j_m}} = \begin{cases} 
  v_r, & \text{if } l \text{ is sink,} \\
  w_r, & \text{if } l \text{ is source,}
\end{cases} \tag{2.31}
\]

with \(r\) given by formulas \[2.22\]

For example, if we take an algebra \(A = kQ\) associated to Figure 2.7, then
\[
S_{(E_6)_{135}} = S_{(E_6)_{246}} = v_5 = 150.
\]

\[\text{Figure 2.7. Quiver of type } E_6.\]
In this chapter, we introduce Dyck paths categories as a combinatorial model of the category of representations of Dynkin quivers of type $A_n$. These categories help us to find an alternative formula of cluster variables of cluster algebras based on Dyck paths. In Section 3.1, we define Dyck paths categories and some of its main categorical properties are given in Section 3.2. In section 3.3, relationships between objects of the categories of Dyck paths, perfect matchings, and cluster algebras are given.

3.1 Dyck Paths Category

In this section, we introduce the category of Dyck paths of length $2n$.

3.1.1 Elementary Shifts

Let $\mathcal{D}_{2n}$ be the set of all Dyck paths of length $2n$ and let $UWD = Uw_1 \ldots w_{n-1}D$ be a Dyck path in $\mathcal{D}_{2n}$ with $w_i \in A = \{UD, DU, UU, DD\}$ for $1 \leq i \leq n-1$.

The support of $UWD$ (denoted by Supp $UWD \subseteq \{1, 2, \ldots, n-1\} = \text{n-1}$) is a set of indices such that

$$\text{Supp } UWD = \{q \in \text{n-1} \mid w_q = UD \text{ or } w_q = UU, 1 \leq q \leq n-1\}.$$ 

A map $f : A \rightarrow A$ such that for any $w \in A$, it holds that $f(w) = f(ab) = w^{-1} = ba$, $a, b \in \{U, D\}$ is said to be a shift.

For $i$ fixed, $1 \leq i \leq n-1$, a unitary shift is a map $f_i : \mathcal{D}_{2n} \rightarrow \mathcal{D}_{2n}$ such that

$$f_i(Uw_1 \ldots w_{i-1}w_iw_{i+1} \ldots w_{n-1}D) = Uw_1 \ldots w_{i-1}f(w_i)w_{i+1} \ldots w_{n-1}D.$$
We will denote a unitary shift by a vector of maps from $\mathbb{D}_{2n}$ to itself of the form $(1, \ldots, 1_{i-1}, f_i, 1_{i+1}, \ldots, 1_{n-1})$, where $1_k$ is the identity map associated to the $i$-th coordinate.

An elementary shift is a unitary shift or composition of different unitary shifts. A shift path of length $m$ $UWD \rightarrow UW_1D \rightarrow \cdots \rightarrow UW_mD \rightarrow UVD$ from $UWD$ to $UVD$ is a composition of elementary shifts. The set of all Dyck paths in a shift path between $UWD$ and $UVD$ will be denoted by $J$. For notation, we introduce the identity shift as the elementary shift $(1, \ldots, 1_{n-1})$.

**Irreversibility condition.** Consider a relation $R \subset \mathbb{D}_{2n} \times \mathbb{D}_{2n}$ consisting of a set of pairs of Dyck paths of the form $(UWD, UVD)$, where $UVD$ is obtained from $UWD$ by applying an elementary shift. Then, $R$ is said to be an irreversible relation, if for any $(UWD, UVD) \in R$, it holds that $(UVD, UWD) \notin R$.

Henceforth, if $(UWD, UVD) \in R$ then we will write $UVD = R(UWD)$.

**Shift Relation.** Suppose that $UWD, UW'D, UW''D, UVD \in \mathbb{D}_{2n}$. And that there are elementary shifts $F : UWD \rightarrow UW'D$, $F' : UWD \rightarrow UW''D$, $G : UW'D \rightarrow UVD$, $G' : UW''D \rightarrow UVD$ in an irreversible relation $R$. Then if the compositions $G \circ F$ and $G' \circ F'$ are shift paths (of length 2) transforming the Dyck path $UWD$ into the Dyck path $UVD$ (see the diagram below),

with $W' \neq W''$. Then $G \circ F$ is said to be related with $G' \circ F'$ (denoted $G \circ F \sim_R G' \circ F'$) whenever $G' = F$ and $G = F'$.

**Category of Dyck paths of length $2n$.** As for the case of diagonals [17], we can also define a $k$-linear additive category $(\mathbb{D}_{2n}, R)$ based on Dyck paths, in this case, objects are $k$-linear combinations of Dyck paths in $\mathbb{D}_{2n}$ with space of morphisms from a Dyck path $UWD$ to a Dyck path $UVD$ over an irreversible relation $R$ being the set

$$\text{Hom}_{(\mathbb{D}_{2n}, R)}(UWD, UVD) = \{g \mid g \text{ is a shift path over } R\})/\langle \sim_R \rangle.$$

The vector space $\text{Hom}_{(\mathbb{D}_{2n}, R)}(UWD, UVD) \neq 0$ if and only if there are shift paths from $UWD$ to $UVD$ and

$$\bigcap_{i \in J} \text{Supp } UW^iD \neq \emptyset,$$

for each shift path, with $UWD$ and $UVD$ in $\mathbb{D}_{2n}$. 
Figure 3.1 shows the elementary shifts over \((D_6, R)\) associated to an irreversible relation \(R\) defined over the set of all Dyck paths of length 6. And such that,

\[
R(UWD) = \begin{cases} 
  f_1(UWD), & \text{if } w_1 = UD, \\
  f_2(UWD), & \text{if } w_2 = DU.
\end{cases}
\] (3.2)

3.1.2 Relations of Type \(R_{i_1...i_k}^{j_1...j_m}\)

Fix an admissible subchain \(C = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq n-1\) (see algorithm 2.1 item (3)), and a Dyck path \(UWD \in D_{2n}\).

Let \(\sigma : \{i_1, j_1\} \to \{0, 1\}\) be a map such that \(\sigma(i_1) = 1\) and \(\sigma(j_1) = 0\). For \(a \in \{i_1, j_1\}\), we assume \(i_r, i_r+\sigma(a) \in \{i_1, \ldots, i_k\}\) and \(j_{r+1-\sigma(a)}, j_r \in \{j_1, \ldots, j_m\}\). The following indices are defined by using intervals \([i_r, j_{r+1-\sigma(a)}]\) (resp. \([j_r, i_{r+\sigma(a)}]\)), where for a fixed admissible chain \(C\), an interval \(I = [x, y]\) is a subset of \(n-1\), for which \(\min I = x \in C\) and \(\max I = y \in C\).

- \(t = \min\{ s \mid i_r \leq s \leq j_{r+1-\sigma(a)}, w_s = UD \} \) \( (t = \max\{ s \mid j_r \leq s \leq i_{r+\sigma(a)}, w_s = UD \} )\),
- \(p = \min\{ s \mid t < s \leq j_{r+1-\sigma(a)}, w_s = DU \} \) \( (p = \max\{ s \mid j_s \leq s < t, w_s = DU \} )\).

We introduce the following elementary shifts:

ES1. If \(w_s = UD\) for all \(s \in [i_r, j_{r+1-\sigma(a)}]\) \( (s \in [j_r, i_{r+\sigma(a)}])\),

\[
[i_{r-\sigma(a)}, i_r][i_r, j_{r+1-\sigma(a)}][j_{r+1-\sigma(a)}, i_{r+1}],
\]

(resp. \([i_{r+\sigma(a)-1}, i_r][j_r, i_{r+\sigma(a)}][i_{r+\sigma(a)}, j_{r+1}]\)),

then

\[
g(UWD) = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UWD),
\]

if there exists \(s \in \mathbb{Z}^+\) such that \(j_{r-\sigma(a)} \leq s \leq i_r, |s - j_r| \geq 1, w_x = UD\) if \(s \leq x \leq i_r\) over \([j_{r-\sigma(a)}, i_r]\) and
Proof. Suppose that there is an elementary shift $f_{r_1} \circ \cdots \circ f_{r_t}$ from a Dyck path $UWD$ to a Dyck path $UD$ and that there is an elementary shift $f_{r_1} \circ \cdots \circ f_{r_t}$ from $UD$ to a $UWD$, then we have five cases:

\begin{align*}
    w_y &= \begin{cases} 
        UD, & \text{if } y = j_{r+1-\sigma(a)}; \\
        DU, & \text{otherwise}, 
    \end{cases} \\
    \text{over } [j_{r+1-\sigma(a)}, i_{r+1}] & \text{ for } j_{r+1-\sigma(a)} \neq n - 1 \text{ or the first condition over } [j_{r-\sigma(a)}, i_r] \text{ for } j_{r+1-\sigma(a)} = n - 1. 
\end{align*}

\begin{align*}
    g(UWD) &= f_{i_{r+\sigma(a)}}(UWD), \\
    \text{if there exists } s \in \mathbb{Z}^+ \text{ such that } i_{r+\sigma(a)} \leq s \leq j_{r+1}, |s - i_{r+\sigma(a)}| \geq 1, \ w_x = UD \text{ if } i_{r+\sigma(a)} \leq x \leq s \text{ over } [i_{r+\sigma(a)}, j_{r+1}] \text{ and } \\
    w_y &= \begin{cases} 
        UD, & \text{if } y = j_r; \\
        DU, & \text{otherwise}, 
    \end{cases} \text{ over } [i_{r+\sigma(a)-1}, j_r] \text{ for } j_r \neq 1 \text{ or the first condition over } [i_{r+\sigma(a)}, j_r+1] \text{ for } j_r = 1, \\
    \text{with } i_r \neq 1 (i_{r+\sigma(a)} \neq n - 1).
\end{align*}

ES2. If $t = 1 \text{ or } n - 1$ then $g(UWD) = f_t(UWD)$.

ES3. If $i_r < t < j_{r+1-\sigma(a)} (j_r < t < i_{r+\sigma(a)})$ then $g(UWD) = f_t(UWD)$.

ES4. If $p = j_{r+1-\sigma(a)}(j_r)$ then

\begin{align*}
    g(UWD) &= \begin{cases} 
        f_{i_{r+1}} \circ \cdots \circ f_{j_{r+1-\sigma(a)}}(UWD) & \text{if } j_{r+1-\sigma(a)} \neq n - 1, \\
        f_{j_{r+1-\sigma(a)}}(UWD) & \text{if } j_{r+1-\sigma(a)} = n - 1. 
    \end{cases} \\
    \text{if } j_{r+1-\sigma(a)} \neq n - 1, \\
    \text{or } j_{r+1-\sigma(a)} = n - 1.
\end{align*}

ES5. If $t < p < j_{r+1-\sigma(a)} (j_r < p < t)$ then $g(UWD) = f_p(UWD)$.

For a given subchain $C = \{j_1, \ldots, j_m, i_1, \ldots, i_k\} \subseteq n-1$, two Dyck paths $D$ and $D'$ of length $2n$ are said to be related by a relation of type $R_{j_1 \ldots j_m}^{i_1 \ldots i_k}$ if there is an elementary shift $E_{i_1}, 1 \leq i \leq 5$ which transforms either $D$ into $D'$ or $D'$ into $D$.

Henceforth, the notation $w_r \ldots w_s$ means that all the elements $w_i$ covered by the brace have the same steps $XY$.

**Proposition 3.1.** *The relation $R_{j_1 \ldots j_m}^{i_1 \ldots i_k}$ is irreversible.*

**Proof.** Suppose that there is an elementary shift $f_{r_1} \circ \cdots \circ f_{r_t}$ from a Dyck path $UWD$ to a Dyck path $UD$ and that there is an elementary shift $f_{r_1} \circ \cdots \circ f_{r_t}$ from $UD$ to a $UWD$, then we have five cases:
(i) If \( f_{r_1} \circ \cdots \circ f_{r_t} \) arises from ES1 over \([i_r, j_{r+1-\sigma(a)}]\). Elementary shifts ES2, ES3 and ES5 allow to conclude that from \(UVD\) to a \(UWD\), \(f_t = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}\) or \(f_p = f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}\) and this is a contradiction. If \(ES1\) is an elementary shift from \(UVD\) to \(UWD\), then two cases arise: If \(j_{r+1-\sigma(a)} \neq n - 1\), thus \(UVD\) equals

\[
Uv_1 \cdots v_{i_r} \cdots v_{i_r-1} v_{i_r} \cdots v_{j_{r+1-\sigma(a)}} (v_{j_{r+1-\sigma(a)}+1}) \cdots v_{i_{r+1}} v_{i_{r+1}+1} \cdots v_{n-1}D,
\]

it turns out that \(f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UVD)\) has the form

\[
Uw_1 \cdots w_{i_r} \cdots w_{i_r-1} w_{i_r} \cdots w_{j_{r+1-\sigma(a)}} (w_{j_{r+1-\sigma(a)}+1}) \cdots w_{i_{r+1}} w_{i_{r+1}+1} \cdots w_{n-1}D,
\]

which is a contradiction. If \(j_{r+1-\sigma(a)} = n - 1\), \(UVD\) is equal to

\[
Uv_1 \cdots v_{i_r} \cdots v_{i_r-1} v_{i_r} \cdots v_{j_{r+1-\sigma(a)}} D,
\]

and \(f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UVD)\) has the shape

\[
Uw_1 \cdots w_{i_r} \cdots w_{i_r-1} w_{i_r} \cdots w_{j_{r+1-\sigma(a)}} D,
\]

again a contradiction. We also get a contradiction if an elementary shift is done by using ES4 from \(UVD\) to \(UWD\), indeed, in these cases it holds that, if \(j_{r+1-\sigma(a)} = n - 1\), there are \(t\) and \(p\) such that \(p = j_{r+1-\sigma(a)} < t \leq i_{r+1}\) and \(UVD\) is equal to

\[
Uv_1 \cdots v_{i_{r-1}} v_{i_r} \cdots v_{j_{r+1-\sigma(a)}} v_{j_{r+1-\sigma(a)}+1} \cdots v_{i_{r+1}} v_{i_{r+1}+1} \cdots v_{n-1}D,
\]

and \(f_{j_{r+1-\sigma(a)}} \circ \cdots \circ f_{i_r}(UVD)\) is

\[
Uw_1 \cdots w_{i_{r-1}} w_{i_r} \cdots w_{j_{r+1-\sigma(a)}} w_{j_{r+1-\sigma(a)}+1} \cdots w_{i_{r+1}} w_{i_{r+1}+1} \cdots w_{n-1}D.
\]

If \(v_{j_{r+1-\sigma(a)}} = n - 1\) \(f_{r+1-\sigma(a)} = f_{r+1-\sigma(a)} \circ \cdots \circ f_{i_r}\), but this is a contradiction.

(ii) If \(f_{r_1} \circ \cdots \circ f_{r_t} \) arises from ES2 over \([i_1, j_1]\) then we cannot use elementary shifts defined in cases ES1, ES4, ES5 or ES3, provided that, \(i_1 \neq 1\), \(t \neq p\) or \(1 < t < j_1\). Therefore, ES2 guarantees the existence of a walk from \(UVD\) to \(UWD\) such that;

\[
U v_{j_1} \cdots v_{n-1}D,
\]

and \(f_1(UWD)\) has the form

\[
U w_{j_1} \cdots w_{n-1}D,
\]

which is a contradiction (if \(t = n - 1\), the proof is dual).

(iii) If \(f_{r_1} \circ \cdots \circ f_{r_t} \) arises from ES3 over \([i_r, j_{r+1-\sigma(a)}]\), provided that, \(i_r < t < p < j_{r+1-\sigma(a)}\), we conclude that it is not possible to use ES1, ES2, ES4 nor ES5. In the
case of ES3 from $UVD$ to a $UWD$, $UVD$ equals

$$Uv_1 \ldots v_{i_r} \ldots v_{t-1} v_t \ldots v_{j_{r+1} - \sigma(a)} \ldots v_{n-1} D,$$

and $f_t(UVD)$ has the shape

$$Uw_1 \ldots w_{i_r} \ldots w_{t} w_{t+1} \ldots w_{j_{r+1} - \sigma(a)} \ldots w_{n-1} D,$$

but this is a contradiction.

(iv) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES4 over $[i_r, j_{r+1} - \sigma(a)]$, provided that $t < p$, we do not use ES2, ES3 nor ES5. If $j + 1 - \sigma(a) = n - 1$, we cannot use ES1. If $j + 1 - \sigma(a) \neq n - 1$ we can use ES1 from $UVD$ to a $UWD$. (Note that, it is not necessary with $v_m = UD$ for all $s \in [j_{r+1} - \sigma(a) + 1, r_{t+1}]$) $UVD$ is equal to

$$Uv_1 \ldots v_{i_r} \ldots v_{t} v_{p} v_{j_{r+1} - \sigma(a)} \ldots v_{i_{r+1}} v_{i_{r+1} + 1} \ldots v_{n-1} D,$$

it turns out that $g(UVD)$ has the form

$$Uw_1 \ldots w_{i_r} \ldots w_{t} w_{p} w_{j_{r+1} - \sigma(a)} \ldots w_{i_{r+1}} w_{i_{r+1} + 1} \ldots w_{n-1} D,$$

which is a contradiction. Using ES5 from $UVD$ to $UWD$, if $j_{r+1} - \sigma(a) \neq n - 1$, $UVD$ is equal to

$$Uv_1 \ldots v_{i_r} \ldots v_{t} \ldots v_{p-1} v_p v_{j_{r+1} - \sigma(a)} \ldots v_{i_{r+1}} v_{i_{r+1} + 1} \ldots v_{n-1} D,$$

and $UWD$ has the shape

$$Uw_1 \ldots w_{i_r} \ldots w_{t} \ldots w_{p} w_{j_{r+1} - \sigma(a)} \ldots w_{i_{r+1}} w_{i_{r+1} + 1} \ldots w_{n-1} D,$$

again a contradiction. If $j_{r+1} - \sigma(a) = n - 1$, $UVD$ is equal to

$$Uv_1 \ldots v_{i_r} \ldots v_{t-1} v_t \ldots v_{p-1} v_p D,$$

it turns out that $UWD$ has the shape

$$Uw_1 \ldots w_{i_r} \ldots w_{t-1} w_t \ldots v_p D,$$

this is a contradiction.

(v) If $f_{r_1} \circ \cdots \circ f_{r_t}$ arises from ES5 over $[i_r, j_{r+1} - \sigma(a)]$. Then we cannot use ES1, ES2, ES3 nor ES4, because $f_p \neq f_{j_{r+1} - \sigma(a)} \circ \cdots \circ f_{i_r}$ and $t < p$. Using ES5 from $UVD$ to a $UWD$, we observe that $UVD$ is equal to

$$Uv_1 \ldots v_{i_r} \ldots v_{t-1} v_t \ldots v_{p-1} v_p \ldots v_{j_{r+1} - \sigma(a)} \ldots v_{n-1} D,$$
and \( f_p(UWD) \) has the form

\[
U w_1 \ldots w_{i_r} \ldots w_{t-1} w_t \ldots w_p v_{p+1} \ldots v_{j_{r+1-\sigma(a)}} \ldots v_{n-1} D,
\]

again this is a contradiction.

Taking into account that if \( f_{r_1} \circ \cdots \circ f_{r_t} \) arises from \( ES1, ES2, ES3, ES4 \) and \( ES5 \) over \([i_r,j_{r+\sigma(a)}]\) then same arguments as described above applied dually allow to conclude the proposition. We are done. \( \Box \)

### 3.1.3 \( A_{n-1} \)-Dyck Paths Categories

For \( n \geq 2 \) fixed, an \( A_{n-1} \)-Dyck paths category is a category of Dyck paths \((D_{2n}, R)\) where \( R \) is a relation of type \( R_{i_1 \ldots i_k}^{j_1 \ldots j_m} \) as described before. As an example we let \((D_8, R_3^1)\) denote the \( A_3 \)-Dyck paths category with the admissible subchain \( 1 < 3 \). Figure 3.2 shows all the elementary shifts of \((D_8, R_3^1)\).

![Figure 3.2. Elementary shifts in an \( A_3 \)-Dyck paths category.](image)

We let \( S \) denote the set of all Dyck paths with exactly \( n-1 \) peaks. The following propositions and lemmas describe some properties of the set \( S \) in the category \((D_{2n}, R_{j_1 \ldots j_m}^{i_1 \ldots i_k})\).

**Proposition 3.2.** Let \( UWD \) be a Dyck path of length \( 2n \), then \( UWD \in S \) if and only if there is a unique sequence \( w_l w_{l+1} \ldots w_{r'}-1 w_{r'} \) such that

\[
  w_i = \begin{cases} 
  UD, & \text{if } l \leq i \leq r', \\
  DU, & \text{otherwise.}
  \end{cases}
\]  

(3.5)
Proof. Firstly, let $\delta$ be a map $\delta : \{\},\{\} \rightarrow \{U, D\}$ where left bracket is associated to the letter $U$ and right bracket is associated to the letter $D$, suppose $UWD \in S$, then there exist bracket-subchains such that $UWD$ can be written in the following form

\[
\begin{pmatrix}
\ldots & (\ )_{1} & (\ )_{2} & \ldots & (\ )_{l-2} & (\ )_{l-1} & (\ )_{l} & (\ )_{r'} & (\ )_{r'+1} & \ldots & (\ )_{n-2} & (\ )_{n-1}
\end{pmatrix},
\]

therefore $w_i = UD$ if $l \leq i \leq r'$ and $w_i = DU$. On the other hand, suppose $UWD$ has a unique subsequence $w_l w_{l+1} \ldots w_{r'} w_{r'}$ that satisfies (3.5), then if we apply $\delta^{-1}$ to $UWD$, the sequence

\[
\begin{pmatrix}
\ldots & (\ )_{1} & (\ )_{2} & \ldots & (\ )_{l-2} & (\ )_{l-1} & (\ )_{l} & (\ )_{r'} & (\ )_{r'+1} & \ldots & (\ )_{n-2} & (\ )_{n-1}
\end{pmatrix},
\]

is obtained, therefore $UWD \in S$. We are done. \hfill \Box

**Lemma 3.1.** Let $UWD$ be a Dyck path in $S$, and integers $r', l$ defined as in Proposition 3.2 with $|r' - l| > 0$, then there exists an elementary shift from $UWD$ to another Dyck path with exactly $n - 1$ peaks.

**Proof** Let $UWD$ be a Dyck path in $S$, let $l$ and $r'$ be positive integers such that $w_m = UD$ for $l \leq m \leq r'$. Let $l \in [i_r, j_{r+1-\sigma(a)}]$, we have the following cases:

1. If $l = i_r = 1$, then
   \[
g(UWD) = U f(w_1) w_2 \ldots w_{r'} w_{r'+1} \ldots w_{n-1} D \in S.
   \]

2. If $l = i_r \neq 1$, then there is a $p = l - 1$ over $[j_{r-\sigma(a)}, i_r]$ such that
   \[
g(UWD) = U w_1 \ldots f(w_p) w_1 \ldots w_m \ldots w_{n-1} D \in S.
   \]

3. If $i_r < l < j_{r+1-\sigma(a)}$, then
   \[
g(UWD) = U w_1 \ldots f(w_l) w_{l+1} \ldots w_{r'} w_{r'+1} \ldots w_{n-1} D \in S.
   \]

4. If $l = j_{r+1-\sigma(a)}$ and $|l - r'| > 0$, then $r' \in [i_{r_1}, j_{r_1+1-\sigma(a)}]$ with $|r_1 - r| > 0$ and the following cases hold:

   4.1. If $i_{r_1} \leq r' < j_{r_1+1-\sigma(a)}$, there is a $p = r' + 1$ such that, if $p \neq j_{r_1+1-\sigma(a)}$ then
   \[
g(UWD) = u w_1 \ldots w_{r'} f(w_p) \ldots w_{n-1} D \in S,
   \]
   if $p = j_{r_1+1-\sigma(a)} = n - 1$, then
   \[
g(UWD) = U w_1 \ldots w_1 \ldots w_{r'} f(w_p) D \in S,
   \]

   4.2. If $i_{r_1} > r'$, there is a $p = n - 1$ such that, if $p \neq j_{r_1+1-\sigma(a)}$ then
   \[
g(UWD) = u w_1 \ldots w_{r'} f(w_p) w_{n-1} D \in S,
   \]
   if $p = j_{r_1+1-\sigma(a)} = n - 1$, then
   \[
g(UWD) = U w_1 \ldots w_1 \ldots w_{r'} f(w_p) D \in S,
   \]
or if \( p = j_{r_1+1-\sigma(a)} \neq n - 1 \) then
\[
g(UWD) = U w_1 \ldots w_l \ldots w_{r'} f(w_p) \ldots f(w_{i_{r_1+1}}) \ldots w_{n-1} D \in S.
\]

(4.2) If \( r' = j_{r_1+1-\sigma(a)} \)
\[
g(UWD) = U w_1 \ldots w_l \ldots w_{i_{r_1+1}} f(w_{i_{r_1+1}}) \ldots f(w_{r'}) \ldots D \in S.
\]

(4.3) Now, if \( |r_1 - r| > 1 \) or \( r_1 = r + 1 \) and \( r' > i_{r_1+1} + 2 \) then
\[
g(UWD) = U w_1 \ldots f(w_l) \ldots f(w_{i_{r_1+1}}) w_{i_{r_1+1}+1} \ldots w_{r'} \ldots D \in S.
\]

For \( r' \in [j_{r_1+1-\sigma(a)}, i_{r_1+1}] \) with \( |r_1 - r| \geq 0 \) we have that:

(4.4) If \( s = t = i_{r_1+1} = n - 1 \), then
\[
g(UWD) = U w_1 \ldots w_l \ldots w_{r'-1} f(w_{r'}) D \in S.
\]

On the other hand, if \( s = t = i_{r_1+1} \neq n - 1 \), then there is a \( p \in [i_{r_1+1}, j_{r_1+2-\sigma(a)}] \) satisfying first condition of (4.1). Thus, if \( j_{r_1+1-\sigma(a)} < s < i_{r_1+1} \), it holds that
\[
g(UWD) = U w_1 \ldots w_l \ldots w_{r'-1} f(w_{r'}) \ldots w_{n-1} D \in S.
\]

(4.5) If \( s = j_{r_1+1-\sigma(a)} \) then \( |r_1 - r| > 0 \) (If \( |r_1 - r| = 0 \), \( |l - f| = 0 \) which is a contradiction)
\[
g(UWD) = U w_1 \ldots w_l \ldots w_{r'-1} f(w_{i_{r_1+1}}) \ldots f(w_{r'}) \ldots w_{n-1} D \in S.
\]

(4.6) Now, suppose that in \( UWD \) \( |r_1 - r| > 0 \), then it satisfies the first condition in (4.3).

In case that \( l \in [j_r, i_{r+\sigma(a)}] \), we have the following cases:

(5) If \( j_r < l \leq i_r + \sigma(a) \), then there exists \( p = l + 1 \) such that, if \( p \neq j_r \) then
\[
g(UWD) = U w_1 \ldots w_j_r \ldots f(w_p) w_l \ldots w_{r'} \ldots w_{n-1} D \in S.
\]

Note that, if \( p = j_r = 1 \) then
\[
g(UWD) = U f(w_p) w_l \ldots w_{r'} \ldots w_{n-1} D \in S,
\]
or if \( p = j_r \neq 1 \), then
\[
g(UWD) = U w_1 \ldots f(w_{i_r-1+\sigma(a)}) \ldots f(w_p) w_l \ldots w_{r'} \ldots w_{n-1} D \in S.
\]

(6) If \( l = j_r \) and \( |l - r'| > 0 \), then \( r' \in [j_{r_1}, i_{r_1+\sigma(a)}] \) with \( |r_1 - r| \geq 0 \), then the following cases hold:

(6.1) If \( j_{r_1} + 2 \leq r' \leq i_{r_1+\sigma(a)} \), then there exists \( p \) satisfying (4.4).

(6.2) If \( j_{r_1} \leq r' < j_{r_1} + 2 \), then \( |r_1 - r| > 0 \) and if \( r = j_{r_1+1} \) satisfies (6.1), or if \( r = j_{r_1} \) then \( UWD \) satisfies (4.5).

(6.3) Now, if \( |r_1 - r| > 0 \) then
\[
g(UWD) = U \ldots f(w_l) \ldots f(w_{i_r+\sigma(a)}) w_{i_r+\sigma(a)+1} \ldots w_s \ldots D \in S;
\]

or \( r' \in [i_{r_1+\sigma(a)}, j_{r_1+1}] \) with \( |r_1 - 1| \geq 0 \) satisfies conditions (4.1), (4.2) and (4.3) for \( i_{r_1+\sigma(a)} \leq r' \leq j_{r_1+1} \).

Same arguments are used for the cases \( r' \in [i_r, j_{r+1-\sigma(a)}]([j_r, i_{r+\sigma(a)}]) \) to conclude the lemma. We are done. \( \square \)

**Lemma 3.2.** Suppose that \( UWD \) is a Dyck path in \( S \) and that integers \( l \) and \( r' \) as defined in Proposition 3.2 are such that \( l = r' \), then the following statements hold:

(a) If \( l \notin \{j_s\} \) then there is an elementary shift to a Dyck path with exactly \( n - 1 \) peaks.

(b) If \( l \in \{j_s\} \) then there is an elementary shift from a Dyck path with exactly \( n - 1 \) peaks to \( UWD \).

**Proof.** Let \( UWD \) be a Dyck path in \( S \), and positive integers \( l \) and \( r' \) with \( l = r' \).

(a) Suppose \( l \notin \{j_s\} \) and \( l \in [i_r, j_{r+1-\sigma(a)}] \). If \( i_r \leq l < j_{r+1-\sigma(a)} \), then \( UWD \) satisfies (4.1) and (4.2) of Lemma 3.1. In particular, if \( l = i_r \neq 1 \) there is a \( p' = l - 1 \) in \([j_{r-\sigma(a)}, i_r]\) that satisfies the first condition of (5) of Lemma 3.1. The case \( l \in [j_r, i_{r+\sigma(a)}] \) is dual.

(b) Suppose \( l = j_{r+1-\sigma(a)} \), we have the following cases:

(i) If \( |i_r - j_{r+1-\sigma(a)}| = 1 \) (or \( |i_{r+1} - j_{r+1-\sigma(a)}| = 1 \)) and \( i_r = 1 \) (or \( i_{r+1} = n - 1 \)), then there is a \( UVD \) which is equal to
\[
\underbrace{U w_1 \ldots w_l \ldots D}_{UD} \in S \quad (or \quad \underbrace{U \ldots w_l \ldots w_{n-1} D \in S}_{UD}),
\]

and
\[
U f(w_l) w_l \ldots D = UWD \quad (or \quad U \ldots w_l f(w_{n-1}) D = UWD).
\]
(ii) If $|i_r - j_{r+1-\sigma(a)}| = 1$ (or $|i_r+1 - j_{r+1-\sigma(a)}| = 1$) and $i_r \neq 1$ (or $i_r+1 \neq n-1$) then there is a $l' = j_{r-\sigma(a)}$ and $r'' = j_{r+1-\sigma(a)}$ (or $l' = j_{r+1-\sigma(a)}$ and $r'' = j_{r+2-\sigma(a)}$) such that $UVD$ is equal to
\[
U \ldots w_{l'} \ldots w_{r''-1}w_l \ldots D \text{ (or } U \ldots w_{l'1}w_{r''-1}w_l \ldots D) \in S,
\]
and
\[
U \ldots f(w_{l'}) \ldots f(w_{r''-1})w_l \ldots D \text{ (or } U \ldots w_{l1}f(w_{r''-1})w_l \ldots D = UWD.
\]
(iii) If $|i_r - j_{r+1-\sigma(a)}| > 1$ (or $|i_r+1 - j_{r+1-\sigma(a)}| > 1$) then there is a $UVD$ which is equal to
\[
U \ldots w_{l1} \ldots w_l \ldots D \text{ (or } U \ldots w_{l1}w_{l1+1} \ldots D \in S),
\]
and
\[
U \ldots f(w_{l1})w_l \ldots D = UWD \text{ (or } U \ldots w_{l1}f(w_{l1}) \ldots D = UWD).
\]

Similar arguments dually applied can be used to obtain the lemma in the case $l = j_r$. We are done. □

**Remark 3.1.** Note that, in general there is an elementary leftshift and an elementary rightshift over $S$, and these elementary shifts are disjoint, i.e. if $f_{p_1} \circ \cdots \circ f_{p_q}$ and $f_{p_1'} \circ \cdots \circ f_{p_{q'}}'$ are elementary left and right shifts, respectively. Then
\[
\{p_1, \ldots, p_q\} \cap \{p_1', \ldots, p_{q'}\} = \emptyset,
\]
these elementary shifts are unique according to Lemma 3.1 and Lemma 3.2. If $F^p = f_{p_1} \circ \cdots \circ f_{p_q}$ is an elementary leftshift (rightshift) we write $F^p_i$ ($F^p_r$).

**Proposition 3.3.** Let $C = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain, then all Dyck paths of $S$ constitute a connected quiver $Q$ whose set of vertices is in correspondence with the set of all Dyck paths in $S$ and there is an arrow from $UWD \in S$ to $UVD \in S$ if there is an elementary shift transforming $UWD$ into $UVD$.

**Proof.** It suffices to prove that $Q$ is connected, to do that, consider Dyck paths $UWD$ and $UVD$ of $S$. Thus, if there is a shift path between $UWD$ and $UVD$ then they are connected. Otherwise, Lemmas 3.1 and 3.2 allow to define a Dyck path $UW^{(1)}D$ and a shift path $F^{(1)} = F^{(1)}_{p_1} \circ \cdots \circ F^{(1)}_{m1}$ with $F^{(1)}_{m1} = f^{(1)}_{m1} \circ \cdots \circ f^{(1)}_{m1}$ such that
\[
UWD \xrightarrow{F^{(1)}_{m1}} \ldots \xrightarrow{F^{(1)}_{p1}} UW^{(1)}D,
\]
and if there is a shift path from $UVD$ to a $UW^{(1)}D$ then they are connected. If there is not a shift path from $UVD$ to $UW^{(1)}D$ then there is a Dyck path $UW^{(2)}D$ and a shift path $F^{(2)} = F^{(2)}_{p2} \circ \cdots \circ F^{(2)}_{m2}$ with $F^{(2)}_{m2} = f^{(2)}_{m1} \circ \cdots \circ f^{(2)}_{m1}$ such that
\[
UW^{(2)}D \xrightarrow{F^{(2)}_{m2}} \ldots \xrightarrow{F^{(2)}_{p2}} UW^{(1)}D \xleftarrow{F^{(1)}_{m1}} \ldots \xleftarrow{F^{(1)}_{p1}} UWD,
\]
Suppose that $\text{Hom}_{\mathcal{C}_{2n}}(UWD, UVD) \neq 0$ then $\dim_k \text{Hom}_{\mathcal{C}_{2n}}(UWD, UVD) = 1$.

**Proof.** Suppose that $\text{Hom}_{\mathcal{C}_{2n}}(UWD, UVD) \neq 0$, then there is a shift path $F$ of the form

$$UWD \xrightarrow{F_0} \cdots \xrightarrow{F_{x_i-1}} UW^{i-1}D \xrightarrow{F_{x_i}} UW^iD \xrightarrow{F_{x_i+1}} UW^{i+1}D \xrightarrow{F_{x_{i+1}}} \cdots \xrightarrow{F_{x_m}} UVD,$$

with $x_i \in \{l, r\}$ and for some $m \in \mathbb{Z}^+$. Now, for each pair $F_{x_i} \circ F_{x_{i-1}}^{-1}$ with $x_{i-1} = l$ and $x_i = r$ (or $x_{i-1} = r$ and $x_i = l$) that satisfies conditions described in Lemma 3.3 there is another shift path $F'$ of the form

$$UWD \xrightarrow{F_0} \cdots \xrightarrow{F_{x_i-1}} UW^{i-1}D \xrightarrow{F_{x_i}} UW^iD \xrightarrow{F_{x_i+1}} \cdots \xrightarrow{F_{x_m}} UVD,$$

transforming $UWD$ and $UVD$. Thus $F \sim_{R^{i_1 \cdots i_k}} F'$.

## 3.2 A Categorical Equivalence

In this section, we establish an equivalence between the full category $\mathcal{C}_{2n}$ and the category of representations of a quiver of Dynkin type $A_n$. 
3.2.1 The Θ Functor

Given an admissible subchain \( C = \{ j_1, \ldots, j_m, i_1, \ldots, i_k \} \), \( C_{2n} \) the full subcategory of \( (D_{2n}, R_{j_1 \ldots j_m}^{1 \ldots k}) \) and \( Q \) a quiver of type \( A_{n-1} \) with \( \{ i_1, \ldots, i_k \} \) and \( \{ j_1, \ldots, j_m \} \) being the sets of sinks and sources, respectively. Then the \( k \)-linear additive functor \( \Theta : C_{2n} \rightarrow \text{rep } Q \) is defined in such a way that, for an object \( UWD \in C_{2n} \), it holds that,

\[
\Theta(UWD) = (\Theta(w_i), \varphi_{\Theta(w_i, w_{i+1})}),
\]

where

\[
\Theta(w_i) = \begin{cases} 
  k, & \text{if } w_i = UD, \\
  0, & \text{if } w_i = DU.
\end{cases} \tag{3.6}
\]

If \( i, i + 1 \in [i_r, j_{r+1}-\sigma(a)] \) \( ([j_r, i_r+\sigma(a)] \) then \( s(\Theta(w_i, w_{i+1})) = i + 1 \), is the starting point of the corresponding arrow, whereas \( t(\Theta(w_i, w_{i+1})) = i \) is the ending vertex of the corresponding arrow \( (s(\Theta(w_i, w_{i+1})) = i, t(\Theta(w_i, w_{i+1})) = i + 1) \) and,

\[
\varphi_{\Theta(w_i, w_{i+1})} : \Theta(w_{s(\Theta(w_i, w_{i+1}))}) \rightarrow \Theta(w_{t(\Theta(w_i, w_{i+1}))}),
\]

\[
\varphi_{\Theta(w_i, w_{i+1})} = \begin{cases} 
  1, & \text{if } w_i = UD = w_{i+1}, \\
  0, & \text{if } w_i = DU \text{ or } w_{i+1} = DU. 
\end{cases} \tag{3.7}
\]

Functor \( \Theta \) acts on morphisms as follows;

Let

\[
f_{q_2} \circ \cdots \circ f_{q_1} = (1, \ldots, 1, q_1-1, f_{q_1}, \ldots, f_{q_2}, 1, q_2+1, \ldots 1_{n-1}),
\]

be an elementary shift between \( UWD \) and \( UVD \), then:

\[
\Theta((1, \ldots, 1, q_1-1, f_{q_1}, \ldots, f_{q_2}, 1, q_2+1, \ldots 1_{n-1})),
\]

\[
(\Theta(1), \ldots, \Theta(1_{q_1-1}), \Theta(f_{q_1}), \ldots, \Theta(f_{q_2}), \Theta(1_{q_2+1}), \ldots, \Theta(1_{n-1})),
\]

where \( \Theta(f_m) = 0 \) and,

\[
\Theta(1_{m_1}) = \begin{cases} 
  1, & \text{if } w_{m_1} = UD = v_{m_1}, \\
  0, & \text{otherwise},
\end{cases} \tag{3.8}
\]

for \( 1 \leq m_1 \leq q_1 - 1, q_1 \leq m \leq q_2 \) and \( q_2 + 1 \leq m_1 \leq n - 1 \).

**Remark 3.2.** Note that, it is easy to see that \( \Theta \) is an additive covariant functor.

**Lemma 3.4.** Let \( UWD \) and \( UVD \) be Dyck paths of \( C_{2n} \). If \( \text{Hom}_{C_{2n}}(UWD, UVD) \neq 0 \) then \( \text{Hom}_{\text{rep } Q}(\Theta(UWD), \Theta(UVD)) \neq 0 \).

**Proof.** Suppose \( \text{Hom}_{C_{2n}}(UWD, UVD) \neq 0 \), and let \( F \) be a shift path \( UW^0D \stackrel{F_0}{\rightarrow} UW^1D \stackrel{F_1}{\rightarrow} \cdots \stackrel{F_{m-2}}{\rightarrow} UW^{m-1}D \stackrel{F_{m-1}}{\rightarrow} UWD \) from \( UWD = UWD \) to \( UVD = UWD \).
$UW^m D$ for some $m \in \mathbb{Z}^+$, then there exist $q_1$ and $q_2$ such that

$$\{q_1, q_1 + 1, \ldots, q_2 - 1, q_2\} = \bigcap_{i \in J} \text{Supp } UW^i D,$$

applying $\Theta$ we obtain the following diagram:

![Diagram 1](attachment:diagram1.png)

where $c_{q_1}^1 - 1, a_{q_1}^1 - 1, a_{q_2}^1, d_{q_2 + 1}^0 \in \{0, k\}$, squares in the diagram are commutative between $q_1$ and $q_2$ (independently of the chosen orientation). For the sub-shift path $F^{(x,y)}$ to $F$ with $0 \leq x \leq y \leq m - 1$ there exist positive integers $q_1^{(x,y)}$ and $q_2^{(x,y)}$ such that

$$S^{(x,y)} = \{q_1^{(x,y)}, q_1^{(x,y)} + 1, \ldots, q_2^{(x,y)} - 1, q_2^{(x,y)}\} = \bigcap_{i \in J^{(x,y)}} \text{Supp } UW^i D,$$

and for the diagrams

![Diagram 2](attachment:diagram2.png)  
![Diagram 3](attachment:diagram3.png)

we have the following cases:

1. If $q_1^{(x,y)} \in [i_r, j_{r+1} - \sigma(a)]$ ($i_r < q_1^{(x,y)} \leq j_{r+1} - \sigma(a)$) four cases must be considered.
   1.1 If $\Theta(w^{x,y}_{q_1^{(x,y)} - 1}) = k$ and $\Theta(w^{y}_{q_1^{(x,y)} - 1}) = k$, $q_1^{(x,y)}$ belong to $S^{(x,y)}$, which is a contradiction.
   1.2 If $\Theta(w^{x}_{q_1^{(x,y)} - 1}) = k$ and $\Theta(w^{y}_{q_1^{(x,y)} - 1}) = 0$, then the Diagram 2 commutes.
(1.3) If $\Theta(w^x_{q_1^{(x,y)}-1}) = 0$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = k$, then there is an elementary shift $f_{q_1^{(x,y)}-1}$ on the interval and this is again a contradiction.

(1.4) If $\Theta(w^x_{q_1^{(x,y)}-1}) = 0$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = 0$, then the Diagram 2 commutes.

(2) If $q_1^{(x,y)} \in [j_{r+1-\sigma(a)}, i_{r+1}]$ such that $q_1^{(x,y)} < q_1$, then the Diagram 2 commutes.

(2.1) If $\Theta(w^x_{q_1^{(x,y)}-1}) = k$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = 0$, then they satisfy condition (1.3).

(2.2) If $\Theta(w^x_{q_1^{(x,y)}-1}) = 0$ and $\Theta(w^y_{q_1^{(x,y)}-1}) = k$, then they satisfy condition (1.2).

(3) Case $q_2^{(x,y)} \in [i_r, j_{r+1-\sigma(a)}]$ is similar to case (2) for the Diagram 3.

(4) Case $q_2^{(x,y)} \in [j_{r+1-\sigma(a)}, i_{r+1}]$ is similar to case (1) for the Diagram 3.

therefore the Diagram 1 commutes. Since the cases over $[j_r, i_{r+\sigma(a)}]$ can be showed by using dual arguments. We are done. \hfill \Box

Lemma 3.5. Functor $\Theta$ is faithful and full.

Proof. Let $\phi$ be the map

$$
\phi : \text{Hom}_{C_{2n}}(UWD, UVD) \to \text{Hom}_{\text{rep } Q}(\Theta(UWD), \Theta(UVD)),
$$

such that $\phi(\lambda F) = \lambda \Theta(F)$ with $F = (1_1, \ldots, 1_{q_1-1}, f_{q_1}, \ldots, f_{q_2}, 1_{q_2+1}, \ldots, 1_{n-1})$, for some $1 \leq q_1, q_2 \leq n-1$ and $\lambda \in k$. Note, $\phi$ is well defined and Lemma 3.4 allows us to observe that the image of a non-zero morphism in $C_{2n}$ is a non-zero morphism in $\text{rep } Q$. Thus, $\phi$ is surjective and injective. \hfill \Box

Theorem 3.1. Functor $\Theta$ is a categorical equivalence between the categories $C_{2n}$ and $\text{rep } Q$.

Proof. Lemma 3.5 implies that functor $\Theta$ is faithful and full. Now, let $(M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be an indecomposable representation in $\text{rep } Q$ of the form

$$
0 \quad \cdots \quad k \quad 1 \quad k \quad 1 \quad \cdots \quad k \quad 1 \quad k \quad \cdots \quad 0
$$

with $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_m\}$ the sets of sinks and sources respectively. Let $\varphi_1 : \{0, k\} \to \{DU, UD\}$ be a map such that $\varphi_1(k) = UD$ and $\varphi_1(0) = DU$. Define the Dyck path $UWD$ such that

$$
UWD = U_{DU} \overbrace{w_1 \ldots w_{q_1-1} w_{q_1} \ldots w_{q_2} w_{q_2+1} \ldots w_{n-1}}^{DU} D.
$$

Proposition 3.2 allows us to observe that $UWD$ has $n-1$ peaks over $\{j_1, \ldots, j_m, i_1, \ldots, i_k\}$ and $\Theta(UWD) = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Thus, $\Theta$ is essentially surjective. \hfill \Box

Corollary 3.1. There exists a bijection $\varphi$ between the set of representatives of indecomposable representations of $\text{rep } Q$ and the set of Dyck paths of length $2n$ with exactly $n - 1$ peaks.
Proof. The Narayana number with exactly \( n - 1 \) peaks over all Dyck paths of length \( 2n \) is the triangular number \( T_{n-1} = \frac{(n-1)n}{2} \), which is equal to the number of indecomposable representations of \( \text{rep} \ Q \), then we define \( \varphi : S \to \text{Ind} \ (\text{rep} \ Q) \) such that \( \varphi(UWD) = \Theta(UWD) \).

\[ \square \]

**Corollary 3.2.** The category \( \mathcal{C}_{2n} \) is an abelian category.

### 3.2.2 Properties of the Category \( \mathcal{C}_{2n} \)

In this section, we introduce some properties of \( \mathcal{C}_{2n} \) regarding simple, projective and injective indecomposable objects, we also construct the Auslander-Reiten quiver for algebras of Dynkin type \( A_{n-1} \). Some conditions for morphisms between objects of the category are introduced as well.

**Theorem 3.2.** Let \( \mathcal{C} = \{ j_1, \ldots, j_m, i_1, \ldots, i_k \} \) be an admissible subchain, and let \( \mathcal{C}_{2n} \) be the corresponding category, then

(i) Indecomposable simple objects of \( \mathcal{C}_{2n} \) are objects of the form

\[
S(x) = US(w^x_1) \ldots S(w^n_x)D,
\]

where

\[
S(w^x_y) = \begin{cases} UD, & \text{if } x = y, \\ DU, & \text{otherwise}. \end{cases}
\]  

(ii) Indecomposable projective objects of \( \mathcal{C}_{2n} \) have the form

\[
P(x) = UP(w^x_1) \ldots P(w^n_x)D
\]

where

\[
P(w^x_y) = \begin{cases} UD, & \text{if } x, y \in [i_r, j_{r+1-\sigma(a)}] \ (\ [j_r, i_{r+\sigma(a)}]) \text{ and } y \leq x \ (x \leq y), \\ DU, & \text{otherwise}. \end{cases}
\]  

(iii) Indecomposable injective objects of \( \mathcal{C}_{2n} \) have the form

\[
I(x) = UI(w^x_1) \ldots I(w^n_x)D
\]

where

\[
I(w^x_y) = \begin{cases} UD, & \text{if } x, y \in [i_r, j_{r+1-\sigma(a)}] \ (\ [j_r, i_{r+\sigma(a)}]) \text{ and } x \leq y \ (y \leq x), \\ DU, & \text{otherwise}. \end{cases}
\]  

Proof. (i) Let \( S(x) = (S(x)_y, \varphi_a) \) be an indecomposable simple object of \( \text{rep} \ Q \) such that \( S(x)_y = k \) if \( x = y \) and \( S(x)_y = 0 \) if \( x \neq y \). Functor \( \Theta \) allows us to observe that, there is an \( UWD \in \mathcal{C}_{2n} \) satisfying the required conditions.

(ii) Let \( P(x) = (P(x)_y, \varphi_a) \) be an indecomposable projective object of \( \text{rep} \ Q \), if \( P(x)_y = k \) then there is a path from \( x \) to \( y \), as well as, a source \( j_{r+1-\sigma(a)} \ (j_r) \) and a sink \( i_r \ (i_{r+\sigma(a)}) \) such that \( i_r \leq y \leq x \leq j_{r+1-\sigma(a)} \ (j_r \leq x \leq y \leq i_{r+\sigma(a)}) \), and \( P(x)_y = 0 \). Thus, there is not a path between \( x \) and \( y \), then functor \( \Theta \) determines an object \( UWD \) of \( \mathcal{C}_{2n} \) with
\[ i_1, \ldots, i_k, j_1, \ldots, j_m \] being an admissible subchain satisfying the required conditions. Case (iii) follows by dually applying the arguments used in the case (ii).

**Corollary 3.3.** The indecomposable simple objects of \( \mathcal{C}_{2n} \) have exactly a subsequence UUDD.

**Proof.** Let \( S(x) \) be an indecomposable simple object of \( \mathcal{C}_{2n} \), then the identity

\[
S(x) = U \cdots S(w^x_{i-1})S(w^x_i)S(w^x_{i+1}) \cdots D = U \cdots DU \underbrace{UD DU \cdots DU}_{x-1 \ x \ x+1} \cdots D
\]

has place as a consequence of Theorem 3.2.

**Remark 3.3.** The Auslander-Reiten translate can be obtained by using the Coxeter transformation and the dimension vector associated to the support of a Dyck path in \( \mathcal{C}_{2n} \).

Figure 3.3 describes the Auslander-Reiten quiver of rep \( Q \) of the quiver \( Q \) given by Figure 1.13.

**Figure 3.3.** Quiver \( Q \) and the Auslander-Reiten quiver of rep \( Q \).

Morphisms in \( \mathcal{C}_{2n} \) also have the following properties.

Let \( UWD \) be a Dyck path of \( \mathcal{C}_{2n} \), then

- \( p_{UWD} = t \) and \( b_{UWD} = \max \{ s \mid i_r \leq s \leq j_{r+1-\sigma(a)}, w_s = UD \} \) over \([i_r, j_{r+1-\sigma(a)}]\),
- \( p^{UWD} = \min \{ s \mid j_r \leq s \leq i_{r+\sigma(a)}, w_s = UD \} \) and \( b^{UWD} = t \) over \([j_r, i_{r+\sigma(a)}]\).

**Theorem 3.3.** The vector space \( \text{Hom}_{\mathcal{C}_{2n}}(UWD, UVD) \neq 0 \) if and only if
(i) \( \text{Supp}(UWD) \cap \text{Supp}(UVD) \neq \emptyset \),

(ii) \( p_{UWD} \leq p_{UVD} \) and \( b_{UWD} \leq b_{UVD} \) over \([i_r, j_{r+1-\sigma(a)}]\),

(iii) \( p_{UWD} \geq p_{UVD} \) and \( b_{UWD} \geq b_{UVD} \) over \([j_r, i_{r+1+\sigma(a)}]\),

for all \([i_r, j_{r+1-\sigma(a)}], [j_r, i_{r+1+\sigma(a)}]\) such that \( i_r \leq q \leq j_{r+1-\sigma(a)} \) and \( j_r \leq q \leq i_{r+1+\sigma(a)} \) with \( q \in \text{Supp}(UWD) \cap \text{Supp}(UVD) \).

**Proof.** The result follows as a consequence of the definition of the functor \( \Theta \) and the construction of Lemma 3.1. \( \square \)

### 3.2.3 A Relationship with Some Nakayama Algebras

In [65] Marczinzik, Rubey and Stump presented a connection between the Auslander-Reiten quiver of Nakayama algebras and Dyck paths. In such a work for a Nakayama algebra \( A \), they associated the vector space dimension of the indecomposable projective modules \( e_i A \) to a Dyck path, this vector is called the Kupisch series. If we take a Nakayama algebra \( A = kQ/I \), with \( I = \langle x_3x_4, x_1x_2x_3 \rangle \),

![Figure 3.4. Quiver Q of type A₅.](image)

then the Kupisch series of \( kQ/I \) is \([3, 3, 2, 2, 1]\), and the Auslander-Reiten quiver of \( kQ/I \) has the shape described in Figure 3.5.

![Figure 3.5. Dyck path associated to kQ/I.](image)

Let \( \mathcal{C}_{2(n+1)} \) be the category with the admissible subchain \( 1 < n, j_1 = 1 \) and \( i_1 = n \), and let \( D_i \) be the sets

\[
D_1 = \{ X \in \text{Ob}(\mathcal{C}_{2(n+1)}) \mid w_1 = UD \},
\]

\[
D_i = \{ X \in \text{Ob}(\mathcal{C}_{2(n+1)}) \mid w_m = DU, \ 1 \leq m \leq i - 1 \},
\]

for \( 1 < i \leq n \). Then, we take the subset \( D_{i,j} \subseteq D_i \),

\[
D_{i,j} = \{ Y \in D_i \mid i \leq r_Y \leq m(i, j_i) + i - 1 \},
\]
such that the vector \( v = (n - (m(i, j) + i - 1))_{i=1}^{n} \) constitutes an integer partition with \( n \) parts. Now, let \( \mathcal{R}_v \) be the subcategory of \( \mathcal{C}_{2(n+1)} \) whose objects are \( k \)-linear combinations of the Dyck paths in the following set

\[
\mathcal{L} = \bigcup_{i=1}^{n} D_{i, j_i},
\]

and morphisms defined by the category \( \mathcal{C}_{2n(n+1)} \).

We assume the following numbering and orientation for a quiver \( Q \) associated to a Nakayama algebra

![Figure 3.6. Quiver Q of type \( A_n \).](image)

The functor \( \Theta' \) between the category \( \mathcal{R}_v \) and the category of representations of \( (Q, I) \) where \( kQ/I \) is a Nakayama algebra with Kupisch series \([m(1, j_1), \ldots, m(n, j_n)]\) is defined in such a way that, \( \Theta'(UWD) = \Theta(UWD) \) and \( \Theta'(F) = \Theta(F) \) for \( UWD \in \mathcal{L} \) and \( F \) being an elementary shift in \( \mathcal{R}_v \).

**Corollary 3.4.** The functor \( \Theta' \) is an equivalence of categories.

**Proof.** It is a direct consequence of Theorem 3.1. \( \square \)

As an example, Figure 3.7 shows the Auslander-Reiten quiver of the Nakayama algebra \( \mathcal{A} = kQ/I \) associated to the quiver \( Q \) shown in Figure 3.4 with \( I = \langle x_3x_4, x_1x_2x_3 \rangle \).

![Figure 3.7. Auslander-Reiten quiver of mod kQ/I.](image)

### 3.3 Cluster Variables Associated to Dyck Paths

In this section, we construct an alphabet associated to Dyck paths. And it is given a formula for cluster variables of cluster algebras associated to Dynkin diagrams of type \( A_n \).
3.3.1 An Alphabet for Dyck Paths

For \( n > 2 \), let \( U^i_1 = u_1 \ldots u_{2n} \) and \( U^i_2 = u'_1 \ldots u'_{2n} \) be Dyck paths in \( \mathcal{D}_{2n} \) with the following form:

\[
  u_j = \begin{cases} 
  U, & \text{if } 1 \leq j \leq i + 1 \text{ or } j = 2(i + 1) + k \leq 2n, \\
  D, & \text{if } i + 2 \leq j \leq 2(i + 1) \text{ or } j = 2(i + 1 + k) \leq 2n,
\end{cases} \quad (3.15)
\]

and

\[
  u'_j = \begin{cases} 
  U, & \text{if } 2i < j \leq i + n \text{ or } j = 1 + 2k \leq 2i, \\
  D, & \text{if } i + n < j \leq 2n \text{ or } j = 2k \leq 2n,
\end{cases} \quad (3.16)
\]

for \( k > 0 \) and \( i \leq n - 2 \). The alphabet \( H_n \) is the union of the set \( \{U^i_j \mid r = 1, 2 \text{ and } 1 \leq i \leq n - 2\} \) and the Dyck path with exactly one peak in \( \mathcal{D}_{2n} \) (denoted \( E_n \)). Figure 3.8 shows the alphabet \( H_3 \).

![Figure 3.8. Alphabet \( H_3 \).](image)

Let \( \mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\} \) be an admissible subchain of \( n-1 \). We fix two different relations of concatenation \( \vartheta_1 \) and \( \vartheta_2 \) over \( H_n \) such that

\[
  \vartheta_1(V_i) = \begin{cases} 
  E_n, & \text{if } V_i = E_n \text{ or } V_i = U^i_1, \\
  U^{i+1}_2, & \text{if } V_i = E_n \text{ or } V_i = U^i_1, \\
  U^{i+1}_1, & \text{if } V_i = U^i_2,
\end{cases} \quad (3.17)
\]

and

\[
  \vartheta_2(V_i) = \begin{cases} 
  E_n, & \text{if } V_i = U^i_2, \\
  U^{i+1}_1, & \text{if } V_i = E_n \text{ or } V_i = U^i_1, \\
  U^{i+1}_2, & \text{if } V_i = U^i_2.
\end{cases} \quad (3.18)
\]

Then, we take the set of words \( V = V_1 \ldots V_{n-2} \) in \( H_n^* \) such that

\[
  V_i = \begin{cases} 
  \vartheta_1(V_{i-1}), & \text{if } i \notin \mathcal{C}, \\
  \vartheta_2(V_{i-1}), & \text{if } i \in \mathcal{C} - \{1, n-1\},
\end{cases} \quad (3.19)
\]

for \( 1 < i \leq n - 2, n \geq 4 \). This set is denoted by \( \mathbb{X}_\mathcal{C} \), in particular case \( \mathbb{X}_{\{1,2\}} = H_3 \).
3.3.2 Dyck Words and Perfect Matchings

Let \( G = (G_1, \ldots, G_{n-1}) \) be a snake graph, then we can associate to \( G \) an admissible subchain \( \mathcal{C} \) of \( n-1 \) in the following way:

If \( G_{i-1}, G_i \) and \( G_{i+1} \) denote tiles of the following snake graph

\[
\begin{array}{ccc}
G_{i-1} & G_i & G_{i+1} \\
\end{array}
\]

then, \( i \in \mathcal{C} \) for \( 1 < i < n - 1 \). For example, for the snake graph \( \mathcal{G} \) shown in Figure 3.9

\[
\begin{array}{ccc}
G_4 & G_5 \\
G_3 \\
G_1 & G_2 \\
\end{array}
\]

Figure 3.9. Snake graph \( \mathcal{G} \).

it holds that the corresponding admissible subchain is given by the identity \( \{1, 3, 5\} = \{i_1, j_1, i_2\} = \{j_1, i_1, j_2\} \). By notation, \( \mathcal{G} \) can be written as \( \mathcal{G}_\mathcal{C} \).

The following result establishes a relationship between the alphabet \( \mathcal{X}_\mathcal{C} \) and perfect matchings of snake graphs.

**Lemma 3.6.** Let \( \mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\} \) be an admissible subchain of \( n-1 \). Then, there is a bijective correspondence between the set \( \mathcal{X}_\mathcal{C} \) and the perfect matchings of \( \mathcal{G}_\mathcal{C} \).

**Proof.** Let \( \mathcal{C} \) be an admissible subchain of \( n-1 \), \( \mathcal{X}_\mathcal{C} \) be a set of words, and \( \mathcal{G}_\mathcal{C} \) be a snake graph associated to \( \mathcal{C} \). Assume a numbering over the edges of \( \mathcal{G}_\mathcal{C} \) in the following way:

For boundary edges of \( G_i \), we have the following four possibilities
with $1 < i < n - 1$ (labeling is given by recurrence). The other edges are labeled with the letter $E_n$. Now, a perfect matching $P$ of $G_c$ can be written as a vector $v = (v_1, \ldots, v_n)$, where each $v_i$ corresponds to an edge of $G_c$ (this vector is unique up to permutation). Define a map $f : X_c \rightarrow \text{Match}(G_c)$ such that $f(V_1 \ldots V_{n-2}) = (E_n, V_1, \ldots, V_{n-2}, E_n)$. Firstly, we will prove that $f$ is well defined by induction over $n$. To start note that for $n = 3$, we have the following three cases:

(I) If $V_1 = E_3$, it turns out that $F(V_1) = (E_3, E_3, E_3)$, which is given by

$$
\begin{array}{c}
G_1 \\
G_2
\end{array}
$$

(II) If $V_1 = U_1^1$, it holds that $f(U_1^1) = (E_3, U_1^1, E_3)$, which is equal to

$$
\begin{array}{c}
G_1 \\
G_2
\end{array}
$$

(III) If $V_1 = U_2^1$, then $f(U_2^1) = (E_3, U_2^1, E_3)$, which is of the form

$$
\begin{array}{c}
G_1 \\
G_2
\end{array}
$$

Suppose that the result holds for $n = k$. Let $n = k + 1$, by hypothesis $(E_{k+1}, V_1, \ldots, V_k)$ are disjoint sets containing all the previous tiles in $G_c$, then there are two possibilities for $k$.

(I) for $k \in C - \{1, k + 1\}$, we have the following conditions:

1.1 If $V_{k-1} = E_{k+1}$, then $f(V_1 \ldots E_{k+1} E_{k+1}) = (E_{k+1}, V_1, \ldots, E_{k+1}, E_{k+1}, E_{k+1})$ and $f(V_1 \ldots E_{k+1} U_2^k) = (E_{k+1}, V_1, \ldots, E_{k+1}, U_2^k, E_{k+1})$, which are given by

$$
\begin{array}{c}
G_{k-1} \\
G_k \\
G_{k+1}
\end{array}
$$

and

$$
\begin{array}{c}
G_{k-1} \\
G_k \\
G_{k+1}
\end{array}
$$

1.2 If $V_{k-1} = U_1^{k-1}$, then $f(V_1 \ldots U_1^{k-1} E_{k+1}) = (E_{k+1}, V_1, \ldots, U_1^{k-1}, E_{k+1}, E_{k+1})$ and $f(V_1 \ldots U_1^{k-1} U_2^k) = (E_{k+1}, V_1, \ldots, U_1^{k-1}, U_2^k, E_{k+1})$, which are equal to

$$
\begin{array}{c}
G_{k-1} \\
G_k \\
G_{k+1}
\end{array}
$$

and

$$
\begin{array}{c}
G_{k-1} \\
G_k \\
G_{k+1}
\end{array}
$$

1.3 If $V_{k-1} = U_2^{k-1}$, then $f(V_1 \ldots U_2^{k-1} U_1^k) = (E_{k+1}, V_1, \ldots, U_2^{k-1}, U_1^k, E_{k+1})$ which is of the form

$$
\begin{array}{c}
G_{k-1} \\
G_k \\
G_{k+1}
\end{array}
$$

(II) for $k \notin C$, there are the following cases:
(2.1) If $V_{k-1} = E_{k+1}$, then $f(V_1 \ldots E_{k+1} U_1^k) = (E_{k+1}, V_1, \ldots, E_{k+1} U_1^k, E_{k+1})$, which is given by

(2.2) If $V_{k-1} = U_1^{k-1}$, then $f(V_1 \ldots U_1^{k-1} U_1^k) = (E_{k+1}, V_1, \ldots, U_1^{k-1} U_1^k, E_{k+1})$, which is equal to

(2.3) If $V_{k-1} = U_2^{k-1}$, then $f(V_1 \ldots U_2^{k-1} U_2^k) = (E_{k+1}, V_1, \ldots, U_2^{k-1} U_2^k, E_{k+1})$, and $f(V_1 \ldots U_2^{k-1} E_2) = (E_{k+1}, V_1, \ldots, U_2^{k-1} E_2, E_{k+1})$, which are of the form

Dual arguments prove the result for the other labelings. We also note that by definition map $f$ is injective and surjective. □

Remark 3.4. Each perfect matching of $\mathcal{G}_C$ is in correspondence with just only one object of the $\mathbb{A}_{n-1}$-Dyck paths category associated to the admissible subchain $C = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$.

For each Dyck path $Y = y_1 \ldots y_{2n}$ with $n - 1$ peaks, we construct a family of words $Y \cap X_C \in H_n^*$ such that:

$$Y \cap X_C = \{Y \cap V^z \mid V^z \in X_C\},$$

(3.20)

where

$$Y \cap V^z = \begin{cases} V^z, & \text{if there exists } j \text{ such that } y_j = v^z_j \text{ for } 1 < j < 2n, \\ E_n, & \text{otherwise,} \end{cases}$$

(3.21)

with $V^z = v_1^z \ldots v_{2n}^z$ in $X_C$. For the set $Y \cap X_C$, it can be defined a relation $\sim$ such that

$$Y \cap V^{z_1} \sim Y \cap V^{z_2} \text{ if and only if } Y \cap V^{z_1} \text{ and } Y \cap V^{z_2} \text{ are the same word.}$$

(3.22)

In this case, $\sim$ is an equivalence relation and $(Y \cap X_C)/\sim$ is denoted by $[Y \cap X_C]$. Also, we remind that a Dyck path $Y$ can be written as the word $UWD = U w_1 \ldots w_{n-1} D$, where $y_1 = U$, $y_{2n} = D$ and, $w_i = y_2 y_{2i+1}$. 
Lemma 3.7. Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $\mathbf{n}$ and let $Y$ a Dyck path of length $2n$ with exactly $n - 1$ peaks. Then, there is a bijective correspondence between the set $[Y \cap X_{\mathcal{C}}]$ and the set of perfect matchings of the snake graph belonging to $\mathcal{G}_\mathcal{C}$ and induced by the words $w_t = UD$ in $Y$.

Proof. Let $\mathcal{C}$ be an admissible subchain of $\mathbf{n}$ and $Y = UWD$ be a Dyck path in $S$, then by Proposition 3.2 there are $l, r \in \mathbb{Z}_{>0}$ with $1 \leq l \leq r \leq n - 1$ such that $w_t = UD$ for $l \leq t \leq r$ and $w_t = DU$ otherwise. Now, let $G_{\mathcal{G}_\mathcal{C}}[l, d]$ be a snake graph belonging to $G_{\mathcal{C}}$ induced by $Y$. Define a map $g : [Y \cap X_{\mathcal{C}}] \to \text{Match}(G_{\mathcal{G}_\mathcal{C}[l, d]})$ such that:

(I) If $1 < l \leq r < n - 1$, then $g([Y \cap V^i]) = g(E_n \ldots E_n V^i_{l-1} \ldots V^i_l E_n \ldots E_n) = (V^i_{l-1}, \ldots, V^i_r)$.

(II) If $l = 1$ and $1 \leq r < n - 1$, then $g([Y \cap V^i]) = g(V^i_1 \ldots V^i_r E_n \ldots E_n) = (E_n, V^i_1, \ldots, V^i_r)$.

(III) If $r = n - 1$ and $1 \leq l \leq r = n - 1$, then $g([Y \cap V^i]) = g(E_n \ldots E_n V^i_{l-1} \ldots V^i_{n-2}) = (V^i_{l-1}, \ldots, V^i_{n-2}, E_n)$.

(IV) If $l = 1$ and $r = n - 1$, then $g = f$.

Since in the four cases $g$ is a restriction of $f$. It follows that $g$ is a bijection as a consequence of Lemma 3.6 □

3.3.3 Cluster Variables Formula Based on Dyck Paths Categories

In this section, Dyck paths categories are used to give a formula for cluster variables of cluster algebras of Dynkin type $\mathbf{A}_n$, to do that, we use the category of Dyck paths associated to an admissible subchain. We also present a connection between cluster variables of algebras of type $\mathbf{A}_{n-1}$ and Dyck paths with $n - 1$ peaks.

Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $\mathbf{n}$ and let $Y = UWD$ be a Dyck path in $S$, then we define the monomials

$$\eta_Y = \prod_{UD = w_t \in Y} x_i,$$

and

$$X_V = \prod_{m \in M_V} x_m,$$

with $M_V$ being the set of indices $m$ such that

$$m = \begin{cases} i + 1, & \text{if } U^i_j \in V, \\ i, & \text{if } U^i_j \in V, \\ 0, & \text{if } E_n \in V, \end{cases}$$

(3.25)
$V \in [Y \cap X_C]$. For this case $x_0 = 1$.

The following theorem gives the cluster variable associated to a Dyck path in the set $S$ and its connection with cluster algebras of type $A_{n-1}$.

**Theorem 3.4.** Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $n{-}1$, $Y = UWD$ a Dyck path with $n{-}1$ peaks and $M$ the set of all cluster variables of a cluster algebra of type $A_{n-1}$ with $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_m\}$ the sets of sinks and sources, respectively. Then:

(i) The cluster variable associated to $Y$ in the category $C_{2n}$ is given by

$$X_Y = (\eta_Y)^{-1} \left( \sum_{V \in [Y \cap X_C]} X_V \right).$$

(ii) There exists a bijective correspondence between Dyck paths with $n{-}1$ peaks and the set $M \setminus x_0$ with $x_0$ the initial seed.

**Proof.** Let $\mathcal{C} = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$ be an admissible subchain of $n{-}1$, and let $T_C$ be the triangulation of the polygon with $n+2$ vertices given by $\mathcal{C}$.

Let $\alpha_{l,r}$ be a diagonal that is not in $T_C$ that cuts the diagonals $\alpha_1, \ldots, \alpha_r \in T_C$. We define a functor $\chi : \mathcal{C}_T \to C_{2n}$ such that $\chi(\alpha_{l,r}) = UW_{l,r}D$, where

$$w_j = \begin{cases} UD, & \text{if } l \leq j \leq r, \\ DU, & \text{otherwise,} \end{cases}$$

and for any pivoting elementary move $E : \alpha_{r,l} \to \alpha'_{r',l'}$, $\chi(E)$ is the elementary shift $F = f_{l_1} \circ \cdots \circ f_{l_k}$ from $UW_{l,r}D$ to $UW_{l',r'}D$. Theorems 1.9 and 3.1 allow us to establish the following sequence of equivalences:

$$\mathcal{C}_T \cong \text{Mod } Q_{T_C} \cong C_{2n},$$

therefore $\chi$ is a categorical equivalence. Thus,

(i) Functor $\chi$ and Lemma 3.7 allow to establish that $x_\gamma = X_Y$.

(ii) The map $\psi : S \to M \setminus x_0$ such that $\psi(Y) = X_Y$ is a bijection as a consequence of Theorem 1.8 and the definition of functor $\chi$. We are done. $\square$
For instance, let $\mathcal{C} = \{j_1 = 1, i_1 = 2, j_2 = 4\}$ be an admissible subchain of 4, the set $X_C$ is in correspondence with the objects of $\mathcal{C}_{10}$ shown in Figure 3.10.

\begin{figure}[h]
\centering
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{a}
\caption{$E_5 \oplus E_5 \oplus U_1^3$}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{b}
\caption{$E_5 \oplus U_2^3 \oplus E_5$}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{c}
\caption{$E_5 \oplus U_2^3 \oplus U_3^3$}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{d}
\caption{$U_1^3 \oplus E_5 \oplus U_3^3$}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{e}
\caption{$U_1^3 \oplus U_2^3 \oplus E_5$}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{f}
\caption{$U_1^3 \oplus U_2^3 \oplus U_3^3$}
\end{subfigure}
\begin{subfigure}{0.25\textwidth}
\centering
\includegraphics[width=\textwidth]{g}
\caption{$U_2^3 \oplus U_1^3 \oplus U_3^3$}
\end{subfigure}
\caption{Objects in $\mathcal{C}_{10}$.}
\end{figure}

Then, for $Y = UDUUDDUDDUD$, we define the set $Y \cap X_C$ such that

$$[Y \cap X_C] = \{E_5 E_5 U_1^3, E_5 U_2^3 E_5, U_2^3 U_1^3 U_3^3\}.$$  

Thus, identities (3.23), (3.24) and (3.25) define the polynomials

$$\eta_Y = x_2 x_3, \; X_{E_5 E_5 U_1^3} = x_0 x_0 x_4, \; X_{E_5 U_2^3 E_5} = x_0 x_2 x_0, \; X_{U_2^3 U_1^3 U_3^3} = x_3 x_1 x_4,$$

therefore, the cluster variable associated to the Dyck path $Y$ is given by the expression

$$X_Y = \frac{x_4 + x_2 + x_3 x_1 x_4}{x_2 x_3}.$$
In this chapter, we describe the way that Dyck paths are used in different kind of algebraic structures. In section 4.1, we prove that frieze patterns arise from Dyck paths, to do that, diamonds of \( \mathcal{A}_n \) are introduced, in particular, we prove that some new diamonds are in bijective correspondence with Dyck paths, triangulations of an \((n + 3)\) polygon, and a family of frieze vectors. This approach allows us to write frieze patterns as a direct sum of indecomposable objects of the category of Dyck paths and it is also given a categorification of the Catalan triangle in the sense of Ringel and Fahr [90]. In section 4.2, we define Brauer configuration algebras whose indecomposable projective modules are in bijective correspondence with Dyck paths, some combinatorial properties of the Catalan triangle allow us to establish formulas for the dimension of these algebras and its corresponding centers.

### 4.1 Frieze Patterns Arising from Dyck Paths

In this section, we introduce a basic set called diamond which is used to build frieze patterns associated to triangulations of a polygon.

#### 4.1.1 Diamonds of \( \mathcal{A}_n \)

Let \( \mathbf{R} \) be an integral domain, a *diamond* \( A = (a_{i,j}) \) of \( \mathcal{A}_n \) is an array

\[
\begin{array}{cccc}
  & a_{2,0} & & \\
 a_{1,1} & & a_{2,1} & \\
 & a_{1,2} & & \ddots & \\
 & & \ddots & a_{2,n-1} \\
 & & & a_{1,n} & a_{2,n} \\
 & & & & a_{1,n+1} \\
\end{array}
\]

that satisfies the following conditions:
(D1) \(a_{2,0} = a_{1,n+1} = 1\),

(D2) \(a_{1,j}a_{2,j} - a_{2,j-1}a_{1,j+1} = 1\) for \(1 \leq j \leq n\),

with \(a_{i,j} \in \mathbb{R}\) and 1 the identity element of \(\mathbb{R}\).

If \(\mathbb{R} = \mathbb{Z}\), \(A\) is called integral diamond, if it also satisfies

(D3) \(a_{1,1} = a\) (or \(a_{1,1} = a + m_a\)), \(a_{2,1} = a + m_a\) (or \(a_{2,1} = a\)) and \(a_{1,2} = a^2 + am_a - 1\), with \(1 \leq a \leq \left\lfloor \frac{n+2}{2} \right\rfloor\), \(1 \leq m_1 \leq n\) and \(0 \leq m_a \leq n + 2(1 - a)\) if \(a > 1\), 

\(A\) is called positive integral diamond.

Two diamonds \(A\) and \(B\) of \(\mathbb{A}_n\) are a couple if and only if \(a_{2,j} = b_{1,j}\) for \(1 \leq j \leq n\) (denoted by \(A \models B\)). A set \(\{A^t\}_{t \geq 0}\) is an \(\mathbb{A}_n\)–sequence of couples of \(\mathbb{A}_n\) if and only if \(A^r \models A^{r+1}\) for \(r \geq 0\). An \(\mathbb{A}_n\)–sequence of couples \(\{A^t\}_{t \geq 0}\) is a \(p\)–cycle if there is a \(p \in \mathbb{N}\) such that \(A^t = A^{t+p}\).

For example, let \(\mathbb{R} = \mathbb{Z}\), the sets \(\{A^t\}_{t \geq 0}\) and \(\{B^t\}_{t \geq 0}\) are \(\mathbb{A}_1\)–sequences of couples which are 2–cycles with \(A^{2k} = B^{2k+1} = A\), \(A^{2k+1} = B^{2k} = B\) and \(k \geq 0\).

\[
\begin{array}{ccc}
A & 1 & 2 \\
B & 2 & 1 \\
1 & 1 & 1
\end{array}
\]

In general, it can be written an \(\mathbb{A}_n\)–sequence of couples \(\{A^t\}_{t \geq 0}\) as an \(\mathbb{A}_n\)–array \(C_{A^t} = (c_{i,j})\) such that \(c_{t+1,j} = a_{1,j}^t\) and \(c_{t+1,0} = c_{t+1,n+1} = 1\), for \(t \geq 0\). For the previous example,

\[
\begin{array}{cccccc}
C_{A^t} & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots
\end{array}
\quad
\begin{array}{cccccc}
C_{B^t} & 2 & 1 & 2 & 1 & \ldots & 1 & 1 & 1 & \ldots \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots
\end{array}
\]

\(C_{A^t}\) and \(C_{B^t}\) are \(\mathbb{A}_1\)–arrays associated to \(\{A^t\}_{t \geq 0}\) and \(\{B^t\}_{t \geq 0}\), respectively.

If the \(\mathbb{A}_n\)–sequence of couples is finite of length \(m\), it can be associated an infinity \(\mathbb{A}_n\)–array, \(C_{A^t}^m = (c_{i,j}^m)\) such that

\[
c_{(t+1)+km,j}^m = a_{1,j}^t, \quad c_{(t+1)+km,0}^m = c_{(t+1)+km,n+1}^m = 1, \quad \text{for } k \in \mathbb{Z}.
\]

(4.1)

for \(k \in \mathbb{Z}\). For any \(\mathbb{A}_n\)–sequence of couples \(\{A^t\}_{t \geq 0}\), we can take an \(\mathbb{A}_n\)–subsequence \(\{B^z\}_{z \geq 0}\) for \(B^z = A^{2z+x}\) and some \(x \geq t\). In particular, if \(\{A^t\}_{t \geq 0}\) is a \(p\)–cycle, we take the subsequence \(\{B^{s_0}\}_{0 \leq s_0 \leq p-1}\) such that \(B^{s_0} = A^t\). This subsequence is called minimal \(p\)–cycle of \(\{A^t\}_{t \geq 0}\).

Henceforth, we present main results regarding diamonds of \(\mathbb{A}_n\).
**Proposition 4.1.** Let \( \{A^i\}_{i \geq 0} \) be a \( p \)-cycle and let \( B = \{B^s\}_{0 \leq s \leq p-1} \) be its minimal \( p \)-cycle. Then, the \( C_B^0 \) is a frieze pattern of order \( n + 3 \). In particular, \( p \) divides \( n + 3 \).

**Proof.** Let \( C_B^0 = (c_{ij}^0) \) be the infinity \( \mathbb{A}_n \)-array associated to \( \{B^s\}_{0 \leq s \leq p-1} \), identity (4.1) implies that

\[
(c_{(s_0+1)+kp,j}^0 = a_{1,j}^{s_0}, c_{(s_0+1)+kp,0}^0 = c_{(s_0+1)+kp,n+1}^0 = 1,
\]

for \( k \in \mathbb{Z} \), given that \( \{A^i\}_{i \geq 0} \) is a \( p \)-cycle, then, \( C_B^0 \) is a frieze pattern. \( \square \)

**Proposition 4.2.** Let \( \{A^i\}_{i \geq 0} \) be a \( p \)-cycle of length \( 2p \), then the subsequences \( \{B^s\}_{0 \leq s \leq p-1} \) generate the same frieze pattern of order \( n + 3 \), for \( 0 \leq i \leq p-1 \), and \( B^s = A^{i+s} \).

**Proof.** Let \( \{A^i\}_{i \geq 0} \) be a \( p \)-cycle of length \( 2p \), let \( C_A^0 = (c_{ij}^0) \) and \( C_B^0 = (c_{ij}^{0'}) \) be the infinity arrays of the subsequences \( A = \{B^s\}_{0 \leq s \leq p-1} \) and \( B = \{B^s\}_{0 \leq s \leq p-1} \) of \( \{A^i\}_{i \geq 0} \) for \( 0 \leq i < i' \leq p-1 \). Applying the translation \( s'_i = s_i - |i' - i| \),

\[
c_{s'_i+1+kp,j}^0 = a_{1,j}^{s'_i} = a_{1,j}^{s_i-|i'-i|+i'} = a_{i'_j}^{s_i+i} = c_{s_i+1+kp,j}^0. \quad \text{We are done.} \quad \square \]

**Lemma 4.1.** Let \( \{A^i\}_{i \geq 0} \) be a sequence of couples, then \( \{A^i\}_{i \geq 0} \) is generated by \( A^0 \). In particular, \( A^0 \) generates a \( p \)-cycle for some \( p > 0 \).

**Proof.** Let \( \{A^i\}_{i \geq 0} \) be a sequence of couples, then

\[
a_{2,j}^x = \frac{1 + (a_{2,j-1}^x)(a_{2,j+1}^{x-1})}{a_{2,j}^{x-1}}, \tag{4.2}
\]

for \( 1 \leq j \leq n \), and \( x \geq t \), then \( a_{2,j}^x \) can be written by using the set \( \{a_{2,j}^0\}_{1 \leq j \leq n} \) for \( x > 0 \). In particular, the set \( \{a_{2,1}^0, \ldots, a_{2,n}^0\} \) is a seed of the cluster algebra associated to the quiver shown in Figure 3.6. Since the cluster variables are finite in the case \( \mathbb{A}_n \), then there is \( p = n + 3 \) (in some cases, it is not minimal) such that \( A^0 = A^{n+3} \).

**Theorem 4.1.** Let \( A \) be a diamond of \( \mathbb{A}_n \), then \( A \) generates a frieze pattern.

**Proof.** It is a direct consequence of Lemma 4.1 and Proposition 4.1 \( \square \)

For instance, the diamonds \( A \) and \( B \) generate the following frieze pattern.

\[
\begin{array}{cccccccc}
\cdots & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & 1 & 2 & 1 & 2 & \cdots \\
\cdots & 1 & 1 & 1 & 1 & \cdots 
\end{array}
\]

### 4.1.2 Seed Vectors

In this section, we give an algorithm to build a family of positive integral frieze vectors associated to the quiver shown in Figure 3.6. These vectors help to find a connection
between the positive integral diamonds of $A_n$, triangulations, and Dyck paths.

Let $A$ be a diamond of $A_n$, we can write the first column as a vector $v_A = (a_1, \ldots, a_n)$ where $a_j = a_{1,j}$.

**Proposition 4.3.** If $v = (a_1, \ldots, a_n)$ is a vector associated to a positive integral diamond of $A_n$ with $a_n = 1$, then the vector $v' = (a_1, a_i, a_i + a_{i+1}, a_{i+1}, \ldots, a_{n-1})$ generates a positive integral diamond of $A_n$, for $1 \leq i < n$.

**Proof.** Let $v_A = (a_1, \ldots, a_n)$ be a vector associated to a positive integral diamond $A = (a_{j,m})$ of $A_n$, we take the vector $v_A + i = (a_1, \ldots, a_i, a_i + a_{i+1}, a_{i+1}, \ldots, a_n - 1)$ and the array $A + i$ of the following form:

\[
b_{1,m} = \begin{cases} a_{1,m}, & \text{if } m \leq i, \\ a_{1,i} + a_{1,i+1}, & \text{if } m = i + 1, \\ a_{1,m-1}, & \text{if } m > i + 1, \end{cases}
\]

and

\[
b_{2,m} = \begin{cases} a_{2,m}, & \text{if } m \leq i - 1, \\ a_{2,i-1} + a_{2,i}, & \text{if } m = i, \\ a_{2,m-1}, & \text{if } m \geq i + 1, \end{cases}
\]

then $b_{1,m}b_{2,m} - b_{2,m-1}b_{1,m+1} = 1$, for $1 \leq m \leq n$ and $1 \leq i < n$. Therefore $A + i$ is a positive integral diamond of $A_n$. \hfill \square

**Proposition 4.4.** The vector $v_{n,z} = (a_1, \ldots, a_n)$ with

\[
a_i = \begin{cases} z + 1 - i, & \text{if } i < z, \\ 1, & \text{if } i \geq z, \end{cases}
\]

is in bijective correspondence with a positive integral diamond of $A_n$, for $z \in \{1, \ldots, n+1\}$.

**Proof.** Let $v_{n,z}$ be a vector and let $z$ be a natural number between $1$ and $n + 1$, we define a positive integral diamond $A$ with $a_{1,i} = a_i$ and $a_{2,i} = b_i$ where

\[
b_i = \begin{cases} 1, & \text{if } i < z, \\ i + 2 - z, & \text{if } i \geq z, \end{cases}
\]

then $a_{1,i}a_{2,i} - a_{2,i-1}a_{2,i+1} = 1$ for $1 \leq i \leq n$. \hfill \square

**Remark 4.1.** $v_{n,z}$ is called a seed vector. The vector $v_{n,z} = (b_1, \ldots, b_n)$ defines a positive integral diamond $B$ of $A_n$ such that $b_{2,i}$ satisfies the following identity

\[
b_{2,i} = \begin{cases} i - 1, & \text{if } i < z - 1, \\ (b_{1,i} + 1)z - 1, & \text{if } z - 2 < i < n, \\ z, & \text{if } i = n, \end{cases}
\]

and $b_i = b_{1,i}$ is defined as in (4.6).
**Proposition 4.5.** The positive integral diamonds $A$ and $B$ generated by $v_{n,z}$ and $v^{n,z}$ respectively are a couple.

**Proof.** It is a direct consequence of Proposition 4.4 and Lemma 4.1. □

The number of ways of applying recursively Proposition 4.3 to a vector $w_A = (a_1, \ldots, a_{z-1}, 1, \ldots, 1) \in \mathbb{N}^n$ is given by the next identity (denoted by $f_{n,z}$),

$$f_{n,z} = \begin{cases} 
\sum_{i=z-1}^{n} f_{n-1,i}, & \text{if } z > 1, \\
\sum_{i=1}^{n} f_{n-1,i}, & \text{if } z = 1,
\end{cases} \quad (4.8)$$

where it is included the trivial move $w_{A+0} = w_A$, for $n > 1$, and any $z \in \{1, \ldots, n+1\}$. In fact, we represent these numbers by the following triangle

\[
\begin{array}{ccccccc}
  & f_{1,1} & f_{1,2} & f_{2,1} & f_{2,2} & f_{3,1} & f_{3,2} & f_{3,3} & f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} & f_{4,5} & \cdots \\
  f_{1,2} & f_{2,1} & f_{2,2} & f_{3,1} & f_{3,2} & f_{3,3} & f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} & f_{4,5} & \cdots \\
  f_{2,3} & f_{3,2} & f_{3,3} & f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} & f_{4,5} & \cdots \\
  f_{3,4} & f_{4,3} & f_{4,4} & f_{4,5} & \cdots \\
  f_{4,5} & \cdots \\
\end{array}
\] \quad (4.9)

for any vector as before. Since the first possibilities are $v_{1,1} = (1)$ and $v_{1,2} = (2)$, then $f_{1,1} = 1$ and $f_{1,2} = 1$. The previous triangle appears in the OEIS as A009766 (Catalan triangle [90]). In particular, we generate all positive integral diamonds of $A_n$ via the seed vectors $v_{n,z}$. For example, for $n = 3$, all vectors that generate positive integral diamonds of $A_3$ are:

\[
\begin{array}{ccccccc}
(1,1,1) & (2,1,1) & (3,2,1) & (4,3,2) \\
(1,1,2) & (2,1,2) & (3,2,3) \\
(1,2,1) & (2,3,1) & (3,5,2) \\
(1,2,3) & (2,3,4) \\
(1,3,2) & (2,5,3) \\
\end{array}
\]

Let $G = UD \ldots UD \ldots$ be a Dyck path of length $2n$ and let $m_i$ be the number of $U$'s before of $i$-th $D$ in $G$, then, $G$ can be written as a vector $v_G = (v_1, \ldots, v_{n-1})$ where $v_i = m_i - i + 1$. If $G$ is the Dyck path shown in Figure 4.1 then $G$ has associated the vector $v_G = (5, 4, 3, 3, 5, 4, 3, 2)$. 
If we take the case \( n = 3 \), all the vectors are

\[
(1, 1, 1) \quad (2, 1, 1) \quad (3, 2, 1) \quad (4, 3, 2) \\
(1, 1, 2) \quad (2, 1, 2) \quad (3, 2, 2) \\
(1, 2, 1) \quad (2, 2, 1) \quad (3, 3, 2) \\
(1, 2, 2) \quad (2, 2, 2) \\
(1, 3, 2) \quad (2, 3, 2)
\]

Note that, the number of generating vectors is given by the Catalan numbers.

In what follows, it is defined a map between the vectors associated to positive integral diamonds of \( A_n \) and Dyck paths by using a relation over the coordinates of a vector \( u = (a_1, \ldots, a_m) \). The map \( T_i \) is defined in such a way that, \( T_i : \mathbb{N}^m \rightarrow \mathbb{N} \) and:

- If \( a_i - a_k > 0 \) for some \( k \in \{1, \ldots, i\} \), we take \( \max \{k\} \) and we write \( r_1 = a_i - a_k \).
  
  Again, we take \( \max \{k\} \) such that \( r_1 - a_k > 0 \) and we write \( r_2 = r_1 - a_k \), this process ends when there is no a \( k \) such that \( r_t - a_k > 0 \), then, \( T_i(u) = r_t + t \). for some \( t \).

- If \( a_i - a_k \leq 0 \) for all \( k \in \{1, \ldots, i\} \), then \( T_i(u) = a_i \).

For instance, we take a vector \( u = (14, 52, 4, 23, 9, 2) \), then \( T_1(u) = 14 \), \( T_2(u) = 13 \), \( T_3(u) = 4 \), \( T_4(u) = 8 \), \( T_5(u) = 3 \), and \( T_6(u) = 2 \).

**Proposition 4.6.** Let \( v_{n,z} \) be a seed vector, then \((T_1(v_{n,z}), \ldots, T_n(v_{n,z}))\) describes a Dyck path of length \( 2(n+1) \).

**Proof.** For any \( z \in \{1, \ldots, n+1\} \), \( T_i(v_{n,z}) = a_i \) with \( a_i \) given by identity \((4.5)\), then there is a word \( G_{v_{n,z}} = w_1 \ldots w_{2(n+1)} \in \{U, D\}^* \) such that

\[
G_{v_{n,z}} = U \underbrace{\ldots U \overbrace{D \ldots D}^{z-1} U \ldots U \overbrace{D \ldots D}^{z-1}}_{z-1} D \ldots D,
\]

for any left factor \( u_s \) in \( G_{v_{n,z}} \) of length \( s \in \{1, \ldots, 2(n+1)\} \), \( 0 \leq |u_s|_U - |u_s|_D \leq z - 1 \), therefore \( G_{v_{n,z}} \in D_{2(n+1)} \). \( \square \)
Proof. Let \( v_A = (a_1, \ldots, a_n) \) be a vector associated to a positive integral diamond \( A \) of \( \mathbb{A}_n \) with \( a_n = 1 \), such that \((T_1(v_A), \ldots, T_n(v_A))\) describes a Dyck path in \( D_{2(n+1)} \). Then \((T_1(v_{A+1}), \ldots, T_n(v_{A+1}))\) describes a Dyck path in \( D_{2(n+1)} \).

Let \( v_A = (a_1, \ldots, a_n) \) be a vector associated to a positive integral diamond \( A \) with \( a_n = 1 \), then there exists a Dyck path \( G_{v_A} \in D_{2(n+1)} \) such that any left factor \( u_s \) of length \( s \) satisfies \(|u_s|_U \geq |u_s|_D \) for \( 1 \leq s \leq 2(n+1) \). Let \( v_{A+i} \) be a vector associated to the positive integral diamond \( A+i \) with

\[
T_m(v_{A+i}) = \begin{cases} 
    T_m(v_A), & \text{if } 1 \leq m \leq i, \\
    T_m(v_A) + 1, & \text{if } m = i + 1, \\
    T_{m-1}(v_A), & \text{if } m > i + 1,
\end{cases}
\tag{4.11}
\]

then there is a word \( G_{A+i} = w'_1, \ldots, w'_{2(n+1)} \) in \( \{U, D\}^* \), we take the index \( m_1 \) of the \( i \)-th \( D \) in \( G_{A+i} \), any left factor \( u'_s \) in \( G_{A+i} \) satisfies the identities

\[
|u'_s|_U = \begin{cases} 
    |u_s|_U, & \text{if } 1 \leq s \leq m_1, \\
    |u_{m_1} - 1|_U, & \text{if } s = m_1 + 1, \\
    |u_{s-2}|_U + 1, & \text{if } s \geq m_1 + 2,
\end{cases}
\tag{4.12}
\]

and

\[
|u'_s|_D = \begin{cases} 
    |u_s|_D, & \text{if } 1 \leq s \leq m_1, \\
    |u_{m_1} - 1|_D, & \text{if } s = m_1 + 1, \\
    |u_{s-2}|_D + 1, & \text{if } s \geq m_1 + 2,
\end{cases}
\tag{4.13}
\]

then, we have the following possibilities:

- If \( 1 \leq s \leq m_1 \), \( |u'_s|_U = |u_s|_U \geq |u_s|_D = |u'_s|_D \).
- If \( s = m_1 + 1 \), \( |u'_s|_U = |u_{m_1} - 1|_U > |u_{m_1}|_D = |u'_{m_1 + 1}|_D \).
- If \( m_1 + 2 \leq s \leq 2(n+1) \), \( |u'_s|_U = |u_{s-2}|_U + 1 \geq |u_{s-2}|_D + 1 = |u'_{s}|_D \).

Therefore, \( G_{A+i} \in D_{2(n+1)} \). \( \square \)

Lemma 4.2. There is a bijective correspondence between the set of all vectors associated to positive integral diamonds of \( \mathbb{A}_n \) and the set of all Dyck paths of length \( 2(n+1) \).

Proof. Let \( D_{\mathbb{A}_n} \) be the set of all vectors associated to positive integral diamonds of \( \mathbb{A}_n \) and let \( D_{2(n+1)} \) be the set of all Dyck paths of length \( 2(n+1) \), then, we define a map \( f : D_{\mathbb{A}_n} \rightarrow D_{2(n+1)} \) with \( f(u_A) = (T_1(u_A), \ldots, T_n(u_A)) \), Propositions 4.6 and 4.7 allow us to establish that \( f \) is well defined. We should prove that the map \( f \) is one to one. Suppose that \( u_A \) different from \( v_B \), we take the minimum \( k \) such that \( u_k \neq v_k \). If \( k = 1 \) then \( T_1(u_A) \neq T_1(v_B) \). If \( k > 1 \), \( u_k = m(u_{k-1}) + a \) and \( v_k = m'(u_{k-1}) + a \) with \( m \neq m' \) is a consequence of Proposition 4.3 then \( r_{t_{uk}} \neq r_{t_{vk}} \), therefore \( T_k(u_A) \neq T_k(v_B) \). \( \square \)

Figure 4.2 shows a positive integral diamond of \( \mathbb{A}_n \) and its corresponding Dyck path.
CHAPTER 4. SOME APPLICATIONS OF CATALAN NUMBERS

Figure 4.2. Diamond (left) and its corresponding Dyck path (right).

An alternative way of writing a Dyck path \( G \in \mathcal{D}_{2(n+1)} \) can be defined by using a vector \( \lambda_G = (\lambda_1, \ldots, \lambda_n) \) where \( \lambda_i \) is the number of \( D \)'s before of \( (n+2-i) \)-th \( U \) in \( G \) (see [64, 92]), for example, Dyck path of Figure 4.2 has associated the following vector \( \lambda_G = (1, 1, 0, 0) \).

In the case \( n = 3 \), all vectors are

\[
(3, 2, 1) \quad (3, 2, 0) \quad (3, 0, 0) \quad (0, 0, 0) \\
(2, 2, 1) \quad (2, 2, 0) \quad (2, 0, 0) \\
(3, 1, 1) \quad (3, 1, 0) \quad (1, 0, 0) \\
(2, 1, 1) \quad (2, 1, 0) \\
(1, 1, 1) \quad (1, 1, 0)
\]

Let \( \lambda \) be a vector associated to a Dyck path of length \( 2(n+1) \), a triangulation of an \( n+3 \) polygon can be defined through the use of \( \lambda \) as follows:

- Fix a labeling in the vertices of polygon \( K_0^{n+3} = (v_0^{n+3}, \ldots, v_{n+2}^{n+3}) \) with \( v_i^{n+3} = i \), for \( 0 \leq i \leq n+2 \).
- For \( \lambda_i \), we draw a diagonal \( l_i^{\lambda_i} \) between \( \lambda_i \) and \( \lambda_i + 2 \). After that, we label the last polygon with \( n+3-i \) vertices \( K_i^{n+3-i} = (v_0^{n+3-i}, \ldots, v_{n+2-i}^{n+3-i}) \), and

\[
v_j^{n+3-i} = \begin{cases} 
  v_j^{n+3-(i-1)}, & \text{if } j \leq \lambda_i, \\
  v_{j+1}^{n+3-(i-1)} - 1, & \text{if } j > \lambda_i,
\end{cases}
\]

for \( i = 1, \ldots, n \).

Figure 4.3. Triangulation of an hexagon.
The previous algorithm describes that if \( t_i^\lambda \) is a diagonal then it does not cross the diagonals \( t_i^\lambda, \ldots, t_{i-1}^\lambda \) for \( 1 \leq i \leq n \). For instance, let \( \lambda_G = (2, 2, 1) \) be the vector associated to \( G = UDUDDUDUDD \), then the triangulation of \( \lambda_G \) is shown in Figure 4.3.

If we fix a labeling \( K \) over all vertices of a polygon with \( n + 3 \) vertices, a triangulation \( T \) is written as a sequence \( T = (l_1^v, \ldots, l_n^v) \), where \( v_i \) belongs to the set of vertices.

**Lemma 4.3.** There is a bijective correspondence between the set of all triangulations of a polygon with \( n + 3 \) vertices and the set of all Dyck paths of length \( 2(n + 1) \).

**Proof.** Let \( T_n \) be the set of all triangulations of a polygon with \( n + 3 \) vertices, then, we can define a map \( g : D_{2(n+1)} \to T_n \) with \( g(\lambda) = T_\lambda \). We should prove that \( g \) is one to one. Fix a labeling \( K \) and suppose \( g(\lambda_G) = g(\sigma_G) \), then \( (l_1^\lambda, \ldots, l_n^\lambda) = (l_1^{\sigma}, \ldots, l_n^{\sigma}) \), provided that \( l_j^\lambda = l_j^{\sigma} \), there are diagonals \( \lambda_j \to (\lambda_j + 2) \) and \( \sigma_j \to (\sigma_j + 2) \), therefore \( \lambda_j = \sigma_j \) for \( j = 1, \ldots, n \). \( \square \)

The next theorem presents the main result regarding the positive integral diamonds of \( \mathbb{A}_n \) and the triangulations of an \( n + 3 \) polygon.

**Theorem 4.2.** There is a bijective correspondence between the set of all vectors associated to positive integral diamonds of \( \mathbb{A}_n \) and triangulations of a polygon with \( n + 3 \) vertices.

**Proof.** We fix a labeling \( K \) in a polygon with \( n + 3 \) vertices, the map \( F : \mathbb{D}_{\mathbb{A}_n} \to T_n \) defined by \( F(v_A) = (g \circ f)(v_A) \) is a bijection (Lemmas 4.2 and 4.3). \( \square \)

Figure 4.4 presents the bijective correspondence between a positive integral diamond of \( \mathbb{A}_4 \), a Dyck path of length 10, and a triangulation of a polygon with 7 vertices.

![Figure 4.4](image-url)

**Figure 4.4.** Connection via the map \( F \).
4.1.3 Frieze Patterns and Dyck Paths

In this section, we describe an algebraic interpretation of frieze patterns as a direct sum of Dyck Paths.

**Lemma 4.4.** The vectors $v_{n,z}$ and $v^{n,z}$ generate the same triangulation except for one anti-clockwise rotation.

**Proof.** Let $v_{n,z}$ and $v^{n,z}$ be frieze vectors, fixed a labeling $K_1$ in an $n+3$ polygon, applying map $F$

\[
F(v_{n,z}) = \left( \frac{n}{1}, \ldots, \frac{z}{n-z}, \frac{0}{n-z+1}, \ldots, \frac{0}{n} \right) \quad \text{and} \quad F(v^{n,z}) = \left( \frac{n-1}{1}, \ldots, \frac{z-1}{n-z}, \frac{z-1}{n-z+1}, \ldots, \frac{1}{n} \right),
\]

if we change $K_1$ by $K_2$ in the following way:

- the vertex $k \in K_1$ is $k-1 \in K_2$ for $1 \leq k \leq n+2$,
- the vertex $0 \in K_1$ is $n+2 \in K_2$,

the diagonals from $0$ to $r_1$ in $K_1$ are diagonals from $r_1-1$ to $n+2$ in $K_2$, and the diagonals from $r_2$ in $K_1$ are diagonals from $r_2-1$ in $K_2$, for $0 \leq r_1 \leq z \leq r_2 \leq n$. Therefore $F(v_{n,z}) \in K_1$ is equal to $F(v^{n,z}) \in K_2$. \qed

Note that, there exists a permutation

\[
\sigma = \begin{pmatrix} 1 & 2 & \ldots & n-z-1 & n-z & n-z+1 & n-z+2 & \ldots & n-1 & n \\ 1 & 2 & \ldots & n-z-1 & n-z & n & n-1 & \ldots & n-z+2 & n-z+1 \end{pmatrix},
\]

in $S_n$ that describes a bijection between the coordinates of the vector $F(v_{n,z}) = (u_1, \ldots, u_n)$ and the vector $F(v^{n,z}) = (u'_1, \ldots, u'_n)$ such that $\sigma(F(v_{n,z})) = (u'_{\sigma(1)}, \ldots, u'_{\sigma(n)}) = (u'_1, \ldots, u'_n) = F(v^{n,z})$ in $K_2$. In general, if $v$ and $w$ generate the same triangulation except for one anti-clockwise rotation, then there exists a permutation $\sigma' \in S_n$ such that $\sigma'(F(v)) = F(w)$ in $K_2$. 

---

(a) $f(v_{n,z})$

(b) $f(v^{n,z})$
Lemma 4.5. Let $A$ and $B$ be positive integral diamonds of $A_n$, $A$ and $B$ are a couple, and $v_A = (a_1, \ldots, a_z, \ldots, a_n)$ with $a_t = 1$ for $z \leq t \leq n$. If $v_A$ and $v_B$ generate the same triangulation except for one anti-clockwise rotation. Then $v_{A+i} = (a_1, \ldots, a_{i-1}, a_{i-1} + a_i, a_{i+1}, \ldots, a_{n-1})$ and $v_{B+i-1} = (b_1, \ldots, b_{i-2}, b_{i-2} + b_{i-1}, b_{i-1}, \ldots, b_{n-1})$ generate the same triangulation except for one anti-clockwise rotation for $z - 1 \leq i \leq n$, $i \geq 2$.

Proof. Let $v_A$ and $v_B$ are vectors, since $v_A$ and $v_B$ generate the same triangulation except for one anti-clockwise rotation, then, there exists a permutation $\sigma \in S_n$ such that $\sigma(F(v_A)) = F(v_B)$ in $K_2$. The following options arise from the map $f$, such that:

1. If $i > z \geq 1$, $f(v_A) = (\ldots, \underbrace{1}_{i-1}, \underbrace{1}_{i}, \ldots)$, and $f(v_B) = (\ldots, \underbrace{d}_{i-2}, \underbrace{2}_{i-1}, \underbrace{2}_{i}, \ldots)$ (see Figure 4.5).

   (1.1) If $d = 1$, $F(v_A) = (\ldots, \underbrace{i}_{n-i}, \underbrace{i-1}_{n+1-j}, \ldots)$, $F(v_B) = (\ldots, \underbrace{i-1}_{n-i}, \underbrace{i-2}_{n+1-j}, \ldots)$, and $\sigma$ satisfies the expression,

   \[
   \sigma(r) = \begin{cases} 
   r, & \text{if } r \leq n + 1 - i, \\
   m, & \text{otherwise, } \quad (4.14)
   \end{cases}
   \]

   for some $m > n + 1 - i$. Applying $F$ to $v_{A+i}$ and $v_{B+i-1}$,

   \[
   F(v_{A+i}) = (\ldots, \underbrace{i-1}_{n-i}, \underbrace{i-1}_{n+1-j}, \ldots), \quad F(v_{B+i-1}) = (\ldots, \underbrace{i-2}_{n-i}, \underbrace{i-2}_{n+1-j}, \ldots),
   \]

   then there exits $\sigma' \in S_n$ such that $\sigma' = \sigma$ and $\sigma'(F(v_{A+i})) = F(v_{B+i-1})$ in $K_2$ (see Figure 4.6).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure45.png}
\caption{Dyck paths associated to $v_A$ and $v_B$ for $i > z$.}
\end{figure}

1.2) If $d = 2$, this case is equal to the previous case.

1.3) If $d = 3$, $A$ and $B$ do not generate the same triangulation.

   Note that, if $z = 1$, this case satisfies the condition (1.1) and (1.2) without $d$.

2. If $i = z \geq 2$, $f(v_A) = (\ldots, \underbrace{2}_{i-1}, \underbrace{1}_{i}, \ldots)$ and $f(v_B) = (\ldots, \underbrace{b}_{i-2}, \underbrace{a}_{i-1}, \underbrace{2}_{i}, \ldots)$ (see Figure 4.7).

   (2.1) If $a = 1$ and $b = 1$, $F(v_A) = (\ldots, \underbrace{i}_{n-i}, \ldots)$, $F(v_B) = (\ldots, \underbrace{i-1}_{n-i}, \underbrace{i-1}_{n+1-j}, \ldots)$ and $\sigma_1$ is equal to
Figure 4.6. Dyck paths associated to $v_{A+i}$ and $v_{B+i-1}$ for $i > z$.

\[
\sigma_1(r) = \begin{cases} 
  r, & \text{if } r \leq n - i, \\
  n + 1 - i, & \text{if } r = n, \\
  m, & \text{otherwise}, 
\end{cases}
\]

for some $m > n + 1 - i$. Applying $F$, we take

\[
F(v_{A+i}) = (\ldots, i - 1, \ldots) \text{ and } F(v_{B+i-1}) = (\ldots, i, i - 2, \ldots),
\]

then there exists $\sigma'_1 \in S_n$ that satisfies

\[
\sigma'_1(r) = \begin{cases} 
  n - i, & \text{if } r = n, \\
  n + 1 - i, & \text{if } r = n - i, \\
  \sigma_1(r), & \text{otherwise}, 
\end{cases}
\]

therefore $\sigma'_1(F(v_{A+i})) = F(v_{B+i-1})$ in $K_2$ (see Figure 4.8).

Figure 4.7. Dyck paths associated to $v_A$ and $v_B$ for $i = z$.

(2.2) If $a = 1$ and $b = 2$, this case satisfies the conditions of (2.1).
(2.3) If $a = 2$ and $b = 1$ or $b = 2$, these cases are contradictions.
(2.4) If $a = 2$ and $b = 3$, $F(v_B) = (\ldots, i - 1, \ldots)$ and $\sigma_2 = \sigma$. Applying $F$ to $v_{B+i-1}$,

it holds that $F(v_{B+i-1}) = (\ldots, i - 2, \ldots)$ then there exists $\sigma'_2 \in S_n$ such that

$\sigma'_2 = \sigma$ and $\sigma'_2(F(v_{A+i})) = F(v_{B+i-1})$ in $K_2$ (see Figure 4.8).
(2.5) If $a = 3$ and $b = 1, 2, 3$, these cases are equal to case (2.3).
Note that, if $z = 2$, this case satisfies the same conditions for $a = 1, 2$ without $b$.

(3) If $i = z - 1 \geq 3$, \( f(v_A) = (\ldots, \underbrace{2}_{i}, 1, \ldots) \) and \( f(v_B) = (\ldots, \underbrace{b}_{i-1}, \underbrace{a}_{i}, 2, \ldots) \) (see Figure 4.9).

(3.1) If $a = 1$ and $b = 1$, \( F(v_A) = (\ldots, \underbrace{i+1}_{n-i-1}, \ldots) \), \( F(v_B) = (\ldots, \underbrace{i}_{n-i}, \underbrace{i-1}_{n+1-i}, \ldots) \), and

\[
\sigma_3(r) = \begin{cases} 
  r, & \text{if } r \leq n - i - 1, \\
  n - i, & \text{if } r = n, \\
  n + 1 - i, & \text{if } r = n - 1, \\
  m, & \text{otherwise,}
\end{cases}
\]

for some $m > n + 1 - i$. Provided that

\[
F(v_{A+i}) = (\ldots, \underbrace{i-1}_{n-i-1}, \ldots) \quad \text{and} \quad F(v_{B+i-1}) = (\ldots, \underbrace{i}_{n-i}, \underbrace{i-2}_{n+1-i}, \ldots),
\]

then, there exists \( \sigma'_3 \in S_n \) such that

\[
\sigma'_3(r) = \begin{cases} 
  n - i - 1, & \text{if } r = n, \\
  n - i, & \text{if } r = n - 1, \\
  n + 1 - i, & \text{if } r = n - i - 1, \\
  \sigma_3(r), & \text{otherwise,}
\end{cases}
\]

then \( \sigma'_3(F(v_{A+i})) = F(v_{B+i-1}) \) in $K_2$ (see Figure 4.10).

(3.2) If $a = 1$ and $b = 2$. Applying $F$ to $v_B$, it holds that $F(v_B) = (\ldots, \underbrace{i}_{n-i-1}, \underbrace{i}_{n-i}, \ldots)$, \( \sigma_4 \) is described by

\[
\sigma_4(r) = \begin{cases} 
  r, & \text{if } r \leq n - i - 1, \\
  n - i, & \text{if } r = n, \\
  m, & \text{otherwise,}
\end{cases}
\]
for some \( m > n - i \). Applying \( F \) to \( v_{B+i-1} \), \( F(v_{B+i-1}) = (\ldots, i + 1, i - 2, \ldots) \), then there exist \( \sigma'_4 \) with

\[
\sigma'_4(r) = \begin{cases} 
  n - i - 1, & \text{if } r \leq n, \\
  n - i, & \text{if } r = n - i - 1, \\
  \sigma_4(r), & \text{otherwise},
\end{cases}
\]  

(4.20)

therefore \( \sigma'_4(F(v_{A+i})) = F(v_{B+i-1}) \) in \( K_2 \) (see Figure 4.10).

(3.3) If \( a = 2 \) and \( b = 3 \). \( F(v_B) = (\ldots, i, \ldots, n - i - 1) \), since \( \sigma_5 \) is

\[
\sigma_5(r) = \begin{cases} 
  r, & \text{if } r \leq n - i - 1, \\
  m, & \text{otherwise},
\end{cases}
\]  

(4.21)

for some \( m > n - i \). Applying \( F \) to \( v_{B+i-1} \), \( F(v_{B+i-1}) = (\ldots, i - 2, \ldots) \), and there exits \( \sigma'_5 = \sigma_5 \) such that \( \sigma'_5(F(v_{A+i})) = F(v_{B+i-1}) \) in \( K_2 \) (see Figure 4.10).

Same arguments are used for the remaining cases (see item (2) of this proof). \( \square \)

**Proposition 4.8.** Two positive integral diamonds of \( \mathcal{A}_n \) are in the same minimal \( p \)-cycle if their triangulations are in the same mutation class.

**Proof.** It is a direct consequence of Theorem 1.12, Proposition 4.3, Lemmas 4.4 and 4.5. \( \square \)
The following result shows a way to build frieze patterns.

**Theorem 4.3.** Let $A^0$ be a positive integral diamond of $A_n$ and let \( \{A^t\}_{0 \leq t \leq p-1} \) be the minimal $p$–cycle generated by $A^0$. Then:

(i) $A^0$ and $F(v_{A^0})$ generate the same frieze pattern.

(ii) \( \{A^t\}_{0 \leq t \leq p-1} \) is in surjective correspondence with a direct sum of $p$ indecomposable objects of a Dyck paths category.

**Proof.** Let $\mathbb{D}_{A_n}$ be the set of all vectors associated to positive integral diamonds of $A_n$, let $A^0$ be a positive integral diamond of $A_n$, and let \( \{A^t\}_{0 \leq t \leq p-1} \) be the minimal $p$–cycle generated by $A^0$.

(i) Let $K$ be a labeling of an $(n + 3)$ polygon, Theorem 4.2 implies that

$$F(v_{A^0}) = g((a_{01}^0, T_2(v_{A^0}), \ldots, T_n(v_{A^0}))) = g(\lambda(a_{01}^0, T_2(v_{A^0}), \ldots, T_n(v_{A^0}))) = g((\lambda_1, \ldots, \lambda_{n+1-a_{11}^0}, 0, \ldots, 0))$$

$$= ((l_1^{n+1-a_{11}^0}, l_{n+1-a_{11}^0}, l_n, \ldots, l_0),)$$

then, there are $a_{11}^0 - 1$ diagonals from the vertex 0 to other vertices, i.e., there are $a_{11}^0$ triangles incident with vertex 0. Proposition 4.8 allows us to establish that $a_{11}^0$ is the number of triangles incident with the vertex $i$, for $1 \leq i \leq n + 3$, $i = pm$ and $1 \leq m \leq p \mid (n + 3)$. Therefore $A^0$ and $F(v_{A^0})$ generate the same frieze pattern.

(ii) Let $(\mathbb{D}_{2(n+1)}, R)$ be any Dyck paths category, we take objects of $(\mathbb{D}_{2(n+1)}, R)$ defined by the following identity

$$\overline{Ob} (\mathbb{D}_{2n}, R) = \left\{ \bigoplus_{G_i \in \mathbb{D}_{2n}} G_i \mid g(\lambda_{G_i}) \text{ and } g(\lambda_{G_i}) \text{ are in the same mutation class} \right\},$$

we define the map $\varphi : \mathbb{D}_{A_n} \to \overline{Ob} (\mathbb{D}_{2n}, R)$, such that

$$\varphi(v_{A^0}) = f(v_{A^0}) \oplus \cdots \oplus f(v_{A^{p-1}}),$$

with \( \{A^t\}_{0 \leq t \leq p-1} \), $\varphi$ is surjective as a consequence of Theorem 4.2 and Proposition 4.8.

For example, let $D$ be an object of any Dyck paths category $(\mathbb{D}_{2(n+1)}, R)$ shown in Figure 4.11.

![Figure 4.11. Examples of objects in a Dyck paths category.](image-url)
then, $D$ has associated the following frieze pattern

\[
\begin{align*}
&\ldots 1 1 1 1 1 \ldots \\
&\ldots 1 3 2 1 3 2 \ldots \\
&\ldots 2 5 1 2 5 \ldots \\
&\ldots 1 3 2 1 3 2 \ldots \\
&\ldots 1 1 1 1 1 \ldots
\end{align*}
\]

4.2 Dyck-Brauer Configuration Algebras

In this section, we present a Brauer configuration and its Brauer configuration algebra whose indecomposable projective modules are in bijective correspondence with Dyck paths.

4.2.1 Brauer Configuration and its Brauer Configuration Algebra Associated to Dyck Paths

For $n$ fixed, let $M_n = \{ \alpha_{n_1}^{n_2} \}_{0 \leq n_1 \leq n-1, n_1 < n_2 \leq n}$ and $N_n = \{ \beta_{n_1}^{n_2} \}_{0 \leq n_1 \leq n-1, n_1 < n_2 \leq n}$ be the sets of letters, we define an alphabet $\Gamma_n^0$ such that

\[ \Gamma_n^0 = \{ \delta \mid \delta \in M_n \text{ or } \delta \in N_n \}. \quad (4.24) \]

For $\Gamma_n^0$, we define a concatenation $c$ in the following way:

\[ c(\delta) = \begin{cases} 
\alpha_{j+1}^i, & \text{if } \delta = \alpha_j^i, \\
\beta_{j+1}^i, & \text{if } \delta = \alpha_j^i, \\
\alpha_{j+1}^{i+1}, & \text{if } \delta = \beta_j^i, \\
\beta_{j+1}^{i+1}, & \text{if } \delta = \beta_j^i,
\end{cases} \quad (4.25) \]

for some $t,i$ and $j$. We take the set of the words $V = \delta_1 \ldots \delta_{2n}$ where $\delta_1 = \alpha_0^1$ and $\delta_j = c(\delta_{j-1})$ (this set is denoted by $\Gamma_n^1$). We will say that $V <_O V'$ if and only if there exist $r \in \mathbb{Z}_{>0}$ such that

\[ N_{V'}^{r_g} = N_V^{r_g}, \text{ if } 0 < r_g < r, \]
\[ N_{V'}^{r_g} > N_V^{r_g}, \text{ if } r_g = r, \quad (4.26) \]

where $N_{V'}^{r_g}$ is the number of alpha words before of the $r_g$-beta word in $V$ ($\langle \Gamma_1, <_O \rangle$) is a linear order. For notation, the words $V$ in $\Gamma_1$ are labeling respect to $<_O$, i.e., $V_1 = \alpha_0^1 \ldots \alpha_0^{n-1} \beta_0^n \ldots \beta_{n-1}$, $V_2 = \alpha_0^1 \ldots \alpha_0^{n-1} \beta_0^{n-1} \alpha_1^n \beta_2^n \ldots \beta_{n-1}$, etc.

Let $\Gamma_n = (\Gamma_n^0, \Gamma_n^1, \mu, O)$ be a Brauer configuration, where $\Gamma_n^0, \Gamma_n^1$ as before, $O$ is induced by $\langle \Gamma_1, <_O \rangle$, and the multiplicity function $\mu$ is defined as follows:
\[ \mu : \Gamma_0 \to \mathbb{N} \]
\[ \delta \mapsto \mu(\delta) = \begin{cases} 1, & \text{if } \text{val}(\delta) > 1, \\ 2, & \text{if } \text{val}(\delta) = 1, \end{cases} \]  
(4.27)

for \( \delta \in \Gamma_0 \). \( \Gamma_n \) is called a Dyck-Brauer configuration. The occurrence of \( \delta = \alpha^n_{i_1} \) (resp. \( \delta = \beta^n_{i_1} \)) in a polygon \( V \) is given by the row \( n(n_1) + n_2 \) in the recurrence matrix \( A_n = (a^n_{u,v}) \) (resp. \( B_n = (b^n_{u,v}) \)) of \( T_n \times f_{n,2} \) where \( A_1 = (1) \) (resp. \( B_1 = (1) \)) and \( A_n \) (resp. \( B_n \)) is described in Table A.5 (see identity A.2).

We define paths \( a_{j_1}^i \ldots a_{j_k}^i \) (resp. \( b_{j_1}^{j'} \ldots b_{j_k}^{j'} \)) where \( \{j_1, \ldots, j_k\} \) (resp. \( \{j'_1, \ldots, j'_k\} \)) are indices of the matrix \( A_n \) (resp. \( B_n \)) such that \( a_{i,j} = 1 \) (resp. \( b_{i,j'} = 1 \)) with \( j_r < j_{r+1} \) (resp. \( j'_r < j'_{r+1} \)) for \( 1 \leq j_r \) (resp. \( j'_r \)) \( \leq f_{n,2} \) and \( 1 \leq i \leq T_0 \). If \( \text{val}(\delta) = 1 \), the path is equal to \( a_{j_1}^i j_k^i \) (resp. \( b_{j_1}^{j'} j_k^{j'} \)). Paths \( a_{j_1}^i \ldots a_{j_k}^i \) and \( b_{j_1}^{j'} \ldots b_{j_k}^{j'} \) induce special \( \delta \)-cycles \( \nu_t \) at \( v_t \) in such a way that:

\[ \nu_t = \begin{cases} a_{i_1}^i \ldots a_{i_{t-1}}^i a_{i_1}^i \ldots a_{i_{t-1}}^i, & \text{if } a_{i_t}^i = 1 \text{ and } \delta \in M_n, \\ b_{j_1}^{j_1} \ldots b_{j_{t-1}}^{j_{t-1}} b_{j_1}^{j_1} \ldots b_{j_{t-1}}^{j_{t-1}}, & \text{if } b_{i_t}^i = 1 \text{ and } \delta \in N_n, \\ 0, & \text{otherwise}. \end{cases} \]  
(4.28)

In the same way, the relations in \( \rho_{\Gamma_n} \) are described by the following cases:

- Relations of type I.

\[ a_{j_1}^{i_1} \ldots a_{j_k}^{i_k} a_{j_1}^{i_1} \ldots a_{j_{s_1}-1}^{i_1} = \cdots = a_{j_{s_t}}^{i_t} \ldots a_{j_{s_{t-1}}-1}^{i_t}, \]
\[ a_{j_1}^{i_1} \ldots a_{j_k}^{i_k} a_{j_1}^{i_1} \ldots a_{j_{s_1}-1}^{i_1} = b_{j_1}^{j_1} \ldots b_{j_{s_1}}^{j_{s_1}} b_{j_1}^{j_1} \ldots b_{j_{s_1}-1}^{j_{s_1}}, \]  
(4.29)

if \( j_{s_1} = \ldots = j_{s_t} = j_{s_{1}}' = \ldots = j_{s_{t}}' \) for \( 1 \leq j_{s_1} \leq f_{n,2}, \{i_1, \ldots, i_t, i_1', \ldots, i_{h}'\} \in \{1, \ldots, T_n\} \) and \( t, h \in \mathbb{Z}_{>0} \).

- Relations of type II.

\[ a_{j_1}^{i_r} \ldots a_{j_k}^{i_r} a_{j_1}^{i_r} \ldots a_{j_{s_1}-1}^{i_r}, \]
\[ b_{j_1}^{i_r} \ldots b_{j_k}^{i_r} b_{j_1}^{i_r} \ldots b_{j_{s_1}-1}^{i_r}, \]  
(4.30)

for some \( i_r \in \{i_1, \ldots, i_t\} \) (\( i'_r \in \{i'_1, \ldots, i_{h}'\} \)).

- Relations of type III.

\[ a_{j_1}^{i_r} a_{j_2}^{i_r} a_{j_3}^{i_r} b_{j_4}^{i_r} b_{j_5}^{i_r} b_{j_6}^{i_r} a_{j_7}^{i_r} b_{j_8}^{i_r} b_{j_9}^{i_r} \]  
(4.31)

for all possible combinations.

\( I_{\Gamma_n} \) is generated by \( \rho_{\Gamma_n} \) and Dyck-Brauer configuration algebra \( \Lambda_{\Gamma_n} \) is defined by \( kQ_{\Gamma_n}/I_{\Gamma_n} \).

For instance, let \( \Gamma_2 = (\Gamma_0^2, \Gamma_1^2, \mu, \mathcal{O}) \) be the Dyck-Brauer configuration where \( \Gamma_0 = \).
\{\alpha_0, \beta_0, \alpha_1, \beta_1, \beta_2, \beta^2_2\} and \Gamma_1 = \{V_1 = \{\alpha_0^1, \beta_0^1, \beta_0^2\}, V_2 = \{\alpha_1^1, \beta_2^1, \beta_0^2\}\} with \ V_1 < \otimes V_2. Since the matrices \(A_2\) and \(B_2\) are
\[
A_2 = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
1 & 1
\end{pmatrix},
\]
then \(\mu_{\alpha_0} = \mu_{\beta_1} = 1\) and \(\mu(\delta) = 2\) for \(\delta \in \{\alpha_0^1, \alpha_1^2, \beta_0^2\}\). \(V_1 < \otimes V_2\) is the successor sequence of vertex \(\alpha_0^1\), \(V_2 < \otimes V_1\) is the successor sequence of vertex \(\beta_1^2\), \(V_1\) (resp. \(V_2\)) is the successor sequence of vertices \(\alpha_0^2\) and \(\beta_0^2\) (resp. \(\alpha_1^2\) and \(\beta_0^2\)). Figure 4.12 shows the quiver \(Q_{\Gamma_2}\).

![Figure 4.12. Quiver of Dyck-Brauer configuration algebra \(\Lambda_{\Gamma_2}\).](image)

Identities (4.29), (4.30), and (4.31) induce the following relations,
\[
a_1^1 a_2^1 = (a_1^2)^2 = (b_1^2)^2 = b_2^2 b_2^3, \quad a_2^1 a_1^1 = (a_2^2)^2 = (b_2^1)^2 = b_2^1 b_1^3,
a_1^1 a_2^2 a_1^1, (a_1^2)^3, (b_1^2)^3, b_1^2 b_3^1 b_4^1, a_2^2 a_1^2, (a_2^3)^3, (b_2^3)^3, b_2^3 b_3^1 b_2^3,
a_1^1 a_2^3, a_2^2 a_2^1, (b_2^3)^3, (b_2^1)^3, a_2^1 b_1^2, b_1^1 b_1^2, b_1^2 b_2^4, b_1^3 b_1^2, b_1^3 b_1^2,
a_1^1 b_1^3, a_1^3 b_1^2, a_1^3 b_3^3, a_2^1 b_1^3, a_2^1 b_1^3, b_3^3 b_3^2, a_1^1 b_2^3, b_1^3 b_1^2, a_1^1 b_3^3,
b_2^3 a_2^1, b_1^3 a_1^2, b_1^3 a_2^1, b_2^3 a_1^2, b_2^3 a_1^2, b_1^3 a_1^2, b_1^3 a_1^2.
\]

(4.32)

Dyck-Brauer configuration algebra \(\Lambda_{\Gamma_2}\) is defined in such a way that \(\Lambda_{\Gamma_2} = kQ_{\Gamma_2}/I_{\Gamma_2}\) with \(I_{\Gamma_2} = \langle p_{\Gamma_2} \rangle\). Figure 4.13 shows the indecomposable projective \(\Lambda_{\Gamma_2}\)-modules.

![Figure 4.13. Indecomposable projective \(\Lambda_{\Gamma_2}\)-modules of Dyck-Brauer configuration algebra.](image)
4.2.2 Dimension of a Dyck-Brauer Configuration Algebra and the Dimension of its Center

We define a family of integer sequences that are in relationship with the Catalan triangle. We also show that the Catalan triangle allows us to establish the dimension of Dyck-Brauer configuration algebras and the dimension of its corresponding centers.

Let \( t_{i,j}^n \) be the integer numbers such that

\[
\begin{align*}
  t_{i,j}^1 &= 1, \\
  t_{i,j}^n &= \sum_{\substack{r - s = i - j \\
n - 1 \leq r \leq i}} t_{r,s}^{n-1} \quad \text{if } n > 1, \\
  t_{i,j}^n &= 0 \quad \text{if } j \leq 0,
\end{align*}
\]

for \( i \geq n - 1 \) and \( j \leq i + 1 \). For example, Table A.4 shows integer sequences \( t_{i,j}^n \) for \( n = 2, \ldots, 5 \) (see Appendix).

The following results describe some properties of the integer numbers \( t_{i,j}^n \) and the Catalan triangle.

**Proposition 4.9.** \( t_{i,j}^n = t_{i,j-1}^n + t_{i-1,j}^{n-1} \), for \( i \geq 1 \), \( j \leq i \), and \( 1 < n < i + 1 \).

**Proof.** By induction. If \( i = 1 \), \( t_{1,1}^2 = t_{0,1}^1 + t_{1,0}^1 = 1 \). Suppose that the proposition holds for \( i = k \) and \( 1 < n < k + 1 \). Then for \( i = k + 1 \), if \( n = 2 \),

\[
t_{k+1,j}^2 = \sum_{\substack{k+1-j=r-s \\
1 \leq r \leq k+1}} t_{r,s}^1 = (k-j) + 1 = \sum_{\substack{k-j=r-s \\
1 \leq r \leq k+1}} t_{r,s}^1 + t_{k,j}^1 = t_{k+1,j-1}^2 + t_{k,j}^1,
\]

suppose that the assertion is true for \( n = p - 1 < k + 1 \), then

\[
t_{k+1,j}^p = \sum_{\substack{k+1-j=r-s \\
p-1 \leq r \leq k+1}} t_{r,s}^{p-1} = \sum_{\substack{k+2-j=r-s \\
p-1 \leq r \leq k+1}} t_{r,s}^{p-1} + \sum_{\substack{k-j=r-s \\
p-2 \leq r \leq k}} t_{r,s}^{p-2} = t_{k+1,j-1}^p + t_{k,j}^{p-1}.
\]

**Proposition 4.10.** \( t_{i,i+1}^n = t_{i,i}^n \), for \( i \geq 1 \) and \( 1 < n \leq i + 1 \).

**Proof.** We proceed by induction. If \( i = 1 \), \( t_{1,2}^2 = 1 = t_{1,1}^2 \). Suppose that the assertion is true for \( i = k \) and \( 1 < n < k + 1 \). Then for \( i = k + 1 \), if \( n = 2 \),

\[
t_{k+1,k+2}^2 = \sum_{\substack{-1=r-s \\
1 \leq r \leq k+1}} t_{r,s}^1 = \sum_{\substack{0=r-s \\
1 \leq r \leq k+1}} t_{r,s}^1 = t_{k+1,k+1}^2,
\]
suppose that the property is true for \( n = p - 1 < k + 2 \), then

\[
\sum_{1 \leq r \leq p-1} t_{r,s}^{p-1} = \sum_{p-1 \leq r \leq k+1} t_{r,s}^{p-1} = t_{k+1,k+1}^{p}.
\]

Lemma 4.6. \( f_{n,n+1-m} = \sum_{j=1}^{n} t_{n-1,j}^{m} \), for \( n \geq 1 \), \( 1 \leq m \leq n \).

**Proof.** By induction. If \( n = 1 \), \( f_{1,1} = 1 = t_{0,1}^{1} \). Suppose that the assertion is true for \( n = k \) and \( 1 \leq m \leq k \). Then for \( n = k + 1 \), if \( m = 1 \), \( \sum_{j=1}^{k+1} t_{k,j}^{1} = k + 1 = f_{k+1,k+1} \), suppose that the property holds for \( m = p - 1 < k + 1 \), then

\[
\sum_{j=1}^{k+1} t_{k,j}^{p} = \sum_{r-s=k-1}^{p-1 \leq r \leq k} t_{r,s}^{p-1} + \sum_{r-s=1}^{p-1 \leq r \leq k-1} t_{r,s}^{p-1} + \sum_{j=1}^{k+1} t_{k,j}^{p-1}
\]

= \( f_{k,k+1-p} + f_{k+1,k+3-p} \)

= \( f_{k,k+1-p} + \sum_{i=k+2-p}^{k+1} f_{k,i} \)

= \( \sum_{i=k+1-p}^{k+1} f_{k,i} \)

Proposition 4.11. \( f_{n,n+1-m} = t_{n,n+1}^{1+m} \), for \( n \geq 1 \) and \( 1 \leq m \leq n \).

**Proof.** We proceed by induction. If \( n = 1 \), \( f_{1,1} = 1 = t_{1,2}^{2} \). Suppose that the assertion is true for \( n = k \) and \( 1 \leq m \leq k \). Then for \( n = k + 1 \), if \( m = 1 \),

\[
f_{k+1,k+1} = k + 1 = \sum_{1 \leq r-s \leq k+1} t_{r,s}^{1} = t_{k+1,k+2}^{2}.
\]

suppose that the property holds for \( m = p - 1 < k + 1 \), then
Given a matrix $C = (c_{i,j})$ of $n \times m$, $M(C)$ is the column vector 
\[
\begin{pmatrix}
c_1 \\
\vdots \\
c_n
\end{pmatrix}
\]
where $c_i = \sum_{j=1}^{m} c_{i,j}$.

Also, $M(C)_r$ is the column vector 
\[
\begin{pmatrix}
c'_1 \\
\vdots \\
c'_n
\end{pmatrix}
\]
such that $c'_j = \begin{cases} 0, & \text{if } 1 \leq j \leq r, \\
c_j, & \text{if } r+1 \leq j \leq n,
\end{cases}$ and

$\overline{M(C)}_r$ is the column vector 
\[
\begin{pmatrix}
c_1 \\
\vdots \\
c_n
\end{pmatrix}
\]
with $\overline{c}_j = \begin{cases} 0, & \text{if } 1 \leq j \leq r, \\
c_{j-r}, & \text{if } r+1 \leq j \leq n.
\end{cases}$

For example,
\[
M(A_2)_0 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad M(A_2)_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and } \overline{M(A_2)}_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.
\] (4.34)

Henceforth, we introduce a formula for the valency of the vertices of Dyck-Brauer configuration $\Gamma_n$ via the vectors $M(A_n)$ and $M(B_n)$, in the following way:

**Lemma 4.7.** Let $A_n$ and $B_n$ be the matrices given by Table A.5. Then

(i) $M(A_n^{i,f_{n,i+1}}) = \begin{cases} M(A_n), & \text{if } i = 0, \\
M(A_n^{i-1,f_{n,i}}), & \text{if } i = 1, \\
M(A_n^{i-1,f_{n,i}}) + M(A_n^{i-2,f_{n-1,i-1}})_{i}, & \text{if } 2 \leq i \leq n-1,
\end{cases}$

(ii) $M(B_n^{i,f_{n,i+1}}) = \begin{cases} M(B_n), & \text{if } i = 0 \text{ or } i = -1, \\
M(B_n^{i-1,f_{n,i+1}}) + M(B_n^{i-2,f_{n-1,i}})_{i}, & \text{if } 1 \leq i \leq n-1,
\end{cases}$

for $n > 0$.

**Proof.** (i) Let $A_n = (a_{n,u})$ be the matrix of $T_n \times f_{n,2}$, and let $A_n^{i,f_{n,i+1}} = (a_{n,u,w})$ be the matrix of $T_n \times f_{n,i+1}$ such that satisfying identity A.2

(1.1) If $i = 0$, $A_n^{0,f_{n,1}}$ is a matrix with $f_{n,1}$ columns, since $f_{n,1} = f_{n,2}$, $a_{n,u,w} = a_{n,u}^{n,0}$, then $a_{n,u}^{n,0} = a_u$ for $1 \leq u \leq T_n$. 

\[f_{k+1,k+2-p} = \sum_{j=1}^{k+1} t_{k,j}^p
= \sum_{j=1}^{k+1} \sum_{r=1}^{p-1} \sum_{s=1}^{r-1} \sum_{t=1}^{r-2} \cdots \sum_{v=1}^{r-k+1} t_{r,s}^{p-1}
= \sum_{j=1}^{k+1} \sum_{j=1}^{p-1} t_{j,j}^{p-1} + \sum_{j=1}^{p-1} t_{j,j}^{p-1}
= f_{k+1,k+1-p} + \cdots + f_{p,1} \quad \text{(Lemma 4.6)}
= t_{p,p}^{k+1} + \cdots + t_{p,p+1}
= t_{k+1,k+2}.
\]
(1.2) If \( i = 1 \), \( A_n^{1,f_{n,2}} \) is a matrix with \( f_{n,2} \) columns, \( a_{u,w}^{1} = a_{u,w}^{n} \) for \( 2 \leq u \leq T_n \), i.e., \( a_u^{n_1} = a_u \) for \( 2 \leq u \leq T_n \), and \( a_1^{n_1} = 0 \).

(1.3) If \( 2 \leq i \leq n - 1 \), \( A_n^{i,f_{n,i+1}} \) is a matrix with \( f_{n,i+1} \) columns, this matrix is equal to matrix \( A_n^{i-1,f_{n,i}} \) by removing the row \( i - 1 \) and the columns between \( f_{n,i+1} \) and \( f_{n,i} \); matrix \( A_n \) implies that in these columns the elements between the rows \( n + 1 \) and \( T_n \) are given by the matrix \( A_n^{i-2,f_{n-1,i-1}} \), i.e.,

\[
a_u^{n,i} = \sum_{j=1}^{f_{n,i}} a_{u,j}^{n,i-1} - \sum_{j=1}^{f_{n-1,i-1}} a_{u-n,j}^{n-1,i-2} = a_u^{n,i-1} - a_u^{n-1,i-2},
\]

for \( n + 1 \leq u \leq T_n \) and \( a_u^{n,i} = 0 \) otherwise.

The case (ii) is similar to case (i).

\[ \square \]

**Proposition 4.12.** Let \( A_n \) and \( B_n \) be the matrices given by Table A.5. Then:

(i) \( a_u^{n,i} = f_{k_1,u-(T_n-T_k)}^{n+1-k} n-k+u-(T_n-T_k+1), n-k+u-(T_n-T_k+1)-(i-1), \) for \( 0 \leq i \leq n - 1 \),

(ii) \( b_u^{n,i} = f_{k-1,u-(T_n-T_k)}^{n+1-k} n-k+u-(T_n-T_k+1), n-k+u-(T_n-T_k+1)-(i-1), \) for \( -1 \leq i \leq n - 1 \),

with \( T_n - T_k \leq u \leq T_n - T_{k-1} \) for \( 1 \leq k \leq n \), and \( n > 1 \).

**Proof.** (i) (Induction) For \( n = 2 \). If \( i = 0 \), by Lemma 4.7, \( M(A_2^{0,f_{2,1}}) = M(A_2) \), then

\[
\begin{align*}
a_1^{2,0} &= 2 = f_{2,2} t_{0,1}^1, \\
a_2^{2,0} &= 1 = f_{2,2} t_{1,2}^1, \\
a_3^{2,0} &= 1 = f_{2,2} t_{2,2}^1.
\end{align*}
\]

If \( i = 1 \), by Lemma 4.7, \( M(A_2^{1,f_{2,2}}) = M(A_2)_1 \), then

\[
\begin{align*}
a_1^{2,1} &= 0 = f_{2,2} t_{0,0}^1, \\
a_2^{2,1} &= 1 = f_{2,2} t_{1,1}^1, \\
a_3^{2,1} &= 1 = f_{2,2} t_{2,1}^1.
\end{align*}
\]

(see identity 4.34). Suppose that the assertion is true for \( n = m \) and \( 0 \leq i \leq m \). Then for \( n = m + 1 \), if \( i = 0 \), \( M(A_m^{0,f_{m+1,1}}) = M(A_{m+1}) \) (Lemma 4.7), if \( 0 < u < m + 1 \),

\[
a_u^{m+1,0} = f_{m+1,i+u} = f_{m+1,i+u} t_{u-1,u}^1.
\]

For the rows between \( m + 2 \) and \( T_{m+1} \),

\[
a_u^{m+1,0} = \sum_{i=0}^{m-1} f_{k_1,u-(m+1)-(T_m-T_k)}^{l_{m+1}-k} m-k+u-(m+1)-(T_m-T_{k-1}+1), m-k+u-(m+1)-(T_m-T_{k-1}+1)-(i-1)
\]

\[
= f_{k_1,u-(T_m+1-T_k)}^{l_{m+2}-k} m-k+u-(T_m+1-T_k)+u-(T_m+1-T_k) \quad \text{(Lemma 4.6)}
\]

\[
= f_{k_1,u-(T_m+1-T_k)}^{l_{m+2}-k} m-k+u-(T_m+1-T_k), m-k+u-(T_m+1-T_k)+1 \quad \text{(Proposition 4.11)}.
\]
with \(1 \leq k \leq m\).

If \(i = 1\), \(M(A_{m+1}^{1,f_{m+1}^1}) = M(A_{m+1}^{0,f_{m+1}^1})_1\) (Lemma 4.7), then, \(a_{1}^{m+1,1} = 0 = f_{m+1,2}^1 t_{0,0}^1\), and

\[
a_{u}^{m+1,p} = a_{u}^{m+1,0} = f_{k,1+u-(T_{m+1}-T_k)} t_{m-k+u-(T_{m+1}-T_k),u-k+u-(T_{m+1}-T_k)+1}^{m+2-k}
\]

(Proposition 4.10),

for \(T_{m+1}-T_k < u \leq T_{m+1}-T_{k-1}\) with \(u \neq 1\), and \(1 \leq k \leq m + 1\).

Suppose that the property is true for \(i = p - 1 < m\), then for \(i = p\), then \(M = (A_{m+1}^{p,f_{m+1}^{p-1}}) = M(A_{m+1}^{p-1,f_{m+1}^{p-1}})_p + M(A_{m+1}^{p-2,f_{m+1}^{p-2}})_m\) (Lemma 4.7), for \(1 \leq u \leq p\)

\[
a_{u}^{m+1,p} = 0 = f_{m+1,1+u} t_{u-1,u-p}^1,
\]

for \(p + 1 \leq u \leq m + 1\)

\[
a_{u}^{m+1,p} = a_{u}^{m+1,p-1} = f_{m+1,1+u} t_{u-1,u-p}^1 = f_{m+1,1+u} t_{u-1,u-p}^1,
\]

for \(u \geq m + 1\),

\[
a_{u}^{m+1,p} = f_{k,1+u-(T_{m+1}-T_k)} t_{m-k+u-(T_{m+1}-T_k),u-k+u-(T_{m+1}-T_k)+1}^{m+2-k} - f_{k,i+u-(m+1)-(T_{m+1}-T_k)} t_{m-k+u-(m+1)-(T_{m+1}-T_k),u-k+u-(m+1)-(T_{m+1}-T_k)+1}^{m+2-k}
\]

(Proposition 4.9 implies that)

\[
a_{u}^{m+1,p} = f_{k,i+u-(m+1)-(T_{m+1}-T_k)} t_{m-k+u-(T_{m+1}-T_k),u-k+u-(T_{m+1}-T_k)+1}^{m+2-k}
\]

(Proposition 4.9 implies that)

\[
a_{u}^{m+1,p} = f_{k,i+u-(m+1)-(T_{m+1}-T_k)} t_{m-k+u-(T_{m+1}-T_k),u-k+u-(T_{m+1}-T_k)+1}^{m+2-k}
\]

for \(1 \leq k \leq m\). The case \((ii)\) is similar to case \((i)\). \(\square\)

For notation \(\omega : \mathbb{N} \to \{1, 2\}\) is a map where

\[
\omega(n) = \begin{cases} 
1, & \text{if } n \neq 2, \\
2, & \text{if } n = 2.
\end{cases}
\]

The following result regards dimension of \(\Lambda_{\Gamma_n}\) and its corresponding center.

**Theorem 4.4.** Let \(\Lambda_{\Gamma_n}\) be a Dyck-Brauer configuration algebra. Then

\[
(i) \quad \dim_k(\Lambda_{\Gamma_n}) = 2(C_n + \omega(n)) + \sum_{u=1}^{T_n} \left(a_{u}^{n,0} \right)^2 + \left(b_{u}^{n,0} \right)^2 - \left(a_{u}^{n,0} + b_{u}^{n,0} \right),
\]

\[
(ii) \quad \dim_k(Z(\Lambda_{\Gamma_n})) = 1 + 2\omega(n) + C_n,
\]

for \(n > 0\).

**Proof.** (i) Firstly, we note that the number of vertices in \(Q_{\Gamma_n}\) is the \(n\)-th Catalan number. Secondly, we note that \(val(a_{n_1}^{n_2})\) (resp. \(val(\beta_{n_1}^{n_2})\)) is given by \(a_{n_1+n_2}^{n,0}\) (resp.
As a consequence of Proposition 4.12, we have that $a_n^{n,0} = b_n^{0,n}$ for any $n$. In particular case, when $n = 2$, also, $a_2^{0,1} = b_2^{0,1} = 1$. Finally, recall that identity 4.27 describes the multiplicity function. (ii) The number of loops in $Q_{Γ_n}$ is equal to the number of elements in the set $C_{Γ_n}$. □

For example, for $n = 2$, $a_1^{2,0} = 2 = b_3^{2,0}$, and $a_2^{2,0} = a_3^{2,0} = b_4^{2,0} = b_2^{2,0} = 1$, then

$$\dim_k(Λ_{Γ_2}) = 2(C_2 + ω(2)) + 12 - 8 = 2(2 + 2) + 4 = 12,$$

and

$$\dim(Z(Λ_{Γ_2})) = 1 + 2ω(2) + C_2 = 7.$$
APPENDIX A

Appendix

| $n \backslash m$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 2             | 0  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 3             | 0  | 1  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 4             | 0  | 1  | 2  | 3  |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 5             | 0  | 1  | 2  | 4  | 4  | 5  | 6  | 7  |    |    |    |    |    |    |    |    |    |    |
| 6             | 0  | 1  | 2  | 5  | 4  | 6  | 8  | 11 | 8  | 9  | 10  | 12 | 12 | 13 | 14 | 15 |    |    |
| 7             | 0  | 1  | 2  | 6  | 4  | 7  | 10 | 16 | 8  | 10 | 12  | 17 | 16 | 19 | 22 | 26 | 16 | 17 | 18 |

Table A.1. Elements of the sequence $C^m_n$.

<table>
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<tr>
<th>$n \backslash r$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>142</td>
<td>120</td>
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</table>

Table A.2. Number of sections in the case $\mathcal{A}_n$.

Table A.2. The number $S_{(\mathcal{A}_n)}^i_{(o_j, j_m)}$ (see, formula 2.4 and Remark 2.5) of sections in the Auslander-Reiten quiver of the path algebra $k\overrightarrow{\mathcal{A}_n}$ where $\overrightarrow{\mathcal{A}_n}$ is an oriented Dynkin diagram of type $\mathcal{A}_n$ with $i_r < n$ sinks.
Table A.3. Number of sections in the case $D_n$.

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<td>668</td>
<td>671</td>
<td>670</td>
<td>643</td>
<td>588</td>
</tr>
</tbody>
</table>

Table A.4. Examples of integer numbers $t_{i,j}$.

\[
\begin{align*}
\text{rows} & = \frac{\sum_{t=w-r+2}^{w} 2^{j_{w}-1} - \sum_{t=1}^{r-1} 2^{i_{w}-1}}{2}, & \text{if } j_{w} > 1, \\
0, & \quad & \text{if } j_{w} = 0.
\end{align*}
\]
Table A.5. Matrix $A_n$ (left) and matrix $B_n$ (right).

Table A.5 $A_{n}^{i,f(n,i+1)} = (a'_{u,w})$ (resp. $B_{n}^{i,f(n,i+2)} = (b'_{u,w})$) is a matrix of $T_n \times f_{n,i+1}$ (resp. $T_n \times f_{n,i+2}$) such that

$$a_{u,w}^{n,i} = \begin{cases} a_{u,w} & \text{if } u > i, \\ 0 & \text{if } u \leq i, \end{cases}$$

(A.2)
for $0 \leq i \leq n - 1$ (resp. $-1 \leq i \leq n - 1$), $A_n = (a_{u,w})_{T_n \times f_{n,2}}$ (resp. $B_n = (b_{u,w})_{T_n \times f_{n,2}}$).

The number $f_{i,j}$ belongs to the triangle described in (4.9) with $f_{0,1} = 1$. 
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